Mathematics. — *"Weights of the most probable values of the Unknowns in the case of Direct Conditioned Observations".* By Prof. M. J. VAN UVEN. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of October 30, 1926).

In adjusting direct conditioned observations, the reliableness of the "solutions", i. e. of the values obtained by adjustment, is indicated by their weights.

In the following paper we propose deriving for these weights an expression which gives them in a closed form as a function of the data, and which moreover holds good under an arbitrary number of conditions.

The quantities observed (m in number) will be designated by $x, y, z, \dots$; the observed values may be $\xi, \eta, \zeta, \dots$, resp., their weights $g_\xi, g_\eta, g_\zeta, \dots$

For simplification's sake we will assume that these m quantities $x, y, z, \dots$ must satisfy a priori μ linear conditions:

$$\alpha_j\, x + \beta_j\, y + \gamma_j\, z + \dots = \varkappa_j \qquad j = 1, 2, \dots \mu, \quad \mu < m \quad . \quad . \quad (1)$$

We form the differences

$$\varkappa_j - (\alpha_j\, \xi + \beta_j\, \eta + \gamma_j\, \zeta + \dots) = \varkappa_j - [\alpha_j\, \xi] = V_j\,, \quad j = 1, 2, \dots \mu \quad (2)$$

where [ ] denotes a summation over the variables $x, y, z, \dots$. Let $X, Y, Z, \dots$ be m provisionally assumed comparative values for $x, y, z, \dots$, which, by agreement, satisfy the μ conditions, so that

$$[\alpha_j\, X] = \varkappa_j \qquad j = 1, 2, \dots \mu,$$

Then, by forming the differences

$$\xi - X = v_x\,, \quad \eta - Y = v_y\,, \quad \zeta - Z = v_z\,, \dots \quad . \quad . \quad . \quad (3)$$

we have, according to (2):

$$[\alpha_j\, v_x] = -V_j \qquad j = 1, 2, \dots \mu. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We next consider those comparative values $X, Y, Z, \dots$ as the most probable values for $x, y, z, \dots$, for which $[g_\xi\, v_x\, v_x]$ takes the least value allowed by the conditions.

Then the equation $[g_\xi\, v_x\, dv_x] = 0$ must be dependent on the μ equations $[\alpha_j\, dv_x] = 0$ $(j = 1, \dots \mu)$, whence

$$[g_\xi\, v_x\, dv_x] \equiv \sum_{j=1}^{\mu} \lambda_j\, [\alpha_j\, dv_x]\,;$$

here the λ_j are multiplicators to be determined; the summation sign Σ is used here (and will be used also in future) to designate a summation over the μ conditions. Thus the sum Σ runs always from 1 to μ.

Now we have, as a consequence of the above identity:

$$g_\xi\, v_x = \Sigma\, \lambda_j\, \alpha_j\ ,\quad g_\eta\, v_y = \Sigma\, \lambda_j\, \beta_j\ ,\quad g_\zeta\, v_z = \Sigma \lambda_j\, \gamma_j\ , \ldots m \text{ equations} \quad (5)$$

and the μ equations already found:

$$[\alpha_j\, v_x] = -\, V_j \qquad j = 1, \ldots \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

hence altogether $m + \mu$ equations to determine the $m + \mu$ unknowns $v_x, v_y, v_z, \ldots, \lambda_1, \lambda_2, \ldots \lambda_\mu$.

The usual solution runs as follows:

From (4) and (5) we derive the μ equations:

$$\sum_{j=1}^{\mu} \lambda_j \left[\frac{\alpha_j\, \alpha_k}{g_\xi} \right] = -\, V_k\ , \qquad k = 1, \ldots \mu$$

or, putting for the sake of abbreviation:

$$\left[\frac{\alpha_j\, \alpha_k}{g_\xi} \right] = s_{jk}, \qquad (s_{jk} = s_{kj}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\sum_{i=1}^{\mu} s_{kj}\, \lambda_j = -\, V_k\ , \qquad k = 1, \ldots \mu. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

These μ equations determine the μ multiplicators λ_j.

