Mathematics. — *"A Special Congruence of Twisted Cubics".* By Prof. JAN DE VRIES.

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1. The twisted cubics ϱ^3 through the points A_1, A_2 and A_3 which have the straight line b as bisecant and the straight lines c_1, c_2 as secants, form a congruence $[\varrho^3]$. Let the plane of the points A be indicated by α, the transits of b, c_1, c_2 through α by B, C_1, C_2.

To this congruence there belongs the **pencil of nodal curves** k^3 which have a node in B and which pass through A_1, A_2, A_3, C_1, C_2. This pencil contains two cuspidal curves with cusp B and five figures consisting of a k^2 and a straight line.

2. In order to arrive at a *representation* of $[\varrho^3]$, I assume two pencils (p) and (q) in one plane with vertices L and M, and I establish a projective correspondence between the point range (P) of c_1 and (p), and likewise between (Q) of c_2 and (q). As the image of the curve through P and Q I consider the point $R \equiv p\,q$.

If P^* and Q^* are associated to the rays LM and ML, the ϱ^3 defined by them is represented on the point range of LM; this curve is, therefore, *singular*.

All the curves through P^* have their image in M, all the curves through Q^* are represented in L. Hence L and M are *singular points* for the representation.

If p_0 and q_0 are associated to the points C_1 and C_2, their point of intersection S_0 is the image of any curve k^3 (§ 1). Accordingly also S_0 is *singular*.

Let α_1 be the plane through A_1 and b, A' the transit of a_1 $(A_2 A_3)$, C'_1, (C'_2) the transit of c_1 (c_2). Each conic ϱ^2 through A_1, A', C'_1, C'_2 forms with a_1 a configuration belonging to $[\varrho^3]$; all these ϱ^3 have their images in the point of intersection S_1 of the rays p', q' corresponding to C'_1, C_2'. The planes α_2 and α_3 contain analogous systems. Also the points S_1, S_2 and S_3 are *singular*.

3. The surface Γ formed by the ϱ^3 which rest on a line c_3, is represented on a curve γ of which we shall determine the order.

Let P be a point of c_1. Through P and the points A there pass four ϱ^3 which cut b twice and which rest on c_2 and c_3. [1]).

[1]) This well known number may be found in the following way by applying the principle of the conservation of the number. If the four points lie in a plane φ, in the first place the curve k^3 satisfies the conditions which has the transit of b as double point and which rests on the two secants. Further the 3 given lines define a hyperboloid which has three more points in common with the conic in φ passing through the given points and resting on b. With the straight lines of the scroll through the said points of intersection this k^2 forms three configurations ρ^3. Hence in the notation of SCHUBERT $P^4B\nu^2 = 4$.

Accordingly by Γ four points Q of c_2 are associated to P; hence a ray p cuts γ in four points outside L. In the same way four rays p are associated to the ray ML; hence γ has a quadruple point in L. It may, therefore, be indicated by the symbol γ^8 (L^4, M^4).

If we replace c_3 by a line c_3^*, hence γ by γ^*, the points of intersection of the two curves outside the singular points are the images of the curves ϱ^3 which rest on c_3 and on c_3^*. Accordingly through A_1, A_2, A_3 there pass 28 curves which have b as bisecant and which cut four given straight lines; $P^3 B \nu^4 = 28$. Consequently the *surface* Γ is of the *degree* 28.

4. The congruence contains still other composite configurations:

a. Any conic β^2 in α through A_1, A_2, A_3, B forms a ϱ^3 with any of the three lines r (PQ) which rest on β^2, b, c_1 and c_2. The lines r form a quadratic scroll $(r)^2$; to each r there corresponds one β^2. As (P) and (Q) are projective this system has as *image* the *conic* through the *singular* points L, M, S_1, S_2 and S_3.

b. The conic γ_1^2, $(A_1 A_2 A_3 B C_1)$ is completed to configurations ϱ^3 by any transversal r of b and c_2 which rests on it. These lines form a cubic scroll $(r)^3$ of which b is the double directrix. The system (γ_1^2, r) has as *image* the *point range* on p_0 (§ 2).

