Mathematics. — "*Determination of the System (2, 1, 1,) of* ∞^3 *Line Elements of Space*". By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of April 24, 1926).

§ 1. In my paper: "*Determination of the Bilinear System of* ∞^3 *Line Elements of Space*" [1]) the system (1, 1, 2) of ∞^3 line elements of space has been examined. In this paper we shall treat the system (2, 1, 1) of ∞^3 line elements (P, l) of space. For this system S_3 the order of the complex of the lines l is two; it contains one line element for which P lies in an arbitrary given point of space and the points P of the line elements of S_3 which belong to an arbitrary given plane, form a straight line r.

§ 2. The lines l which define line elements of S_3 with the points P of a given line m, form a scroll which has m as a single directrix. A plane through m contains one generatrix of this scroll, the line which, as line l, corresponds to the point P of m which lies in the point of intersection of the line r of the said plane with m.

The lines l which correspond to the points P of a line m, form, therefore, a quadratic scroll μ^2.

The locus of the points P of the lines of a complex cone $\varkappa$ of the quadratic complex C^2 of the lines l of S_3, is a curve which cuts a plane through the vertex T of $\varkappa$ in two points outside T, the points P of the two generatrices l of $\varkappa$ in this plane. Besides this curve passes through T where it touches the line l which forms a line element of S_3 with T.

Accordingly the locus of the points P of the generatrices l of a complex cone $\varkappa$ of C^2 is a twisted cubic which passes through the vertex of $\varkappa$.

The lines l of C^2 which cut two given lines m and n, form a biquadratic scroll λ^4 of the genus one, which has m and n as double directrices. The locus of the points P of the lines l of λ^4 cuts a plane through m outside m in the points P of the two generatrices l of λ^4 in this plane. There are two lines l of λ^4 for which the corresponding point P lies in m; they are the two generatrices of the scroll μ^2 corresponding to m which cut n.

Consequently the points P of the lines l of C^2 which cut the lines m and n, form a biquadratic twisted curve k^4 of the genus one which has m and n as chords.

The lines l of C^2 which cut a line m, form a congruence $M(2, 2)$. The locus of the points P of the lines l of a congruence $M(2, 2)$ is a

[1]) These Proceedings, **27**, 2.

surface which contains m, because among the lines which pass through a point of m, there is one which has its point P in the said point. To the points P where this surface cuts an arbitrary line m', there correspond as lines l the generatrices which cut m of the scroll μ'^2 corresponding to m'.

Accordingly to a congruence $M(2, 2)$ of lines of C^2 which cut a given line m, there corresponds a quadratic surface Ω^2 of points P which contains m.

§ 3. A curve of singular points P would lie on all surfaces Ω^2, as one or more of the ∞^1 lines l associated to a point of this curve, would cut an arbitrary line m. And the intersection of two surfaces Ω^2 is formed by the curve k^4 of the points P of the lines l of C^2 which cut the two lines to which the said surfaces Ω^2 correspond.

Accordingly the system S_3 does not contain any curve of singular points P.

Two of the six points of intersection of the curve k^3 associated to a complex cone $\varkappa$ and the quadratic surface corresponding to a line m, lie in the points P of the lines l of $\varkappa$ which cut m. The other four, to which there must correspond a straight line of $\varkappa$ as well as a line outside $\varkappa$ cutting m, are necessarily singular. As S_3 does not contain any singular curve, any curve k^3 and any surface Ω^2 have the same four singular points in common. Through these four points $H_1, \ldots, H_4$ there pass any surface Ω^2, any curve k^3 and consequently any complex cone $\varkappa$. Accordingly the points $H_1, \ldots, H_4$ are cardinal points of the complex C^2 and also of S_3, as each of these points corresponds as point P to any line l through it.

In a plane through three of the points H there lie three pencils of lines l of C^2, which, therefore, form a curve of the third class. As the lines of C^2 in an arbitrary plane envelop a conic, all the lines of a plane through three points H must belong to C^2 so that the sides of the tetrahedron $H_1 H_2 H_3 H_4$ are cardinal planes of C^2. The quadratic complexes through the congruence (4, 4) consisting of the four sheaves $H_1, \ldots, H_4$ and the four fields in the sides of the tetrahedron $H_1 H_2 H_3 H_4$, form a pencil of which one individual also contains a line l given at random. Consequently C^2 must be the tetrahedral complex of which $H_1 H_2 H_3 H_4$ is the tetrahedron of singularities and which contains a line of C^2 that does not pass through any of the angular points of the said tetrahedron and does not lie in any of the sides either.

