Mathematics. — "*A property of 2-dimensional Elements*" by M. H. A. NEWMAN. (Communicated by Prof. L. E. J. BROUWER).

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In a paper recently published in these Proceedings [1]) it was stated without proof that if $n > 2$ not every n-element can be transformed into an n-simplex by moves of type 2 alone, *i.e.* in certain n-elements there is no unit (n-simplex) whose removal leaves an n-element. It is the purpose of this note to prove this assertion by the production of an example (§ 2), and also to shew (§ 1) that if E_2 and S_2 are respectively a 2-element and a 2-simplex, then $E_2 \underset{2}{\rightarrow} S_2$.

A proof of the 2-dimensional JORDAN's Theorem, (in the combinatory sense), is included.

§ 1.

LEMMA. *If E_2 is a 2-element and $\alpha\beta$ an internal edge [2]) whose extremities belong to $\overline{E_2}$ and divide it into the parts E_1' and E_1'', then E_2 is the sum of two 2-elements with no common units, whose boundaries are $\alpha\beta + E_1'$ and $\alpha\beta + E_1''$.*

THEOREM 1. *If the 2-element E_2 is neither a simplex nor a complete star it has two detachable [3]) units with no common internal vertex.*

THEOREM 2. *If $O_{12}(E_2) = q$, [4]) and S_2 is a detachable unit of E_2, then $O_{12}(E_2 - S_2) = q - 1$.*

It is convenient to prove the lemma and the two theorems together. Let them be assumed true of elements of order less than q, (1, 2); and suppose that $O_{12}(E_2) = q$ in all three cases.

Proof of the Lemma. From the inductive hypothesis about Theorem 2 it follows that $E_2 \underset{2}{\rightarrow} S_2$. For if E_2 were derived from E_2^*, of order $q - 1$, by the *removal* of a unit, $O_{12}(E_2)$ would be $q - 2$, not q. Suppose then that T_2 is such that $O_{12}(E_2 - T_2) = q - 1$.

If T_2 contains $\alpha\beta$, (whose extremities lie in $\overline{E_2}$), the contact of T_2 with the rest of E_2 can only be regular if its remaining vertex is free; and then the required 2-elements are T_2 and $E_2 - T_2$.

If T_2 does not contain $\alpha\beta$, the part of $\overline{T_2}$ in $\overline{E_2 - T_2}$ belongs to one

1) M. H. A. NEWMAN, Proc. Roy. Ac. Amsterdam, **29** (1926) pp. 611 and 627, two parts here called F I and F II. The definitions of regular contact, moves of types 1, 2 and 3, *etc.*, are given in F I.

2) An *edge* is a 1-component.

3) A unit, S_n, of Γ_n, is *detachable* if it has regular contact with $\Gamma_n - S_n$.

4) "$O_{rs}(E) = q$", or "E is of order q, (r, s)", means that q is the smallest number of moves of types r and s by which E can be transformed into a simplex.

of the parts E_1' or E_1''. For if it had an edge in each they would necessarily be the pair of edges of $\overline{E_2 - T_2}$ at α, or the pair at β; and since the contact of T_2 is regular this is contrary to the supposition that α and β are in the boundary of E_2. Suppose then that the part $V.\overline{U}$. of $\overline{T_2}$ belongs to E_1'. $\alpha\beta$ is interior to $E_2 - T_2$ and has its extremities on the boundary, and therefore divides it into two elements, E_2'' and E_2''', with boundaries $E_1'' + \alpha\beta$ and $(E_1' - V.\overline{U} + U.\overline{V}) + \alpha\beta$ respectively. Hence E_2 is divided by $\alpha\beta$ into the elements E_2'' and $E_2''' + T_2$, with boundaries $E_1'' + \alpha\beta$ and $E_1' + \alpha\beta$.

Proof of Theorem 1. Suppose, as before, that T_2 is a unit of E_2 such that $O_{12}(E_2 - T_2) = q-1$, (and therefore $O_2(E_2 - T_2) = q-1$). Let T_2 be $\alpha\beta\gamma$.

If $\alpha\beta$ is internal and γ free, $E_2 - \alpha\beta\gamma$, being of order $q-1$, contains two detachable units, of which at least one does not contain $\alpha\beta$. This simplex and $\alpha\beta\gamma$ are detachable units of E_2; and $\alpha\beta\gamma$ has no internal vertex.