Analogously q_0 is the *image* of the system $(\gamma_2{}^2, r)$.

c. The conics $\delta_{1,1}^2$ through A_2 and A_3 which rest on b, on c_1, and on the transversal $t_{1,2}$ of b and c_2 through A_1, form a cubic dimonoid with double points A_2 and A_3. The line $t_{1,2}$ forms a ϱ^3 with any $\delta_{1,1}^2$. This system has as *image* the *point range* on the ray $q_{1,2} \equiv MS_1$, which corresponds to the point of intersection of c_2 and $t_{1,2}$.

There are five analogous systems; for we may interchange c_1 and c_2 and replace A_1 by A_2 or A_3.

The *images* of these systems are the *point ranges* on rays which may be indicated by $p_{1,1}$, $p_{2,1}$, $p_{3,1}$ and $q_{2,2}$, $q_{3,2}$.

d. The conics $\varepsilon_1{}^2$ through A_2 and A_3 which cut b, c_1 and c_2, form a dimonoid of the fourth degree with triple points A_2, A_3. To each $\varepsilon_1{}^2$ the transversal t_1 through A_1 of b and $\varepsilon_1{}^2$ is associated. Any ray of the plane pencil (t_1) corresponds to three conics. As again (P) and (Q) are projective, the system $(\varepsilon_1{}^2, t_1)$ has as *image* a *conic* through L, M and S_0.

Analogously there are the systems $(\varepsilon_2{}^2, t_2)$ and $(\varepsilon_3{}^2, t_3)$.

5. The degree of the surface Γ (§ 3) may also be determined by the aid of the intersection of Γ with α. This contains in the first place the curve k^3 which rests on c_3 (§ 1). Further the lines a_1, a_2, a_3 (§ 2), a conic β^2 (§ 4, *a*), which is triple, a $\gamma_1{}^2$ and a $\gamma_2{}^2$ (§ 4, *b*), which are likewise triple, and finally two conics β^2 of which the completing lines r rest on c_3. Apparently the complete intersection is of the order 28.

On Γ there also lie 18 conics δ^2 (§ 4, *c*), 21 conics ε^2 (§ 4, *d*) and 3 conics ϱ^2, which are completed by a_1, a_2, a_3 (§ 2). Further Γ contains the 9 straight lines corresponding to the above mentioned conics β^2, $\gamma_1{}^2$, $\gamma_2{}^2$

and the 2 lines r resting on c_3 (§ 4, a); also the 6 lines $t_{k,l}$ (§ 4, c), which are triple, 3 lines t_k (§ 4, d), likewise triple, and 12 single lines t_k.

The curves ϱ^3 which have c_1 as bisecant, lie on the quadratic surface defined by A_1, A_2, A_3, b and c_1. Each of the two points of intersection of this scroll with c_2, with each of the two points of intersection with c_3, defines a ϱ^3 which belongs to the congruence and which cuts c_1 twice. Hence Γ contains twelve cubic *nodal curves*.

The lines c are *quadruple* on Γ (§ 3).

The intersection of Γ with the plane α_1 (through A_1 and b) consists of a ϱ^2 (§ 2), the triple lines $t_{1,1}$ and $t_{1,2}$ (§ 4. c), the triple line t_1, which rests on c_3 (§ 5, d), four rays t_1, which are single, and the line b. Consequently the line b is *thirteen-fold* on Γ.

The consideration of the intersection of Γ with the plane α readily shows that A_1, A_2 and A_3 are *fourteen-fold* points.

6. Let A_4 be a point of c_1. The curves ϱ^3 through the four points A_k which cut b twice and which rest on c_2, form a surface of the 4th degree (§ 3). The intersection of this surface with the plane through three of the points A_k consists of two conics; hence the points A_k are double points.

Consequently through A_4 there pass two curves of the [ϱ^3] which cut a line g through A_1 outside A_1. If A_4 describes the line c_1, these two curves describe a surface which has a curve $\gamma^4\,(L^2, M^2)$ as image. For any ray p contains two image points besides L, and the images of the ϱ^3 through Q^* (§ 2) lie in L. This surface contains two conics, which are represented in S_2 and S_3 (§ 2). $\gamma^8\,(L^4, M^4)$ and γ^4 (L^2, M^2) have 14 points in common besides the singular points L, M, S_2, S_3; accordingly the line g cuts Γ in 14 points besides A_1. This again shows that A_1, A_2 and A_3 are *fourteen-fold* points.