Physics. — P. EHRENFEST and G. E. UHLENBECK: "*On the connection of different methods of solution of the wave equation in multidimensional spaces*".

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Introduction.

The general equation for wave motion in space under the influence of an external force $k(t, x\,y\,z)$:

$$\frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} - \triangle s = k(t, xyz)$$

has, when for $t = -\infty$ everything is at rest $\left(s_{t=-\infty} = \left(\frac{\partial s}{\partial t}\right)_{t=-\infty} = 0 \right)$, the well known solution, the retarded potential:

$$s = \frac{1}{4\pi} \iiint_{-\infty}^{+\infty} \frac{k\left(t - \frac{r}{c}, \xi\eta\zeta\right)}{r} \, d\xi d\eta d\zeta .$$

The pecularities of this solution, especially its close connection to the three dimensional space, become apparent when we consider the analogous problem for the wave equation in multidimensional spaces:

$$\frac{\partial^2 s}{\partial t^2} - \sum_{h=1}^{n} \frac{\partial^2 s}{\partial x^2_h} = k(t, x_1 \ldots x_n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Various methods are available, but they all give the solution in *very different* analytical forms, whose identity unlike the case with three dimensions is not at once clear. In the following we will endeavour to show the connection and the identity of these solutions. [1]) The difference between spaces with an even and an odd number of dimensions will be especially brought to the front. [2])

§ 1. *Method of* HERGLOTZ [3]).

Putting $t = i\omega$, in (1), then the latter is transformed to the equation of POISSON in $n+1$ dimensions. Now the use of the known solution for this, suggests in our case the trial of:

$$s(t, x_1 \ldots x_n) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{1}{2}n - \frac{1}{2}}} \int_{-\infty}^{+\infty} \!\!\ldots\! \int d\xi_1 \ldots d\xi_n . \frac{1}{2\pi i} \int_{-\infty i}^{+\infty i} \frac{k(t-\vartheta, \xi_1 \ldots \xi_n)}{(r^2 - \vartheta^2)^{\frac{n-1}{2}}} \, d\vartheta \quad . \quad (2)$$

as a solution. By transforming the integration path of the last integral, we can write it without complex variables. Here enters the difference between spaces with even and odd numbers of dimensions.

a. When n is *odd* (put $n = 2p + 3$), we transform the path of integration to a small circle round one of the poles $\vartheta = +r$ or $\vartheta = -r$ of the integrand, and obtain with the help of the theorem of residues the solution represented by a "retarded" or an "advanced" potential respectively. In the first case for example:

$$s(t, x_1 \ldots x_{2p+3}) = \frac{\Gamma(p+1)}{2\pi^{p+1}} \int_{-\infty}^{+\infty} \cdots \int d\xi_1 \ldots d\xi_{2p+3}\, H_{2p+3} \quad . \quad . \quad (3)$$

is obtained, where:

$$H_{2p+3} = \frac{1}{p!} \sum_{0}^{p} l \frac{(2p-l)!}{l!\,(p-l)!} \cdot \frac{1}{2^{2p-l+1}} \frac{k^{(l)}(t-r, \xi_1 \ldots \xi_n)}{r^{2p-l+1}} \quad . \quad . \quad . \quad (4)$$

b. When n is *even* (put $n = 2q$), we can no longer close the integration path, as $\vartheta = \pm r$ are now also branching points, and we can only transform it to a loop round $\vartheta = +r$ or $\vartheta = -r$. The real expressions can now be written with the help of the so called "partie finie" of an infinite integral, as defined by HADAMARD.[4]) We obtain:

$$s(t, x_1 \ldots x_{2q}) = \frac{\Gamma\left(q - \frac{1}{2}\right)}{2\pi^{q-\frac{1}{2}}} \int_{-\infty}^{+\infty} \cdots \int d\xi_1 \ldots d\xi_{2q}\, H_{2q} \quad . \quad . \quad (5)$$

where

$$H_{2q} = \frac{(-1)^q}{\pi} \left| \overline{\int_r^\infty d\vartheta \frac{k(t-\vartheta, \xi_1 \ldots \xi_{2q})}{\left(\vartheta^2 - r^2\right)^{q-\frac{1}{2}}}} \right. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the bracket indicating the "partie finie". This becomes then in ordinary symbols:

$$H_{2q} = \frac{(-1)^q}{\pi} \int_r^\infty \frac{f(\vartheta) - \left[f(r) + (\vartheta - r) f'(r) + \ldots + \frac{(\vartheta - r)^{q-2}}{(q-2)!} f^{(q-2)}(r) \right]}{(\vartheta - r)^{q-\frac{1}{2}}} \quad (7)$$

where:

$$f(\vartheta) = \frac{k(t-\vartheta)}{(\vartheta + r)^{q-\frac{1}{2}}}$$

Remarks:

1. If the solution for a certain n is known, then the solution for $n - 1$ can always be found by the consideration of the cylindrical problem (the so called "methode de la descente" of HADAMARD). Formula (6) can be derived in this way from the solution (3).