Denoting the solution of these equations by $\bar{\lambda}_j$, and representing the values of $v_x, v_y, v_z, \ldots$, corresponding with them, according to (5), by $u_x, u_y, u_z, \ldots$, we have:

$$u_x = \frac{1}{g_\xi} \Sigma\, \alpha_j\, \bar{\lambda}_j\ ,\quad u_y = \frac{1}{g_\eta} \Sigma\, \beta_j\, \bar{\lambda}_j\ ,\quad u_z = \frac{1}{g_\zeta} \Sigma\, \gamma_j\, \bar{\lambda}_j\ , \ldots \quad . \quad (8)$$

the most probable values of $x, y, z, \ldots$ being

$$\bar{x} = \xi - u_x\ ,\quad \bar{y} = \eta - u_y\ ,\quad \bar{z} = \zeta - u_z \ldots \quad . \quad . \quad . \quad (9)$$

and $[g_\xi\, u_x\, u_x]$ being the least value which $[g_\xi\, v_x\, v_x]$ may assume, so far as the conditions allow.

The mean (true) error σ of the unity of weight follows from

$$\sigma^2 = \frac{[g_\xi\, u_x\, u_x\,]}{\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The sum $[g_\xi\, u_x\, u_x]$ may also be expressed directly into the data:

$$[g_\xi\, u_x\, u_x\,] = \sum_k \sum_j \left[g_\xi \cdot \frac{\alpha_k\, \bar{\lambda}_k}{g_\xi} \cdot \frac{\alpha_j\, \bar{\lambda}_j}{g_\xi} \right] = \sum_k \sum_j \left[\frac{\alpha_k\, \alpha_j}{g_\xi}\, \bar{\lambda}_k\, \bar{\lambda}_j \right] =$$

$$= \sum_k \sum_j \left[\frac{\alpha_k\, \alpha_j}{g_\xi} \right] \bar{\lambda}_k\, \bar{\lambda}_j = \sum_k \sum_j s_{kj}\, \bar{\lambda}_k\, \bar{\lambda}_j$$

(see (8) and (6)), or, since $\bar{\lambda}_j$, being the solution of (7), satisfies

$$\sum_j s_{kj}\, \bar{\lambda}_j = -\, V_k\ ,$$

$$[\, g_\xi\, u_x\, u_x\,] = -\sum_k V_k\, \bar{\lambda}_k\ .$$

Representing the determinant $|s_{jk}| = \begin{vmatrix} s_{11}, & s_{12}, & \ldots & s_{1\mu} \\ \vdots & & & \\ s_{1\mu}, & s_{2\mu}, & \ldots & s_{\mu\mu} \end{vmatrix}$ by S, and the minor (algebraic complement) of the element s_{jk} by S_{jk} (whence $S = \sum_j s_{jk} S_{jk}$, $0 = \sum_j s_{jk} S_{jl}$ $(l \neq k)$), we have

$$\overline{\lambda}_k = \frac{-\sum_j S_{jk} V_j}{S},$$

hence

$$[g_\xi\, u_x\, u_x] = \frac{\sum_k \sum_j S_{jk} V_j V_k}{S} = -\frac{1}{S} \begin{vmatrix} s_{11}, & \ldots & s_{1\mu}, & V_1 \\ \vdots & & & \\ s_{1\mu}, & \ldots & s_{\mu\mu}, & V_\mu \\ V_1, & \ldots & V_\mu, & 0 \end{vmatrix},$$

or, putting for the sake of abbreviation:

$$\begin{vmatrix} s_{11}, & \ldots & s_{1\mu}, & q_1 \\ \vdots & & \vdots & \vdots \\ s_{1\mu}, & \ldots & s_{\mu\mu}, & q_\mu \\ p_1, & \ldots & p_\mu, & r \end{vmatrix} = \begin{vmatrix} s_{jk} & q_j \\ p_k & r \end{vmatrix}, \qquad (11)$$

$$[g_\xi\, u_x\, u_x] = -\frac{1}{S} \begin{vmatrix} s_{jk}, & V_j \\ V_k, & 0 \end{vmatrix} \qquad (12)$$