If γ is internal and $\alpha\beta$ free, let $\xi\eta$ be a boundary edge of E_2 such that the unit $\xi\eta\zeta$ containing it does not contain γ. (There is such an edge, since E_2 is not a complete star). If ζ is interior to E_2, $\xi\eta\zeta$ is a second detachable unit, not containing γ. If ζ is in $\overline{E_2}$, $\xi\zeta$ divides E_2 into two 2-elements, both of order less than q (since $E_2 \underset{2}{\rightarrow} S_2$), of which one, say E_2', does not contain γ. E_2' has two detachable units, of which one at least does not contain $\xi\zeta$, and is therefore detachable from E_2. This is a second detachable unit, not containing the internal vertex, γ, of $\alpha\beta\gamma$.

Proof of Theorem 2. If E_2 is a complete star the theorem is obviously true. We suppose then that it is not a complete star.

Let T_2 be as before, and let $\varrho\sigma\tau$ be the given detachable unit S_2. If $\varrho\sigma\tau$ has a free vertex, τ, the connected array $E_2 - T_2$, which contains $\varrho\sigma\tau$, must have $\varrho\sigma$ as an internal edge, save in the trivial case where $E_2 - T_2$ is $\varrho\sigma\tau$. Hence $E_2 - T_2 \underset{2}{\rightarrow} E_2 - T_2 - \varrho\sigma\tau$, which is therefore of order $q-2$; and $(E_2 - T_2 - \varrho\sigma\tau) + T_2$, which is $E_2 - \varrho\sigma\tau$, is of order $q-1$.

If τ is internal and $\varrho\sigma$ free, and T_2 does not contain τ, a similar argument shows that $O_2(E_2 - \varrho\sigma\tau) = q-1$. If τ belongs to T_2, let U_2 be a detachable unit of E_2 not containing τ. U_2 is then detachable from $E_2 - T_2$; $(E_2 - T_2) - U_2$ is of order $q-2$, and therefore $O_2(E_2 - U_2) = q-1$. The unit $\varrho\sigma\tau$ is detachable from $E_2 - U_2$, and therefore

$$O_2(E_2 - U_2 - \varrho\sigma\tau) = q-2, \text{ whence } O_2(E_2 - \varrho\sigma\tau) = q-1.$$

JORDAN's Theorem in two dimensions, of which the lemma is a special case, is now easily proved in its unrestricted form.

Theorem 3a. *If e_1 is a 1-element consisting of internal edges of E_2 but having its extremities, α and β, in $\overline{E_2}$, then E_2 is the sum of two 2-elements with no common unit, whose boundaries are $e_1 + E_1'$ and $e_1 + E_1''$, where E_1' and E_1'' are the parts into which $\overline{E_2}$ is divided by α and β.*

Suppose the theorem true of 2-elements of order less than q, and let $O_2(E_2) = q$. Let T_2 be such that $O_2(E_2 - T_2) = q - 1$. Clearly the part, (say $U.\overline{V}$), of $\overline{T_2}$ in $\overline{E_2}$ belongs either to E_1' or to E_1'' — say to E_1'.

If T_2 contains no internal vertex of e_1, $E_2 - T_2$ is divided by e_1 into two 2-elements, with boundaries $E_1' - U.\overline{V} + V.\overline{U} + e_1$ and $E_1'' + e_1$; and on adding T_2 to the first, the division of E_2 by e_1 is established.

T_2 cannot contain two edges of E_1 save in the trivial case where $V.\overline{U}$ is e_1. If it contains the edge $\alpha\xi$ and no other, $e_1 - \alpha\xi$ is interior to $E_2 - T_2$, and the proof is completed as before.

If T_2 contains an interior vertex, ξ, but no edge, of e_1, let e_1^1 and e_1^2 be the parts, (containing α and β respectively), into which ξ divides e_1, and $\mathcal{E}_1^1$ and $\mathcal{E}_1^2$ the corresponding parts into which it divides $E_1' - U.\overline{V} + V.\overline{U}$.

