Mathematics. — *The invariants of linear connexions with different transformations.* By Prof. J. A. SCHOUTEN. (Communicated by Prof. HENDRIK DE VRIES.)

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A RIEMANNian connexion with fundamental tensor $g_{\lambda\mu}$ being transformed conformally

$$'g_{\lambda\mu} = \sigma g_{\lambda\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the quantity of curvature is transformed into

$$'K_{\omega\mu\lambda\nu} = K_{\omega\mu\lambda\nu} - g_{[\omega\,[\lambda}\, s_{\mu]\,\nu]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

being

$$s_{\mu\lambda} = 2\,\nabla_\mu\, s_\lambda - s_\mu\, s_\lambda + {}^1/_2\, s_\alpha\, s^\alpha\, g_{\mu\lambda} \quad ; \quad s_\lambda = \partial_\lambda \log \sigma \text{ [1]} \quad . \quad . \quad (3)$$

Hence the connexion is then and only then conformal-euclidian if the quantity of curvature has the form

$$K_{\omega\mu\lambda\nu} = g_{[\omega\,[\lambda}\, L'_{\mu]\,\nu]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$L'_{\mu\lambda}$ being symmetrical, and if moreover a field s_λ can be found so that

$$2\,\nabla_\mu\, s_\lambda - s_\mu\, s_\lambda + {}^1/_2\, s_\alpha\, s^\alpha\, g_{\mu\lambda} = L'_{\mu\lambda} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now it is remarkable that the conditions of integrability of this equation are a consequence of (4) because of the identity of BIANCHI. Hence the equations (4) by themselves are necessary and sufficient to secure the conformal-euclidean property of the connexion [2]. In other wise we can say that the quantity

$$C_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} = K_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} - g_{[\mu\,[\lambda}\, L_{\mu]\,\nu]} \quad ; \quad L_{\mu\lambda} = -K_{\mu\lambda} + \frac{1}{2\,(n-1)}\, K\, g_{\mu\lambda} \quad . \quad (6)$$

is an invariant with conformal transformations [3]) and that the connexion is then and only then conformal-euclidean if this quantity vanishes.

WEYL [4]) has shown that this peculiarity occurs also in the case of the transformations of an affine connexion that leaves the geodesics invariant. Also in that case the conditions of integrability are a consequence of the algebraical conditions and the identity of BIANCHI and the connexion is then and only then projective-euclidean if an invariant quantity of the fourth degree vanishes.

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[1]) ∂_λ stands for $\frac{\partial}{\partial x^\lambda}$.

[2]) Math. Zeitschr. **11** (1921), p. 104—122; R.K. (Der Ricci Kalkül. Springer 1924), p. 169.

[3]) WEYL, Math. Zeitschr. 2 (1918), p. 384—411; Göttinger Nachrichten (1921), p. 99—112. Comp. R.K. p. 170.

[4]) Göttinger Nachrichten (1921), p. 99—112. Comp. R. K., p. 131.

In the following pages will be shown that the same peculiarity occurs in two other cases.

1. *Transformations leaving invariant all directions.*

By a linear connexion with $C_{\lambda\mu}^{\cdot\cdot\nu}=0$ [1]), but not necessary a symmetrical one, the local E_n in every point P of the X_n is represented linearly on every E_n belonging to a neighbouring point. Now we transform the connexion in such a way that the representation of all directions rests the same.

The formula for a general transformation is

$$'\Gamma^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} + A_{\lambda\mu}^{\cdot\cdot\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$A_{\lambda\mu}^{\cdot\cdot\nu}$ being an arbitrary quantity of the third degree. If a field v^{ν} is choosen in such a way that in an arbitrary point P $\nabla_{\mu} v^{\nu}$ is zero, then the differential

$$'\delta v^{\nu} = \delta v^{\nu} + A_{\lambda\mu}^{\cdot\cdot\nu} v^{\lambda} dx^{\mu} = A_{\lambda\mu}^{\cdot\cdot\nu} v^{\lambda} dx^{\mu} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

must have the same direction as v^{ν} for every choice of v^{ν} and dx^{ν}. This is then and only then possible when $A_{\lambda\mu}^{\cdot\cdot\nu}$ has the form

$$A_{\lambda\mu}^{\cdot\cdot\nu} = A^{\nu}_{\lambda}\, p_{\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

p_{λ} being an arbitrary vectorfield. Hence the general formula for a transformation leaving invariant the representation of directions is (7) with $A_{\lambda\mu}^{\cdot\cdot\nu} = A^{\nu}_{\lambda}\, p_{\mu}$. The general formula for the transformed quantity of curvature is

$$'R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - 2\nabla_{[\omega} A_{|\lambda|\mu]}^{\cdot\cdot\nu} - S_{\omega\mu}^{\cdot\cdot\alpha} A_{\lambda\alpha}^{\cdot\cdot\nu} - 2\, A_{\alpha[\omega}^{\cdot\cdot\nu} A_{|\lambda|\mu]}^{\cdot\cdot\alpha} \quad . \quad (10)$$