The uncertainty of the "solutions" $\overline{x}, \overline{y}, \overline{z}, \ldots$ (see (9)) depends, in the last instance, on the uncertainty of the observed values $\xi, \eta, \zeta, \ldots$; these latter have the weights $g_\xi, g_\eta, g_\zeta, \ldots$, thus the mean errors:

$$\sigma_\xi = \frac{\sigma}{\sqrt{g_\xi}}, \quad \sigma_\eta = \frac{\sigma}{\sqrt{g_\eta}}, \quad \sigma_\zeta = \frac{\sigma}{\sqrt{g_\zeta}}, \ldots$$

If we imagine that, in eventually repeating the series of observations, the observed values $\xi, \eta, \zeta, \ldots$ undergo the variations $\triangle\xi, \triangle\eta, \triangle\zeta, \ldots$ respectively, then the mean square of $\triangle\xi$ (represented by $\overline{\triangle\xi^2}$) is nothing else than the square of the mean error of ξ, etc., in other words:

$$\overline{\triangle\xi^2} = \frac{\sigma^2}{g_\xi}, \quad \overline{\triangle\eta^2} = \frac{\sigma^2}{g_\eta}, \quad \overline{\triangle\zeta^2} = \frac{\sigma^2}{g_\zeta}, \ldots \qquad (13)$$

the mutual independency of the observed values $\xi, \eta, \zeta, \ldots$ having as a consequence that the mean products, two by two, of the variations $\triangle\xi, \triangle\eta, \triangle\zeta, \ldots$ are zero, so that

$$\overline{\triangle\xi \,.\, \triangle\eta} = 0, \quad \overline{\triangle\xi \,.\, \triangle\zeta} = 0, \ldots, \overline{\triangle\eta \,.\, \triangle\zeta} = 0 \text{ etc.} \qquad (14)$$

In consequence of the variations $\triangle\xi, \triangle\eta, \triangle\zeta, \ldots$, the V_j undergo (according to (2)) the variations

$$\triangle V_j = -[a_j \triangle \xi] = -(a_j \triangle \xi + \beta_j \triangle \eta + \gamma_j \triangle \zeta + \ldots).$$

The variations $\triangle\lambda_k$ of $\overline{\lambda}_k$, deriving from these $\triangle V_j$, satisfy (see (7))

$$\sum s_{kj} \triangle \lambda_j = -\triangle V_k . \qquad k = 1, \ldots \mu \qquad (15)$$

Further, from

$$u_x = \frac{1}{g_\xi} \sum_j a_j \overline{\lambda_j} = \xi - \overline{x}: \quad \ldots \ldots \quad (8), (9)$$

follows

$$\Sigma \frac{a_j}{g_\xi} \triangle \lambda_j = \triangle \xi - \triangle \overline{x}. \quad \ldots \ldots \quad (16)$$

We now multiply the μ equations (15) and the equation (16) by $\triangle\xi$. Then, taking the mean value, we obtain (by (13) and (14))

$$\sum_j s_{kj} \overline{\triangle \lambda_j \, . \triangle \xi} = - \overline{\triangle V_k . \triangle \xi} = + (a_k \overline{\triangle \xi^2} + \beta_k \overline{\triangle \xi . \triangle \eta} + \gamma_k \overline{\triangle \xi . \triangle \zeta} + \ldots)$$

$$= \frac{a_k}{g_\xi} \sigma^2 \qquad k = 1, \ldots \mu$$

$$\Sigma \frac{a_j}{g_\xi} \overline{\triangle \lambda_j \, . \triangle \xi} = \overline{\triangle \xi^2} - \overline{\triangle \overline{x} \, . \triangle \xi} = \frac{\sigma^2}{g_\xi} - \overline{\triangle \overline{x} \, . \triangle \xi}.$$