The 1-element e_1^1 is interior to $E_2 - T_2$, save for its extremities, and so divides it into, say, E_2^1 and E_2^3 with boundaries $e_1^1 + \mathcal{E}_1^1$ and $e_1^1 + \mathcal{E}_1^2 + E_1''$. e_1^2 is interior to E_2^3, save for its extremities, and divides it into, say, E_2^2 and E_2'', with boundaries $e_1^2 + \mathcal{E}_1^2$ and $e_1^2 + e_1^1 + E_1''$. T_2 has regular contact with E_2^1, and the 2-elements E_2^2 and $E_2^1 + T_2$ have a single boundary edge in common. Their sum, E_2', is a 2-element, (F II Theorem 8a), whose boundary is $E_1' + e_1$. The two elements E_2' and E_2'' are the parts into which E_2 is divided by e_1.

Theorem 3b. *If Σ_1 is a circle contained in a 2-sphere, Σ_2, Σ_2 is the sum of two 2-elements, without common units, of both of which Σ_1 is the boundary.*

Let a unit, S_2, of Σ_2 containing an edge, $\alpha\beta$, of Σ_1 be replaced by $\xi.\overline{S_2}$, where ξ is a new vertex, (move of type 3). Theorem 3b is easily proved by applying, Theorem 3a to $\Sigma_2 - \xi\alpha\beta$, then replacing $\xi\alpha\beta$, and finally reversing the move of type 3.

§ 2.

Consider the 3-array, Γ_3, with the vertices

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \delta',\ \varepsilon',\ \zeta',\ \eta';$$

and the units

$$\begin{array}{lll} \alpha\,\beta\,\varepsilon'\,\dot{\eta}' & \beta\,\gamma\,\zeta'\,\dot{\eta}' & \alpha\,\beta\,\varepsilon\,\dot{\eta} \\ \alpha\,\beta\,\zeta'\,\eta' & \alpha\,\dot{\beta}\,\zeta\,\zeta' & \alpha\,\beta\,\zeta\,\eta \\ \alpha\,\delta'\,\varepsilon'\,\dot{\eta}' & \dot{\beta}\,\dot{\gamma}\,\zeta\,\zeta' & \alpha\,\delta\,\varepsilon\,\dot{\eta} \\ \alpha\,\delta'\,\zeta'\,\dot{\eta}' & \dot{\beta}\,\gamma\,\zeta\,\dot{\eta} & \alpha\,\delta\,\zeta\,\dot{\eta} \end{array}$$ [1])

(The significance of the dots will be explained later).

[1]) See fig. 1.

Let Δ_3 be the solid ring formed by placing Γ_3 and three arrays congruent to it end to end in cyclical order, the "ends" being $\delta\zeta\eta + \delta\varepsilon\eta$

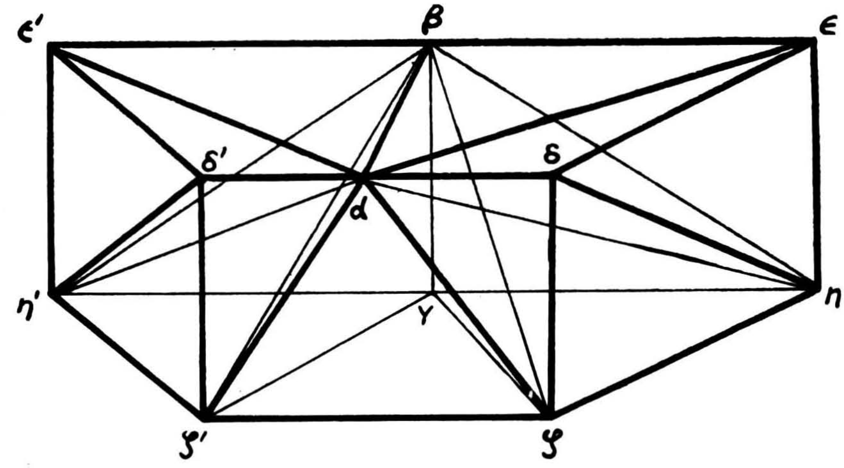

Fig. 1 :- Γ_3.

and $\delta'\zeta'\eta' + \delta'\varepsilon'\eta'$ in Γ_3, and their correlates in the other arrays. Fig. 2 will make the arrangement clear.