2. The application of this method to (2) gives the solution in one dimension lower again in exactly the same form.

3. Still (2) and the transformations of the integration path indicated under *a* and *b* must only be considered as a heuristic method of arriving at the real expressions (3) and (5), which as can be shown are really

the solutions of our problem. It is difficult to give the exact meaning of (2) and the justification of the transformations.

§ 2. *"Polarisationmethod"*. [5])

Physically the solution of our problem can be represented as the superposition of all the „spherical waves" excited in the various points of the phase space by the „force" $k(t, x_1 \ldots x_n)$. It can be easily shown [5]) that, in analogy to $\frac{\Phi(t-r)}{r}$ for three dimensions, a solution of the equation for spherical waves in $(2p+3)$ dimensions is given by :

$$u_{2p+3} = \left(\frac{1}{r}\frac{\delta}{\delta r}\right)^p \frac{\Phi(t-r)}{r} \quad \ldots \ldots \ldots \quad (8)$$

which at once suggests as a solution:

$$s(t, x_1 \ldots x_{2p+3}) = A_{2p+3} \int_{-\infty}^{+\infty} \ldots \int d\xi_1 \ldots d\xi_{2p+3} \left(\frac{1}{r}\frac{\delta}{\delta r}\right)^p \frac{k(t-r, \xi_1 \ldots \xi_{2p+3})}{r} \quad (9)$$

Here A_{2p+3} is a constant. The solution is essentially limited to an *odd* number of dimensions.

The identity of (9) and (3) can be shown as follows:

Let:
$$P_{2p+3} = \left(\frac{1}{r}\frac{\delta}{\delta r}\right)^p \frac{k(t-r)}{r} \quad \ldots \ldots \ldots \quad (10)$$

From the definition of H_n:

$$H_3 = \frac{1}{2\pi i}\int d\vartheta \frac{k(t-\vartheta)}{r^2-\vartheta^2} = \frac{1}{2}\frac{k(t-r)}{r} = \frac{1}{2} P_3$$

and therefore

$$\left.\begin{aligned} P_{2p+3} &= \left(\frac{1}{r}\frac{\delta}{\delta r}\right)^p \frac{k(t-r)}{r} = \left(\frac{1}{r}\frac{\delta}{\delta r}\right)^p (2H_3) = \\ &= \frac{1}{\pi i}\int d\vartheta \frac{k(t-\vartheta)}{(r^2-\vartheta^2)^{p+1}} 2^p (-1)^p p! = (-1)^p 2^{p+1} p!\, H_{2p+3} \end{aligned}\right\} \quad . \quad (11)$$

Equation (11) can also be derived by direct differentiation, H_{2p+3} then being given by (4). From (3) and (11) then follows the constant in (9):

$$A_{2p+3} = \frac{(-1)^p}{2^{p+2}\pi^{p+1}} \quad \ldots \ldots \ldots \quad (12)$$

§ 3. *Method of* RIEMANN-HADAMARD [6]).

This method starts from the identity of GREEN, and is therefore in many ways analogous to the usual method of solving POISSON's equation. Hence in our case it is actually not very different from the method of HERGLOTZ, except that now everything is kept real from the beginning.

HADAMARD, however, has applied it to much more general hyperbolic equations and boundary conditions. Specialising his results to our problem then:

a) for an *even* number of dimensions ($n = 2q$) [7]:

$$s(t, x_1 \dots x_{2q}) = \frac{(-1)^{q+1}\,\Gamma(q-\frac{1}{2})}{2\pi^{q+\frac{1}{2}}} \left| \int_T \!\dots\! \int \frac{1}{\Gamma^{q-\frac{1}{2}}}\, k(\tau, \xi_1 \dots \xi_{2q})\, d\xi_1 \dots d\xi_{2q}\, d\tau \right. \quad (13)$$

where:

$$\Gamma \equiv (t-\tau)^2 - \sum_1^{2q} (x_h - \xi_h)^2 = (t-\tau)^2 - r^2$$

and T is the cone $\Gamma = 0$. Putting $t - \tau = \vartheta$, we get (5) and (6), i. e. the solution of HERGLOTZ for an even number of dimensions.

