Mathematics. — *Fusion of the existing Theories of the Irrational Number into a New Theory.* By Prof. FRED. SCHUH. (Communicated by Prof. D. J. KORTEWEG).

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1. Introduction. In the existing theories of the irrational number as those of CANTOR, DEDEKIND, BAUDET and WEIERSTRASS (cf. my book: "Het Getalbegrip, in het bijzonder het Onmeetbare getal, met toepassingen op de Algebra, de Differentiaal- en de Integraalrekening") definitions are given of irrational numbers, of the operations addition and multiplication of real (rational or irrational) numbers and of the relation greater. It follows from these definitions that *for the real numbers the same calculation-rules* (*and the same proporties connected with the relation greater*) *are valid as for the rational numbers* and that for the system of the real numbers the following *theorem of the upper boundary holds good:*

If S is a non-empty set of real numbers all of which are less than or equal to the same real number (bounded on the right), there exists a real number U (upper boundary of S), with the following two properties:

1. No number of S is $> U$;

2. If U' is a real number $< U$, S contains at least one number which is $> U'$.

It is at once clear that *there can only be one number which corresponds to the definition of upper boundary.*

As I have explained in my "Getalbegrip", for the application of the irrational numbers we have only to do with these results and not with the meaning of addition, etc. For this reason the theorem of the upper boundary is the keystone of each of the above mentioned theories of the irrational number and plays a fundamental part in the Algebra, the Differential and the Integral calculus. Instead of it we might also take a theorem that is equivalent to the theorem of the upper boundary, as the theorem of BOLZANO or the general principle of convergence of CAUCHY. However, the theorem of the upper boundary (or that of the lower boundary, which comes about to the same) seems to me to be the most suitable.

It would be very unpracticle to introduce the irrational number into Algebra and Analysis in another way than via the theorem of the upper boundary (or an equivalent theorem), e. g. by defining a^x for an irrational value of x as a convergent sequence of CANTOR or a section of DEDEKIND, or by proving the theorem about the zero-point of a continuous function which can assume positive and negative values, by producing that zero-point of the function as a convergent sequence of CANTOR or a section of DEDEKIND. This would give the wrong impression that there exists an Algebra according to CANTOR and one according to DEDEKIND. As all the theories of the irrational number come together in the theorem of the upper boundary, in Algebra we have nothing to do with the chosen theory.

The purpose of this paper is to show *that we can make the different theories of the irrational number concur still more,* so that the part played by the theorem of the upper boundary becomes still greater. By so doing the theories of the irrational number approach each other more *because the definitions of addition and multiplication become the same.* The difference between the various theories of the irrational number only remains in the definition of this numbers, in the definition of "greater" and in a few proofs. In this way the theory of the irrational number (if we stop where it coincides with other theories) becomes considerably shorter and simpler, because addition and multiplication no longer belong to it but must be reckoned among the applications [1]).

2. Contents of a theory of the irrational number. We assume that the *rational numbers* have been introduced and that for them the well known fundamental properties (cf. my "Getalbegrip", § 18) have been proved.

A theory of the irrational number means (according to the ideas developed in this paper) an extension of the system of the rational numbers to that of the *real numbers* and such a definition of "greater" in the new system that the following properties are valid:

a. For any two real numbers α and β one and only one of the following three relations holds good:

$\alpha = \beta$ *(both numbers are the same, also written as* $\beta = \alpha$*),*

$\alpha > \beta$ *(also written as* $\beta < \alpha$*),*

$\beta > \alpha$ *(also written as* $\alpha < \beta$*).*

b. If $\alpha > \beta$ and $\beta > \gamma$, we have $\alpha > \gamma$ (transitive property).

c. If α is a real number there always exists a rational number $> \alpha$.

d. If α is a real number, there always exists a rational number $< \alpha$.

e. If α and β are real numbers and if $\alpha < \beta$, there always exists a rational number c, so that $\alpha < c < \beta$.

f. For the system of the real numbers the theorem of the upper boundary holds good.

The property *a.* includes that *the definition of "greater" given for the real numbers, if applied to two rational numbers, leads to the same result as the definition of "greater" for rational numbers.*

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[1]) A more elaborate treatment and a discussion of the connection with the existing theories can be found in the periodical "Christiaan Huygens".