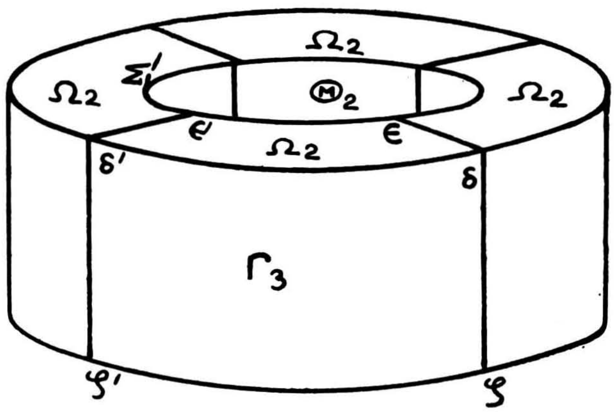

Fig 2 - Δ_3

Part of the boundary of Δ_3 is the "cylinder" Θ_2, formed by adding together the four "rectangles" of the type $\beta\varepsilon\eta + \beta\gamma\eta + \beta\varepsilon'\eta' + \beta\gamma\eta'$; and the boundary of Θ_2 is made up of two circles, viz. Σ_1, (the sum of the four 1-elements of the type $\gamma\eta + \gamma\eta'$), and Σ_1', (the sum of the 1-elements of the type $\beta\varepsilon + \beta\varepsilon'$).

Let σ and τ be two new vertices, and consider the array, E_3, defined to be $\Delta_3 + \sigma(\Theta_2 + \tau\Sigma_1)$.

a. E_3 is a 3-element.

Let Ω_2 be the sum of the four 2-elements of type $\alpha\beta\varepsilon + \alpha\delta\varepsilon + \alpha\beta\varepsilon' + \alpha\delta'\varepsilon'$, and let ω be a new vertex. Then

$$E_3 + \omega(\Omega_2 + \sigma\Sigma_1') \underset{2}{\rightarrow} \omega(\Omega_2 + \sigma\Sigma_1').$$

For the units of $\sigma\tau\Sigma_1$ can be removed according to the order of their edges in Σ_1; then the units of $\sigma\Theta_2$ in the order $\sigma\beta\gamma\eta$, $\sigma\beta\varepsilon\eta$, $\sigma\beta\gamma\eta'$, $\sigma\beta\varepsilon'\eta'$, etc.; and then the units of Γ_3 in the following order:

1.	$\beta\gamma\zeta\eta$	$(\gamma\eta)$	7.	$\alpha\delta\varepsilon\eta$	$(\alpha\eta)$
2.	$\beta\gamma\zeta\zeta'$	$(\gamma\zeta)$	8.	$\beta\gamma\zeta'\eta'$	(γ)
3.	$\alpha\beta\zeta\zeta'$	$(\zeta\zeta')$	9.	$\alpha\beta\zeta'\eta'$	$(\beta\zeta')$
4.	$\alpha\beta\zeta\eta$	$(\beta\zeta)$	10.	$\alpha\delta\zeta'\eta'$	$(\alpha\zeta')$
5.	$\alpha\delta\zeta\eta$	$(\alpha\zeta)$	11.	$\alpha\beta\varepsilon'\eta'$	$(\beta\eta')$
6.	$\alpha\beta\varepsilon\eta$	$(\beta\eta)$	12.	$\alpha\delta'\varepsilon'\eta'$	$(\alpha\eta')$

It will be verified that just before any one of these twelve moves the component which appears after the name of the unit to be removed is free, while the opposite component is interior to what remains of $E_3 + \omega(\Omega_2 + \sigma\Sigma_1')$.

The reduction is completed by dismantling in the same way the other three blocks making up Δ_3.

Now $\Omega_2 + \sigma\Sigma_1'$ is clearly a 2-element: the method of reducing it to a 2-simplex is obvious. It follows that $\omega(\Omega_2 + \sigma\Sigma_1')$ and therefore also $E_3 + \omega(\Omega_2 + \sigma\Sigma_1')$ are 3-elements. But $\omega(\Omega_2 + \sigma\Sigma_1')$ has the 2-element $\Omega_2 + \sigma\Sigma_1'$ in common with the remainder, E_3, of $E_3 + \omega(\Omega_2 + \sigma\Sigma_1')$. Hence, (*F II* Theorem 8*b*), E_3 is a 3-element.

b. E_3 contains no detachable unit.

E_3 has clearly no internal vertex, and it will be verified that no unit has more than one face in the boundary [1]).

[1]) In the original specification of Γ_3 every face belonging to $\overline{E_3}$ is indicated by a dot below the opposite vertex of the unit containing it. All other faces of Γ_3 belong to two units of Γ_3, or to $\sigma\Theta_2$, or to an array congruent to Γ_3,