By eliminating the μ variables $\overline{\triangle\lambda_j \, . \triangle\xi}$ from these $\mu + 1$ equations and using the notation (11), we arrive at

$$\begin{vmatrix} s_{kj}, & \frac{a_k}{g_\xi} \sigma^2 \\ \frac{a_j}{g_\xi}, & \frac{\sigma^2}{g_\xi} - \overline{\triangle \overline{x} \, . \triangle \xi} \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} s_{kj}, & \frac{a_k}{g_\xi} \sigma^2 \\ \frac{a_j}{g_\xi}, & \frac{\sigma^2}{g_\xi} \end{vmatrix} - \begin{vmatrix} s_{kj}, & 0 \\ \frac{a_j}{g_\xi}, & \overline{\triangle \overline{x} \, . \triangle \xi} \end{vmatrix} = 0,$$

or

$$\frac{\sigma^2}{g_\xi} \begin{vmatrix} s_{kj}, & \frac{a_k}{g_\xi} \\ a_j \;, & 1 \end{vmatrix} - S . \overline{\triangle \overline{x} \, . \triangle \xi} = 0,$$

whence

$$\overline{\triangle \overline{x} \, . \triangle \xi} = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \cdot \begin{vmatrix} s_{kj}, & \frac{a_k}{g_\xi} \\ a_j \;, & 1 \end{vmatrix} \quad \ldots \ldots \quad (17)$$

In the same manner we obtain, by multiplying the μ equations (15) and the equation (16) by $\triangle\eta$, and then taking the mean,

$$\sum_j s_{kj} \overline{\triangle \lambda_j \, . \triangle \eta} = - \overline{\triangle V_k \, . \triangle \eta} = + (a_k \overline{\triangle \eta . \triangle \xi} + \beta_k \overline{\triangle \eta^2} + \gamma_k \overline{\triangle \eta . \triangle \zeta} + \ldots)$$

$$= \frac{\beta_k}{g_\eta} \sigma^2 \qquad k = 1, \ldots \mu$$

$$\sum_j \frac{a_j}{g_\xi} \overline{\triangle \lambda_j \, . \triangle \eta} = \overline{\triangle \xi \, . \triangle \eta} - \overline{\triangle \overline{x} \, . \triangle \eta} = - \overline{\triangle \overline{x} \, . \triangle \eta},$$

From these $\mu + 1$ equations we derive by eliminating the μ variables $\overline{\triangle\lambda_j . \triangle\eta}$:

$$\begin{vmatrix} s_{kj}, & \dfrac{\beta_k}{g_\eta}\sigma^2 \\ \dfrac{a_j}{g_\xi}, & 0 - \overline{\triangle \bar{x} . \triangle \eta} \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} s_{kj}, & \dfrac{\beta_k}{g_\eta}\sigma^2 \\ \dfrac{a_j}{g_\xi}, & 0 \end{vmatrix} - \begin{vmatrix} s_{kj}, & 0 \\ \dfrac{a_j}{g_\xi}, & \overline{\triangle \bar{x} . \triangle \eta} \end{vmatrix} = 0,$$

or

$$\frac{\sigma^2}{g_\xi} \begin{vmatrix} s_{kj}, & \dfrac{\beta_k}{g_\eta} \\ a_j, & 0 \end{vmatrix} - S . \overline{\triangle \bar{x} \triangle \eta} = 0,$$

whence

$$\overline{\triangle \bar{x} . \triangle \eta} = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \cdot \begin{vmatrix} s_{kj}, & \dfrac{\beta_k}{g_\eta} \\ a_j, & 0 \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Likewise we find

$$\overline{\triangle \bar{x} . \triangle \zeta} = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \cdot \begin{vmatrix} s_{kj}, & \dfrac{\gamma_k}{g_\zeta} \\ a_j, & 0 \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (18 \text{ bis})$$

etc.