Accordingly the complex C^2 of the lines of S_3 is a tetrahedral complex. The four cardinal points of this complex are also cardinal points of S_3.

§ 4. Of a complex cone $\varkappa$ of C^2 with vertex T we consider two generatrices l_1 and l_2 to which resp. the points P_1 and P_2 correspond in S_3. The curve k^3 of the points P of the lines of $\varkappa$ passes through

$T, P_1, P_2, H_1, H_2, H_3$ and H_4. If we consider the projective correspondence between the sheaves P_1 and P_2 in which to the lines $P_1 H_1, \ldots, P_1 H_4$ there correspond resp. the lines $P_2 H_1, \ldots, P_2 H_4$, k^3 is the locus of the points of intersection of lines associated to each other in this correspondence. To the line l_1 through P_1 the line l_2 through P_2 is conjugated as these cut each other in the point T of k^3. This also holds good if P_2 lies for instance in T. In this case to the line $l_1 \equiv P_1 T$, the tangent to k^3 at T is associated which line l_2 corresponds to the point P_2 in T.

Let us now choose two points P_1 and P_2 so that the lines l_1 and l_2 do not cut each other. If P describes the line l_1, l describes a scroll with l_1 not only as directrix but also as generatrix, i. e. the system of tangents to a complex conic of C^2 in a plane through l_1, to which plane, accordingly, l_1 is associated as line r. Through the point of intersection of l_2 and this plane we draw a tangent l_3 to the said complex conic. To l_3 there corresponds in S_3 the point of intersection P_3 of l_3 and l_1. In the projective correspondence between the sheaves P_1 and P_3 which is defined by the four pairs of rays $(P_1 H_i, P_3 H_i)$, the line l_3 corresponds to l_1 according to the above. In the same way the line l_2 corresponds to l_3 in the projective correspondence between the sheaves P_3 and P_2 which is defined by the four pairs $(P_3 H_i, P_2 H_i)$. Hence in the projective correspondence between the sheaves P_1 and P_2 which is defined by the four pairs $(P_1 H_i, P_2 H_i)$, the line l_2 corresponds to l_1.

Accordingly a system S_3 (2, 1, 1) of ∞^3 line elements may always be produced by the aid of four given points $H_1, \ldots, H_4$ and a given line element (P_1, l_1) by associating to each point P of space the line l through P which in the projective correspondence between the sheaves P_1 and P defined by the four pairs $(P_1 H_i, P H_i)$, corresponds to l_1.

Instead of a line element (P_1, l_1) of S_3 we may also choose five rays $s_1, \ldots, s_5$ which pass through an arbitrary point S, and we may associate to any point P the line l which in the projective correspondence between the sheaves S and P defined by the four pairs (s_i, PH_i) corresponds to s_5.

§ 5. On the line l_1 of the line element $(P_1 l_1)$ belonging to S_3, we choose a point Q_1 different from P_1. We consider the collinear transformation (P,Q) which has its coincidences in the points $H_1, \ldots, H_4$, and in which the point Q_1 corresponds to P_1. The join l of the points P and Q corresponding to each other in this transformation, form a tetrahedral complex which has its cardinal points in $H_1, \ldots, H_4$ and contains l_1, and which is, therefore, identical with C^2. If to any line l of C^2 we associate the point P of the pair (P, Q) on l, there arises a system (2, 1, 1) of ∞^3 line elements which has the cardinal points $H_1, \ldots, H_4$ and the line element $(P_1 l_1)$ in common with S_3 and which is, therefore, identical with S_3.

A system S_3 (2, 1, 1) of ∞^3 line elements can always be derived from a collinear transformation (P, Q) of space if to any point P the join l of P and the point Q corresponding to P in the transformation, is associated.

This leads to the well known theorem [1]), that the quintets of rays which join the points P of space with the coincidences of a collinear correspondence and with the points Q corresponding to P, can be transformed into each other by collinear transformations.

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[1]) STURM, Liniengeometrie, I. p. 354.