Substituting (9) in this equation we get

$$\left.\begin{aligned} 'R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} &= R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - 2\, A^{\nu}_{\lambda} \nabla_{[\omega}\, p_{\mu]} + 2\, S_{\omega\mu}^{\cdot\cdot\alpha}\, p_{\alpha} A^{\nu}_{\lambda} - 2\, A^{\nu}_{\lambda}\, p_{[\omega}\, p_{\mu]} \\ &= R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - 2\, A^{\nu}_{\lambda}\, \partial_{[\omega}\, p_{\mu]} \end{aligned}\right\} \quad (11)$$

and

$$'V_{\omega\mu} = V_{\omega\mu} - 2\, n\, \partial_{[\omega}\, p_{\mu]} \quad ; \quad V_{\omega\mu} = R_{\omega\mu\alpha}^{\cdot\cdot\cdot\alpha} \quad . \quad . \quad . \quad . \quad (12)$$

hence

$$'R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - \frac{1}{n}\, 'V_{\omega\mu}\, A^{\nu}_{\lambda} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - \frac{1}{n}\, V_{\omega\mu}\, A^{\nu}_{\lambda} = \text{invariant.} \quad . \quad . \quad (13)$$

Writing $A_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ for this invariant quantity we deduce immediately

$$A_{\omega\mu\lambda}^{\cdot\cdot\cdot\lambda} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and

$$A_{\alpha\mu\lambda}^{\cdot\cdot\cdot\alpha} = R_{\mu\lambda} - \frac{1}{n} V_{\lambda\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

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[1]) R.K. p. 75.

From (15) we see that there exists also an invariant quantity of the second degree, which is in general not zero.

$A_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ is zero if the transformed connexion is euclidean (= integrable). But vice versa, if $A_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ vanishes it is possible to transform the connexion into an euclidean one by a transformation of the form (7,9).

Proof.

The supposition $A_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}=0$ is aequivalent to the supposition that $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ has the form

$$R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}=M_{\omega\mu}\,A_{\lambda}^{\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

By contraction we get $M_{\omega\mu}=\frac{1}{n}R_{\omega\mu\alpha}^{\cdot\cdot\cdot\alpha}$. Now we try to find a field p_λ so that

$$2\,\partial_{[\omega}\,p_{\mu]}=M_{\omega\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The conditions of integrability are

$$\partial_{[\xi}\,M_{\omega\mu]}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

But by applying the identity of BIANCHI on (16) we get

$$\nabla_{[\xi}\,M_{\omega\mu]}\,A_{\lambda}^{\nu}=-S_{[\omega\mu}^{\cdot\cdot\alpha}\,R_{\xi]\alpha\lambda}^{\cdot\cdot\cdot\nu}=0 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

from which equation follows

$$\partial_{[\xi}\,M_{\omega\mu]}=-S_{[\omega\xi}^{\cdot\cdot\alpha}\,M_{|\alpha|\mu]}-S_{[\mu\xi}^{\cdot\cdot\alpha}\,M_{|\alpha|\omega]}-2\,S_{[\omega\mu}^{\cdot\cdot\alpha}\,M_{\xi]\alpha}=0 \quad (20)$$

Hence the conditions of integrability are a consequence of (16) and the identity of BIANCHI, and we have got the following theorem:

It is then and only then possible to transform a linear connexion (with $C_{\lambda\mu}^{\cdot\cdot\nu}=0$) into an euclidean one by a transformation leaving invariant all representations of directions, if the quantity of curvature has the form $M_{\omega\mu}\,A_{\lambda}^{\nu}$, in other words if the quantity $A_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}=R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}-\frac{1}{n}R_{\omega\mu\alpha}^{\cdot\cdot\cdot\alpha}\,A_{\lambda}^{\nu}$, which is an invariant with these transformations, vanishes.

Besides $A_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ there exists another invariant quantity of the fourth degree, namely the projective quantity of curvature $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ belonging to the symmetrical connexion with the parameters

$$\overset{0}{\Gamma}{}_{\lambda\mu}^{\nu}=\Gamma_{(\lambda\mu)}^{\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

By substituting $A_{\lambda\mu}^{\cdot\cdot\nu}=-S_{\lambda\mu}^{\cdot\cdot\nu}$ in the general formula (10) it follows that the quantity of curvature of this connexion is

$$\overset{0}{R}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}=R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}+2\,\nabla_{[\omega}\,S_{|\lambda|\mu]}^{\cdot\cdot\nu}-2\,S_{\omega\mu}^{\cdot\cdot\alpha}\,S_{\lambda\alpha}^{\cdot\cdot\nu}-2\,S_{\alpha[\omega}^{\cdot\cdot\nu}\,S_{|\lambda|\mu]}^{\cdot\cdot\alpha} \quad . \quad (22)$$

and from $\overset{0}{R}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ we get $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ in the well known way [1]).