b) for an *odd* number of dimensions ($n = 2p + 3$) [8]:

$$s(t, x_1 \dots x_{2p+3}) = \frac{(-1)^{p+1}}{2\pi^{p+1}} \lim_{\gamma \to 0} \left[\frac{d^p}{d\gamma^p} \int_\sigma \!\dots\! \int k(\tau, \xi_1 \dots \xi_{2p+3})\, d\sigma \right] \quad (14)$$

where σ represents the surface of the hyperboloid:

$$\Gamma \equiv (t-\tau)^2 - r^2 = \gamma$$

and the element $d\sigma$ thereof is defined by:

$$d\sigma\, d\Gamma = d\xi_1 \dots d\xi_{2p+3}\, d\tau.$$

Hence:

$$d\sigma = -\frac{d\xi_1 \dots d\xi_{2p+3}}{2(t-\tau)} = -\frac{d\xi_1 \dots d\xi_{2p+3}}{2\sqrt{r^2+\gamma}}$$

$$\tau = t - \sqrt{r^2+\gamma}$$

$$s(t, x_1 \dots x_{2p+3}) = \frac{(-1)^p}{4\pi^{p+1}} \lim_{\gamma \to 0} \left[\frac{d^p}{d\gamma^p} \int_{-\infty}^{+\infty} \!\dots\! \int d\xi_1 \dots d\xi_{2p+3} \frac{k(t-\sqrt{r^2+\gamma})}{\sqrt{r^2+\gamma}} \right] =$$

$$= \frac{(-1)^p}{2^{p+2}\pi^{p+1}} \lim_{\gamma \to 0} \left(\frac{\delta}{r\delta r}\right)^p \int_{-\infty}^{+\infty} \!\dots\! \int d\xi_1 \dots d\xi_{2p+3} \frac{k(t-\sqrt{r^2+\gamma})}{\sqrt{r^2+\gamma}}$$

$$= \frac{(-1)^p}{2^{p+2}\pi^{p+1}} \int_{-\infty}^{+\infty} \!\dots\! \int d\xi_1 \dots d\xi_{2p+3} \left(\frac{\delta}{r\delta r}\right)^p \frac{k(t-r)}{r}.$$

We thus get the polarisationsolution (9).

§ 4. *Method of* FOURIER-POISSON [9]).

From the identity of FOURIER, we have:

$$k(t, x_1 \dots x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \!\dots\! \int d\nu_1 \dots d\nu_n \int_{-\infty}^{+\infty} \!\dots\! \int d\xi_1 \dots d\xi_n e^{i\Sigma\nu_h(x_h-\xi_h)} k(t, \xi_1 \dots \xi_n). (15)$$

Similarly we now *put* for the solution:

$$s(t, x_1 \ldots x_n) =$$

$$= \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \ldots \int d\nu_1 \ldots d\nu_n \int_{-\infty}^{+\infty} \ldots \int d\xi \ldots d\xi_n e^{i\Sigma \nu_h (x_h - \xi_h)} H(t, \xi_1 \ldots \xi_n, \nu_1 \ldots \nu_n) \quad (16)$$

where $H(t, \ldots \xi_1 \ldots \xi_n, \nu_1 \ldots \nu_n)$ is an *unknown* function. Substituting (15) and (16) in (1), we get for H the equation:

$$\frac{\delta^2 H}{\delta t^2} + \nu^2 H = k(t, \xi_1 \ldots \xi_n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where $\nu^2 = \sum_1^n \nu_h^2$. With the limiting values $H_{t=-\infty} = \left(\frac{\partial H}{\partial t}\right)_{t=-\infty} = 0$, we get:

$$H = \int_{-\infty}^{t} d\tau \, \frac{\sin \nu(t-\tau)}{\nu} \, k(\tau, \xi_1 \ldots \xi_n).$$

as the solution of (17).

This in (16) gives the required solution. If we introduce polar coordinates into the ν-space, then we can easily perform all the integrations except those over the radiusvector ν and one angle γ $\left[\text{defined by } \cos \gamma = \sigma = \right.$ $\left. = \frac{1}{\nu r} \Sigma \, \nu_h (x_h - \xi_h) \right]$. Also putting $t - \tau = \vartheta$, then:

$$s(t, x_1 \ldots x_n) = \frac{n-1}{2^{n-1} \pi^{\frac{1}{2} n + \frac{1}{2}} \Gamma\left(\frac{n+1}{2}\right)} \int_{-\infty}^{+\infty} \ldots \int d\xi_1 \ldots d\xi_n \, F_n \quad . \quad (18)$$

where:

$$F_n = \int_0^\infty d\nu \nu^{n-2} \int_0^\infty d\vartheta \, \sin \nu\vartheta k(t-\vartheta, \xi_1 \ldots \xi_n) \int_0^1 d\sigma (1-\sigma^2)^{\frac{n-3}{2}} \cos \nu r\sigma \quad . \quad (19)$$