If in a definite theory (e.g. that of CANTOR) the five above mentioned properties have been proved [1]), this theory may be considered as finished because its continuation is the same as for another theory.

3. Enclosure of a real number between two rational numbers. Relating to this we shall prove the following property (always starting from the properties of N⁰. 2):

If α is a given real number and v a given positive rational number, we can always determine a rational number a so that $a < \alpha < a + v$.

As the validity of this property is at once apparent if α is rational, we shall suppose α to be irrational. According to the properties *c.* and *d.* of N⁰. 2 there are rational numbers b and c so that $b < \alpha < c$. The numbers $b + nv$, in which we can choose for n any postive integer or 0, contain only a finite number $< c$, hence also a finite number $< \alpha$ (according to the property *b.* of N⁰. 2). Hence among the numbers $b + nv$ that are $< \alpha$ there is one greatest. If this is a we have $a < \alpha < a + v$, as $\alpha = a + v$ is impossible, α being irrational.

4. *If α is a given positive real number and v is a given rational number > 1, we can always determine a positive rational number a so that $a < \alpha < av$.*

According to the property *e.* of N⁰. 2 there exists a rational number b, so that $0 < b < \alpha$. According to N⁰. 3 there exists a rational number a, so that $a < \alpha < a + b(v-1)$. Hence $b < \alpha < bv$, if $a < b$, and $a < \alpha < av$, if $b \leqq a$.

5. Definition of the addition of real numbers. Let α and β be two (equal or unequal) real numbers; let A be the set of the rational numbers $a < \alpha$ and B the set of the rational numbers $b < \beta$. According to the property *d.* of N⁰. 2 neither A nor B is empty so that the *set* $A + B$ *of the numbers* $a + b$ is not empty either. Let a' be a rational number $> \alpha$ and b' a rational number $> \beta$. From $a < \alpha$ and $\alpha < a'$ it follows in connection with the transitive property *b.* of N⁰. 2, that $a < a'$. In the same way $b < b'$, hence $a + b < a' + b'$, *so that the set $A + B$ is bounded on the right and therefore has an upper boundary. This upper boundary, which we shall call $\alpha + \beta$,* defines *the sum of the two real numbers α and β.*

6. If $a + b$ is a number of the set $A + B$ of N⁰. 5 and a_1 is a rational number so that $a < a_1 < \alpha$, also $a_1 + b$ is a number of $A + B$. We have, therefore, $a_1 + b \leqq \alpha + \beta$ (as $\alpha + \beta$ is the upper boundary of $A + B$), hence $a + b < \alpha + \beta$ (according to $a + b < a_1 + b$ and the transitivity).

Let inversely c be a rational number $< \alpha + \beta$. If d is a rational

[1]) The proofs of these properties are not given in this paper as they are also given in existing theories. They may be found in a paper destined for "Christiaan Huygens".

number so that $c < d < \alpha + \beta$, according to the property of N^0. 3 we can define the rational numbers a and b such that

$$a < \alpha < a + \tfrac{1}{2}(d - c), \qquad b < \beta < b + \tfrac{1}{2}(d - c).$$

If we represent the third members of these inequalities resp. by a' and b', we have $a' + b' = a + b + d - c$. From $a' > \alpha$, $b' > \beta$ it follows that a' is greater than any number of A and b' greater than any number of B, hence $a' + b'$ greater than any number of $A + B$. Consequently $a' + b' \geqq \alpha + \beta$ (according to the definition of the upper boundary). As $d < \alpha + \beta$, we have, therefore, $a' + b' > d$, hence $a + b > c$. Consequently $c - a < b$, hence $c - a < \beta$ (as $b < \beta$), so that $c - a$ is a number of B. As $c = a + (c - a)$, c is a number of $A + B$. We find accordingly:

The set $A + B$ of N^0. 5 (formed by the numbers $a + b$, where a is an arbitrary rational number $< \alpha$ and b an arbitrary rational number $< \beta$) is the same as that of the rational numbers $< \alpha + \beta$.

As an immediate consequence $a' + b' \geqq \alpha + \beta$ *if a' is a rational number $\geqq \alpha$ and b' a rational number $\geqq \beta$.* For $a' + b'$ is greater than any number of $A + B$; hence it is not a number of $A + B$ and does not satisfy the relation $a' + b' < \alpha + \beta$.