We next multiply (15) by $\triangle\bar{x}$ and take the mean; so we obtain (replacing the index k by i):

$$\sum_j s_{ij} \overline{\triangle\lambda_j . \triangle \bar{x}} = - \overline{\triangle V_i . \triangle \bar{x}} = + (a_i \overline{\triangle \bar{x} . \triangle \xi} + \beta_i \overline{\triangle \bar{x} . \triangle \eta} +$$
$$+ \gamma_i \overline{\triangle \bar{x} . \triangle \zeta} + \ldots), \; i = 1, \ldots \mu$$

or, on account (17), (18), (18 bis) etc.

$$\sum_j s_{ij} \overline{\triangle\lambda_j . \triangle \bar{x}} = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \cdot \left\{ a_i \begin{vmatrix} s_{kj}, & \dfrac{a_k}{g_\xi} \\ a_j, & 1 \end{vmatrix} + \beta_i \begin{vmatrix} s_{kj}, & \dfrac{\beta_k}{g_\eta} \\ a_j, & 0 \end{vmatrix} + \gamma_i \begin{vmatrix} s_{kj}, & \dfrac{\gamma_k}{g_\zeta} \\ a_j, & 0 \end{vmatrix} + \ldots \right\}$$

$$= \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \cdot \begin{vmatrix} s_{kj}, & \left[\dfrac{a_k\, a_i}{g_\xi}\right] \\ a_j, & a_i \end{vmatrix} = \frac{1}{S} \cdot \frac{\sigma^2}{g_\xi} \cdot \begin{vmatrix} s_{kj}, & s_{ki} \\ a_j, & a_i \end{vmatrix}, \; i = 1, \ldots \mu$$

or

$$\sum_j s_{ij}\,\overline{\triangle\lambda_j\,.\,\triangle\bar{x}}=\frac{1}{S}\cdot\frac{\sigma^2}{g_\xi}\begin{vmatrix} s_{11},\dots, & s_{1i},\dots, & s_{1\mu}, & s_{1i}\\ \vdots & \vdots & & \\ s_{\mu1},\dots, & s_{\mu i},\dots, & s_{\mu\mu}, & s_{\mu i}\\ & & & \\ a_1,\dots, & a_i,\dots, & a_\mu, & a_i \end{vmatrix}\equiv 0 \qquad i=1,\dots\mu$$

The determinant $|s_{ij}|=S$ of the μ homogeneous equations

$$\sum_j s_{ij}\,\overline{\triangle\lambda_j\,.\,\triangle\bar{x}}=0 \qquad i=1,\dots\mu$$

which we obtain in this way, being different from zero, we have separately

$$\overline{\triangle\lambda_j\,.\,\triangle\bar{x}}=0. \qquad j=1,\dots\mu \quad\dots\dots\quad (19)$$

Multiplying at last the equation (16) by $\triangle\bar{x}$ and then taking the mean, we find

$$\overline{\triangle u_x\,.\,\triangle\bar{x}}=\sum\frac{a_j}{g_\xi}\,\overline{\triangle\lambda_j\,.\,\triangle\bar{x}}=\overline{\triangle\xi\,.\,\triangle\bar{x}}-\overline{\triangle\bar{x}^2}$$

thus, by (19),

$$\overline{\triangle u_x\,.\,\triangle\bar{x}}=\overline{\triangle\xi\,.\,\triangle\bar{x}}-\overline{\triangle\bar{x}^2}=0 \quad\dots\dots\quad (20)$$

or, on account of (17),

$$\overline{\triangle\bar{x}^2}=\overline{\triangle\xi\,.\,\triangle\bar{x}}=\frac{1}{S}\cdot\frac{\sigma^2}{g_\xi}\begin{vmatrix} s_{kj} & \dfrac{a_k}{g_\xi}\\ & \\ a_j & 1\end{vmatrix} \quad\dots\dots\quad (21)$$