[1]) C.f. f.i. R.K. p. 131.

2. *Transformations of the form* $'\Gamma^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} + q_{\lambda} A^{\nu}_{\mu}$.

In the second place we consider the case where the representation of the vectors in P on the E_n in a neighbouring point Q is changed in such a way that all extremities of vectors move parallel to δx^{ν}. If a field v^{ν} is choosen so that $\nabla_{\mu} v^{\nu}$ is zero in P, than $'\delta v^{\nu}$ must have the same direction as dx^{ν}. From this follows that $A_{\lambda\mu}^{\cdot\cdot\nu}$ has the form

$$A_{\lambda\mu}^{\cdot\cdot\nu} = q_{\lambda} A^{\nu}_{\mu}. \quad \ldots \ldots \ldots \quad (23)$$

q_{λ} being an arbitrary vectorfield. The transformed quantity of curvature is

$$\left.\begin{aligned} 'R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} &= R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - 2 A^{\nu}_{[\mu} \nabla_{\omega]} q_{\lambda} + 2 S_{\omega\mu}^{\cdot\cdot\nu} q_{\lambda} - 2 A^{\nu}_{[\omega} q_{\mu]} q_{\lambda} \\ &= R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} + 2 A^{\nu}_{[\omega} (\nabla_{\mu]} q_{\lambda} - q_{\mu]} q_{\lambda}) + 2 S_{\omega\mu}^{\cdot\cdot\nu} q_{\lambda} \end{aligned}\right\} \quad (24)$$

hence

$$\left.\begin{aligned} 'R_{\mu\lambda} &= R_{\mu\lambda} + n \nabla_{\mu} q_{\lambda} - \nabla_{\mu} q_{\lambda} - n\, q_{\mu} q_{\lambda} + q_{\mu} q_{\lambda} + S_{\alpha\mu}^{\cdot\cdot\alpha} q_{\lambda} \\ &= R_{\mu\lambda} + (n-1)(\nabla_{\mu} q_{\lambda} - q_{\mu} q_{\lambda}) + 2 S_{\alpha\mu}^{\cdot\cdot\alpha} q_{\lambda} \end{aligned}\right\} \quad (25)$$

and

$$'S_{\lambda\mu}^{\cdot\cdot\nu} = S_{\lambda\mu}^{\cdot\cdot\nu} + q_{[\lambda} A^{\nu}_{\mu]} \quad ; \quad 'S_{\alpha\mu}^{\cdot\cdot\alpha} = S_{\alpha\mu}^{\cdot\cdot\alpha} - {}^1\!/_2 (n-1) q_{\mu} \quad . \quad . \quad (26)$$

From (24—26) we deduce that the quantity

$$\left.\begin{aligned} T_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - \frac{2}{n-1} A^{\nu}_{[\omega} R_{\mu]\lambda} + \\ + \frac{4}{n-1} S_{\omega\mu}^{\cdot\cdot\nu} S_{\alpha\lambda}^{\cdot\cdot\nu} - \frac{8}{(n-1)^2} A^{\nu}_{[\omega} S_{|\alpha|\mu]}^{\cdot\cdot\alpha} S_{\beta\lambda}^{\cdot\cdot\beta} \end{aligned}\right\} \quad (27)$$

is invariant. From (27) follows

$$T_{\alpha\mu\lambda}^{\cdot\cdot\cdot\alpha} = 0 \quad \ldots \ldots \ldots \quad (28)$$

and

$$T_{\omega\mu\alpha}^{\cdot\cdot\cdot\alpha} = V_{\omega\mu} + \frac{2}{n-1} R_{[\omega\mu]} + \frac{4}{n-1} S_{\omega\mu}^{\cdot\cdot\beta} S_{\alpha\beta}^{\cdot\cdot\alpha} \quad . \quad . \quad . \quad (29)$$

From (29) we see that there exists also an invariant quantity of the second degree, which is in general not zero. If the given connexion is halfsymmetrical

$$S_{\lambda\mu}^{\cdot\cdot\nu} = S_{[\lambda} A^{\nu}_{\mu]} \quad \ldots \ldots \ldots \quad (30)$$

then the transformed connexion has the same property and $T_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ and $T_{\omega\mu\alpha}^{\cdot\cdot\cdot\alpha}$ take the simple forms

$$T_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - \frac{2}{n-1} A^{\nu}_{[\omega} R_{\mu]\lambda} \quad \ldots \ldots \quad (31)$$

$$T_{\omega\mu\alpha}^{\cdot\cdot\cdot\alpha} = V_{\omega\mu} + \frac{2}{n-1} R_{[\omega\mu]} \quad \ldots \ldots \quad (32)$$

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[1]) R. K. p. 69.