For an *odd* number of dimensions the identity of (18) with (3) and (4) can be shown by direct calculation of (19). For optional values of n the demonstration of the identity with (2) can also be given (perhaps not mathematically irreproachable *)) as follows: The comparison of (2) with (18) and (19) shows, that we must demonstrate:

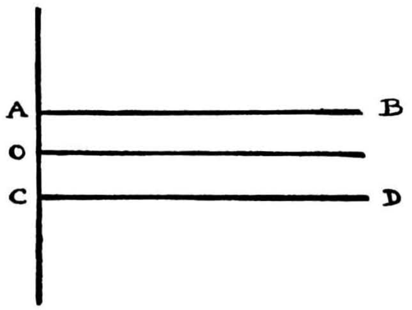
$$\left.\begin{aligned} &\frac{1}{2\pi i} \int_{-\infty i}^{+\infty i} \frac{k(t-\vartheta)}{(r^2-\vartheta^2)^{\frac{n-1}{2}}} d\vartheta = \\ &= \frac{1}{2^{n-3} \pi \Gamma^2\left(\frac{n-1}{2}\right)} \int_0^\infty d\nu \nu^{n-2} \int_0^\infty d\vartheta \, \sin \nu\vartheta k(t-\vartheta) \int_0^1 d\sigma \, (1-\sigma^2)^{\frac{n-3}{2}} \cos \nu r\sigma \end{aligned}\right\} \quad (20)$$

*) In the transformations of the integration paths we have the same incertainties as in § 1. (See remark 3).

We start from the identity ($\alpha > o$):

$$\int_0^\infty d\nu\nu^{n-2}e^{-\alpha\nu}\int_0^1 d\sigma(1-\sigma^2)^{\frac{n-3}{2}}\cos\nu r\sigma = \frac{2^{n-3}\Gamma^2\left(\frac{n-1}{2}\right)}{(r^2+\alpha^2)^{\frac{n-1}{2}}} \quad . \; . \; . \quad (21)$$

Now putting in (20) $\sin\nu\vartheta = \frac{1}{2i}\left(e^{i\nu\vartheta} - e^{-i\nu\vartheta}\right)$, we can divide the right hand side into two integrals. Instead of the interval $(0.\infty)$ we now take in the first integral for ϑ the path OAB, and in second the path OCD. Changing in each integral the integration over ϑ with that over ν, then the resulting integrals are convergent and can be calculated by (21). Finally, by combining both integrals into one over the loop $DCAB$, (20) is demonstrated.



LITERATURE AND FURTHER REMARKS.

1. The analogue for the equation of heat conduction has recently been shown by M. C. GRAY (Proc. Edinb. Roy. Soc. XLV, 230, 1925).

2. The principle of HUYGENS is, as is well known, no longer valid in spaces of an even number of dimensions. See for two dimensions RAYLEIGH, The Theory of Sound (Sec. edition 1896) Vol. II, Ch. XIV, § 275, and for further differences: P. EHRENFEST: In what way does it become manifest in the fundamental laws of physics, that space has three dimensions? Proc. Acad. Amst. **20**, 200, 1917.

3. HERGLOTZ: Gött. Nachr. p. 549, 1904.

4. HADAMARD: Acta Math. **31**, 339, 1908; Lectures on Cauchy Problem (London, Humphrey Milford, 1923) p. 133.

5. Comp. J. COULON: Sur l'intégration des équations aux dérivées partielles par la méthode des caractéristiques (Paris, Hermann, 1902) p. 56.

6. RIEMANN: Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. Ges. Werke p. 157.

KIRCHHOFF: Zur Theorie der Lichtstrahlen. Ges. Abh., Nachtrag No. 3, p. 22.

VOLTERRA: Atti Lincei 1892; Acta Math. **18**, 214, 1894.

TEDONE: Annali di Mathematica, Serie 3, T. **1** (1898) p. 1.

HADAMARD: Ann. Sc. de l'Ec. Norm. Sup. **21**, 535, 1904; **22**, 101, 1905.
Journ. de Phys. 1906 (survey); Acta Math. **31**, 333, 1908.

HADAMARD has given a survey of all these researches in his book: Lectures on Cauchy Problem in linear' hyperbolic partial differential equations (London, Humpfrey Milford, 1923). This is cited below.

7. Follows from form. (39), p. 166 l.c.

8. Follows from form. (28), p. 232 l.c., which is got by the „méthode de la descente" from form. (39) p. 166.

9. For the papers of POISSON, see: H. BURKHARDT, Entwicklung nach oscillierenden Funktionen (Teubner, 1908) p. 606. Comp. further: RAYLEIGH, Theory of Sound (Sec. Edit. 1896), Ch. XIV. RIEMANN-HATTENDORF, Partielle Differentialgleichungen § 106.