Now, $\overline{\triangle\bar{x}^2}$ being nothing but the square of the mean error $\sigma_{\bar{x}}$ of $\bar{x}$, we have

$$\sigma_{\bar{x}}^2=\frac{1}{S}\cdot\frac{\sigma^2}{g_\xi}\cdot\begin{vmatrix} s_{kj}, & \dfrac{a_k}{g_\xi}\\ & \\ a_j, & 1\end{vmatrix} \quad\dots\dots\dots\quad (22)$$

We shall moreover put this latter determinant into another form. In

$$\begin{vmatrix} s_{kj}, & \dfrac{a_k}{g_\xi}\\ & \\ a_j, & 1\end{vmatrix}=\begin{vmatrix} s_{11},\dots, & s_{ij},\dots, & s_{1\mu}, & a_1:g_\xi\\ \vdots & \vdots & \vdots & \vdots\\ s_{k1},\dots, & s_{kj},\dots, & s_{k\mu}, & a_k:g_\xi\\ \vdots & \vdots & \vdots & \vdots\\ s_{\mu1},\dots, & s_{\mu j},\dots, & s_{\mu\mu}, & a_\mu:g_\xi\\ a_1,\dots, & a_j,\dots, & a_\mu, & 1\end{vmatrix}$$

we take from the 1[st] column a_1 times the last column

$$\begin{array}{lllllll} \vdots & & & & & & \\ \text{,,}\ j^{th} & \text{,,} & a_j & \text{,,} & \text{,,} & \text{,,} & \text{,,}\\ \vdots & & & & & & \\ \text{,,}\ \mu^{th} & \text{,,} & a_\mu & \text{,,} & \text{,,} & \text{,,} & \text{,,}\ . \end{array}$$

Then we retain for the element of the j^{th} column and the k^{th} row:

$$s_{kj}-\frac{\alpha_j\,\alpha_k}{g_\xi}=\left[\frac{\alpha_k\,\alpha_j}{g_\xi}\right]-\frac{\alpha_k\,\alpha_j}{g_\xi}=\frac{\beta_k\,\beta_j}{g_\eta}+\frac{\gamma_k\,\gamma_j}{g_\zeta}+\ldots=a_{kj}\quad . \qquad (23)$$

the j^{th} element of the last row becoming zero ($j=1,\ldots\mu$). Representing the determinant $|a_{kj}|$ by A, we obtain

$$\begin{vmatrix} s_{kj}\ , & \dfrac{\alpha_k}{g_\xi} \\ \alpha_j\ , & 1 \end{vmatrix}=\begin{vmatrix} a_{kj}\ , & \dfrac{\alpha_k}{g_\xi} \\ 0\ , & 1 \end{vmatrix}=|a_{kj}|=A.$$

So we find

$$\sigma_{\bar{x}}^{\ 2}=\frac{A}{S}\cdot\frac{\sigma^2}{g_\xi}$$

and for the weight $g_{\bar{x}}$ of $\bar{x}$:

$$g_{\bar{x}}=\frac{S}{A}\cdot g_\xi\ .$$

Putting in a similar manner:

$$s_{kj}-\frac{\beta_j\,\beta_k}{g_\eta}=\left[\frac{\alpha_k\,\alpha_j}{g_\xi}\right]-\frac{\beta_k\,\beta_j}{g_\eta}=\frac{\alpha_k\,\alpha_j}{g_\xi}+\frac{\gamma_k\,\gamma_j}{g_\zeta}+\ldots=b_{kj}\quad . \qquad (23\text{ bis})$$

$$s_{kj}-\frac{\gamma_j\,\gamma_k}{g_\zeta}=\left[\frac{\alpha_k\,\alpha_j}{g_\xi}\right]-\frac{\gamma_k\,\gamma_j}{g_\zeta}= \qquad\qquad =c_{kj}\quad . \qquad (23\text{ ter})$$