From (24) follows that $T_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ is zero when the transformed connexion is euclidean. But vice versa, if $T_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ vanishes for a halfsymmetrical connexion it is for $n > 2$ always possible to transform the connexion into an euclidean one by a transformation with an $A_{\lambda\mu}^{\cdot\cdot\nu}$ of the form (23).

PROOF.

The supposition $T_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 0$ is aequivalent to the supposition that $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ has the form

$$R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = \frac{2}{n-2} A_{[\omega}^{\nu} R_{\mu]\lambda} \quad \ldots \ldots \quad (33)$$

Now we try to find a field q_λ so that

$$R_{\mu\lambda} = -(n-1)(\nabla_\mu q_\lambda - q_\mu q_\lambda - S_\mu q_\lambda) \quad \ldots \quad (34)$$

The conditions of integrability are

$$\left.\begin{aligned} 2\nabla_{[\omega} R_{\mu]\nu} = &-(n-1) R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} q_\nu - 2(n-1) S_{[\omega} \nabla_{\mu]} q_\lambda + \\ &+ 2(n-1) \nabla_{[\omega} q_{\mu]} q_\lambda + 2(n-1) \nabla_{[\omega} S_{\mu]} q_\lambda \end{aligned}\right\} \quad (35)$$

which equation passes into

$$\left.\begin{aligned} 2\nabla_{[\omega} R_{\mu]\lambda} = &- 2(n-1) q_{[\omega} S_{\mu]} q_\lambda - 4(n-1) S_{[\omega} \nabla_{\mu]} q_\lambda + \\ &+ 2(n-1) q_\lambda \nabla_{[\omega} q_{\mu]} + 2(n-1) q_\lambda \nabla_{[\omega} S_{\mu]} \end{aligned}\right\} \quad (36)$$

by applying (34) and (35). But by applying the identity of BIANCHI on (33) we get

$$-2 S_{[\omega} R_{\xi\mu]\lambda}^{\cdot\cdot\cdot\nu} = \frac{2}{n-1} A_{[\omega}^{\nu} \nabla_\xi R_{\mu]\lambda} \quad \ldots \ldots \quad (37)$$

or, regarding (33)

$$-\frac{4}{n-1} S_{[\omega} A_\xi^\nu R_{\mu]\lambda} = \frac{2}{n-1} A_{[\omega}^{\nu} \nabla_\xi R_{\mu]\lambda} \quad \ldots \quad (38)$$

from which follows for $n \neq 2$

$$2\nabla_{[\omega} R_{\mu]\lambda} = 4 S_{[\omega} R_{\mu]\lambda} \quad \ldots \ldots \quad (39)$$

or, in consequence of (34)

$$2\nabla_{[\omega} R_{\mu]\lambda} = -4(n-1) S_{[\omega} \nabla_{\mu]} q_\lambda + 4(n-1) S_{[\omega} q_{\mu]} q_\lambda \quad . \quad (40)$$

Now the second identity [1]) for a halfsymmetrical connexion is

$$R_{[\omega\mu\lambda]}^{\cdot\cdot\cdot\nu} = 2\nabla_{[\omega} S_\mu A_{\lambda]}^{\nu} \quad \ldots \ldots \quad (41)$$

[1]) R. K. p. 88.

from which follows, regarding (33)

$$\frac{2}{n-1} A^{\nu}_{[\omega} R_{\mu\lambda]} = 2 A^{\nu}_{[\omega} \nabla_{\mu} A^{\nu}_{\lambda]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

for $n > 2$ aequivalent to

$$\nabla_{[\mu} S_{\lambda]} = \frac{1}{n-1} R_{[\mu\lambda]} = - \nabla_{[\mu} q_{\lambda]} + S_{[\mu} q_{\lambda]} \quad . \quad . \quad . \quad (43)$$

In consequence of this equation the conditions of integrability (36) pass into the equation (39) which follows from the identity of BIANCHI, and we have got the following theorem:

It is for $n > 2$ then and only then possible to transform a half-symmetrical linear connexion (with $C_{\lambda\mu}^{\cdot\cdot\,\nu} = 0$) into an euclidean one by a transformation of the form $'\Gamma^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} + q_{\lambda} A^{\nu}_{\mu}$ if the quantity of curvature has the form $\frac{2}{n-1} A^{\nu}_{[\omega} R_{\mu]\lambda}$, in other words if the quantity $T_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} - \frac{2}{n-1} A^{\nu}_{[\omega} R_{\mu]\lambda}$ vanishes.

The quantity $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}$ (§ 1) is also invariant with transformations characterised by (23).