7. Fundamental properties of addition. These are:

I. *From any two real numbers α and β we can derive one and only one real number $\alpha + \beta$ by addition.*

II. $\alpha + \beta = \beta + \alpha$ *(commutativity of addition).*

III. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ *(associativity of addition).*

IV. $\alpha + 0 = \alpha$ *(modulus-property of addition).*

V. *If $\alpha < \beta$, than $\alpha + \gamma < \beta + \gamma$.*

The validity of I appears from the fact *that the definition of addition of real numbers given in N^0. 5, applied to two rational numbers, leads to the same result as the earlier addition of rational numbers.*

The validity of II is at once evident from the corresponding property of the rational numbers.

The same holds good for III. For if we represent a rational number $< \alpha$, $< \beta$ or $< \gamma$ resp. by a, b, c, according to the property of N^0. 6 a rational number $< \alpha + \beta$ is the same as a number of the form $a + b$. Consequently $(\alpha + \beta) + \gamma$ is the upper boundary of the set of the numbers $(a + b) + c$. In the same way $\alpha + (\beta + \gamma)$ is the upper boundary of the set of the numbers $a + (b + c)$. Both sets are the same and have, therefore, the same upper boundary.

If α is a real number and if a and b are rational numbers so that $a < \alpha$, $b < 0$, we have $a + b < a$, hence $a + b < \alpha$. If, inversely, c is a rational number $< \alpha$ and a a rational number so that $c < a < \alpha$, we have $c = a + (c - a)$ where $a < \alpha$ and $c - a < 0$. The set of the numbers $a + b$ (a rational and $< \alpha$, b rational and < 0), that has $\alpha + 0$ as upper boundary, is, therefore, the same as the set of the rational numbers $< \alpha$. Hence $\alpha + 0 = \alpha$. In this way IV is proved.

If $\alpha < \beta$, we can determine the rational numbers p and q so that $\alpha < p < q < \beta$. If γ is a given real number, according to the property of N⁰. 3 we can determine the rational number c so that $c < \gamma < c + q - p$. According to the property of N⁰. 6 $\beta + \gamma > q + c$, and from the remark at the end of N⁰. 6 it follows that $\alpha + \gamma \leqq p + (c + q - p) = q + c$, whence $\alpha + \gamma < \beta + \gamma$. In this way V is proved.

8. Subtraction of real numbers. The *possibility and unambiguity of subtraction* rests on the fundamental properties I—IV of N⁰. 7 and

VI. *If α is a given real number, there exists at least one real number ξ which satisfies $\alpha + \xi = 0$ (possibility of subtraction from 0).*

If we represent a rational number $> \alpha$ by a', the set S of the numbers $-a'$ is bounded on the right (as all these numbers are $< -a$, if a is a rational number $< \alpha$). Hence the set has an upper boundary. We shall call this ξ. If a' and p are rational numbers so that $a' > p > \alpha$, we have $-p \leqq \xi$ (as $-p$ is a number of S); as $-a' < -p$ we may write $-a' < \xi$. If inversely b is a rational number $< \xi$, there exists such a rational number a' that $a' > \alpha$ and $-a' > b$ (according to the definition of upper boundary); consequently $-b > a'$, hence $-b > \alpha$, so that b is the opposite of a rational number $> \alpha$. *The set of the rational numbers $< \xi$ is, therefore, the same as that of the numbers of the form $-a'$ where a' is rational and $> \alpha$.* From this follows that $\alpha + \xi$ is the upper boundary of the set of the numbers $a + (-a') = -(a' - a)$ (a rational and $< \alpha$, a' rational and $> \alpha$). As $a' - a$ is positive and can assume any positive value (according to the property of N⁰. 3), $\alpha + \xi$ is the upper boundary of the set of all negative rational numbers, whence $\alpha + \xi = 0$. In this way VI is proved.

From VI we can further deduce that $\beta - \alpha = \beta + (-\alpha)$ where $-\alpha$ (the opposite of α) is an abbreviation for $0 - \alpha$.