etc.,
and further:

$$|b_{kj}|=B\quad ,\quad |c_{kj}|=C,\ \text{etc.},$$

we arrive at last at:

$$\sigma_{\bar{x}}^{\ 2}=\frac{A}{S}\cdot\frac{\sigma^2}{g_\xi}\quad ,\quad \sigma_{\bar{y}}^{\ 2}=\frac{B}{S}\cdot\frac{\sigma^2}{g_\eta}\quad ,\quad \sigma_{\bar{z}}^{\ 2}=\frac{C}{S}\cdot\frac{\sigma^2}{g_\zeta},\ \text{etc.},\quad . \quad . \qquad (24)$$

or

$$g_{\bar{x}}=\frac{S}{A}\cdot g_\xi\quad ,\quad g_{\bar{y}}=\frac{S}{B}\cdot g_\eta\quad ,\quad g_{\bar{z}}=\frac{S}{C}\cdot g_\zeta\ ,\ \text{etc.}\quad . \quad . \quad . \qquad (25)$$

Thus we have expressed the weights of the solutions in a closed form into the data g_ξ, g_η, g_ζ, ...; α_j, β_j, γ_j, ... ($j=1,\ldots\mu$)

If there is given, in particular, but one condition

$$\alpha x+\beta y+\gamma z+\ldots=\varkappa$$

so that $\mu=1$, then we have

$$S=\left[\frac{\alpha\,\alpha}{g_\xi}\right],\ A=\left[\frac{\alpha\,\alpha}{g_\xi}\right]-\frac{\alpha\,\alpha}{g_\xi}=\frac{\beta\,\beta}{g_\eta}+\frac{\gamma\,\gamma}{g_\zeta}+\ldots,$$

$$B=\left[\frac{\alpha\,\alpha}{g_\xi}\right]-\frac{\beta\,\beta}{g_\eta}=\frac{\alpha\,\alpha}{g_\xi}+\frac{\gamma\,\gamma}{g_\zeta}+\ldots,\quad C=\ldots,\ \text{etc.},$$

thus

$$\sigma_x^2 = \frac{\sigma^2}{g_\xi} \cdot \frac{\frac{\beta\beta}{g_\eta} + \frac{\gamma\gamma}{g_\zeta} + \ldots}{\frac{\alpha\alpha}{g_\xi} + \frac{\beta\beta}{g_\eta} + \frac{\gamma\gamma}{g_\zeta} + \ldots}, \quad \sigma_y^2 = \frac{\sigma^2}{g_\eta} \cdot \frac{\frac{\alpha\alpha}{g_\xi} + \frac{\gamma\gamma}{g_\zeta} + \ldots}{\frac{\alpha\alpha}{g_\xi} + \frac{\beta\beta}{g_\eta} + \frac{\gamma\gamma}{g_\zeta} + \ldots}, \quad \sigma_z^2 = \ldots, \text{etc.} \qquad (24\text{bis})$$

The determinants $S, A, B, C, \ldots$ may yet be put into another form. Of the m variables $x, y, z, \ldots$, represented by $g_\xi, g_\eta, g_\zeta, \ldots$ and by $\alpha_j, \beta_j, \gamma_j, \ldots$, we consider the combinations μ by μ. Be $x, z, \ldots t$ ($g_\xi, g_\zeta, \ldots g_\tau$; $\alpha_j, \gamma_j, \ldots \theta$) such a combination.

Then we may form determinants such as

$$\triangle(x, z, \ldots t) = \begin{vmatrix} \alpha_1, & \gamma_1, & \ldots & \theta_1 \\ \alpha_2, & \gamma_2, & \ldots & \theta_2 \\ \vdots & & & \\ \alpha_\mu, & \gamma_\mu, & \ldots & \theta_\mu \end{vmatrix}$$

According to the theory of determinants, we may write S as follows:

$$S = |s_{kj}| = \left| \left[\frac{\alpha_k \alpha_j}{g_\xi} \right] \right| = \Sigma \frac{\triangle^2(x, z, \ldots t)}{g_\xi \cdot g_\zeta \cdot \ldots \cdot g_\tau} \quad . \quad . \quad . \quad . \qquad (27)$$

where the sum Σ runs over all the $C_\mu(m) = \frac{m!}{\mu!\,(m-\mu)!}$ combinations μ by μ of the m elements $x, y, z, \ldots$.

In the same manner we may also reduce the determinant A.

Here all the m elements $x, y, z, \ldots$ appear, except the element x. With these $m-1$ elements $y, z, \ldots$ we can build determinants of the form:

$$\triangle_x(y, z, \ldots t) = \begin{vmatrix} \beta_1, & \gamma_1, & \ldots & \theta_1 \\ \vdots & & & \\ \beta_\mu, & \gamma_\mu, & \ldots & \theta_\mu \end{vmatrix}, \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

wherin the elements α_j are always missing.

So we have

$$A = |a_{kj}| = \left| \left[\frac{\alpha_k \alpha_j}{g_\xi} \right] - \frac{\alpha_k \alpha_j}{g_\xi} \right| = \Sigma_x \frac{\triangle_x^2(y, z, \ldots t)}{g_\eta \cdot g_\zeta \cdot \ldots \cdot g_\tau}, \quad . \quad . \qquad (29)$$

where the sum Σ_x extends over all the $C_\mu\,(m-1) = \frac{(m-1)!}{\mu!\,(m-\mu-1)!}$ combinations μ by μ of the $m-1$ elements $y, z, \ldots$ (x absent).

Similarly:

$$B = |b_{kj}| = \Sigma_y \frac{\triangle_y^2(x, z, \ldots t)}{g_\xi \cdot g_\zeta \cdot \ldots \cdot g_\tau} \quad . \quad . \quad . \quad . \quad . \qquad (29\text{ bis})$$

$$C = |c_{kj}| = \Sigma_z \frac{\triangle_z^2(x, y, \ldots t)}{g_\xi \cdot g_\eta \cdot \ldots \cdot g_\tau} \quad . \quad . \quad . \quad . \quad . \qquad (29\text{ ter})$$

etc.

So the determinants $S, A, B, C, \ldots$ are written as sums of positive terms. The terms of the sum A also appear in the sum S, which contains besides those terms depending upon the element $x(g_\xi, a_j)$. Thus the value A is certainly less than that of S. Likewise $B, C, \ldots$ are altogether less than S.

The fractions $\frac{A}{S}, \frac{B}{S}, \frac{C}{S}, \ldots$ are therefore altogether less than unity, so that the mean errors $\sigma_{\bar{x}}, \sigma_{\bar{y}}, \sigma_{\bar{z}}, \ldots$ of the solutions are less than the mean errors of the respective observations, viz:

$$\sigma_\xi = \frac{\sigma}{\sqrt{g_\xi}}, \quad \sigma_\eta = \frac{\sigma}{\sqrt{g_\eta}}, \quad \sigma_\zeta = \frac{\sigma}{\sqrt{g_\zeta}}, \ldots .$$

thus:

$$\sigma_{\bar{x}} < \sigma_\xi, \quad \sigma_{\bar{y}} < \sigma_\eta, \quad \sigma_{\bar{z}} < \sigma_\zeta, \quad \ldots \ldots \qquad (30)$$

If all the terms of the sum S (27) are nearly equal, then the sums A and S are almost proportional to the numbers of their terms, thus as $C_\mu(m-1)$ to $C_\mu(m)$, that is: as $m-\mu$ to m. In this case the mean errors of the solutions are about $\sqrt{\frac{m-\mu}{m}}$ times as large as the mean errors of the corresponding observations. The larger μ is in comparison to m, the more the solution surpasses the observation in accuracy.