9. From $\alpha + \xi = 0$ and $\alpha > 0$ follows $0 = \alpha + \xi > 0 + \xi = \xi$, hence $0 > \xi$, *so that the opposite of a positive real number is negative.* In the same way it appears that *the opposite of a negative real number is positive.*

From

$$\{-\alpha + (-\beta)\} + (\alpha + \beta) = (-\alpha + \alpha) + (-\beta + \beta) = 0 + 0 = 0$$

there follows:

$$-(\alpha + \beta) = -\alpha + (-\beta),$$

so that the opposite of the sum of two real numbers is the sum of the opposites of these numbers.

10. Definition of the multiplication of real numbers. Let α and β be two positive real numbers, A the set of the positive rational numbers $a < \alpha$ and B the set of the positive rational numbers $b < \beta$. According to the property e. of N⁰. 2 neither A nor B is empty so that the set AB of the numbers ab is not empty either. If a' and b' are rational numbers that are $> \alpha$ resp. $> \beta$, we have $a < a'$ and $b < b'$, hence

$ab < a'b'$. Accordingly *the set AB is bounded on the right and hence has an upper boundary. We shall call this $\alpha\beta$* and *define in this way the* p r o d u c t *of two positive real numbers.*

Supplementary definitions. If α is positive and β negative (hence $-\beta$ positive), we define $\alpha\beta$ as $-\{\alpha(-\beta)\}$. Likewise we define $\alpha\beta = -\{(-\alpha)\beta\}$, as *$\alpha$ negative and β positive. If α and β are both negative* we define $\alpha\beta = (-\alpha)(-\beta)$.

If $\alpha = 0$ or $\beta = 0$, $\alpha\beta$ is defined as 0.

11. In a similar way as in N^0. 6 it appears that a number ab of the set AB of N^0. 10 is less than $\alpha\beta$ (where α and β are positive real numbers). Let c be a positive rational number $< \alpha\beta$. If d is a rational number so that $c < d < \alpha\beta$ and e is a rational number so that $1 < e < \frac{d}{c}$, we have $\frac{d}{c} = ef$ where f is a rational number > 1. According to the property of N^0. 4 we can determine the positive rational numbers a and b such that

$$a < \alpha < ae, \qquad b < \beta < bf.$$

Consequently ae is greater than any number of A and bf greater than any number of B, hence $abef$ is greater than any number of AB so that $abef \geqq \alpha\beta$. As $d < \alpha\beta$ we have $abef > d$, hence $ab > c$. Accordingly $\frac{c}{a} < b$, hence $\frac{c}{a} < \beta$, so that $\frac{c}{a}$ is a number of B. Owing to $c = a.\frac{c}{a}$, c is a number of AB.

Accordingly:

If α and β are positive real numbers, the set AB of the numbers ab (where a and b are arbitrary positive rational numbers resp. $< \alpha$ and $< \beta$) is the same as that of the positive rational numbers $< \alpha\beta$.

12. Fundamental properties of multiplication. These are:

VII. *From any two real numbers α and β one and only one real number $\alpha\beta$ may be derived through multiplication.*

VIII. $\alpha\beta = \beta\alpha$ (*commutativity of multiplication*).

IX. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (*associativity of multiplication*).

X. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ (*distributive property*).

XI. $\alpha \,.\, 1 = \alpha$ (*modulus-property of multiplication*).

XII. *If $\alpha > 0$ and $\beta > 0$, than $\alpha\beta > 0$.*

Proof of VII. The definition of multiplication of positive real numbers given in N^0. 10, if applied to two positive rational numbers, leads to the same result as the earlier multiplication of positive rational numbers. As also the supplementary definitions of N^0. 10 are in accordance with the properties of the rational numbers, multiplication is an unambiguous operation.

Proof of VIII. This rests on the corresponding property of rational numbers.

Proof of IX. If α, β, γ are all positive, the proof is quite analogous to that of the fundamental property III (cf. N⁰. 7). If one (or more) of the numbers α, β, γ is zero, $(\alpha\beta)\gamma$ as well as $\alpha(\beta\gamma)$ is zero. If α, β, γ are all $\neq 0$ but not all positive, the validity of $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ follows from the validity for positive real numbers and the supplementary definitions of N⁰. 10.

Proof of X. We shall first suppose that α, β and γ *are all three positive*. It appears from N⁰. 6 that $\alpha(\beta+\gamma)$ is the upper boundary of the set S of the numbers $a(b+c)$ where a, b and c are arbitrary positive rational numbers resp. $< \alpha$, $< \beta$ and $< \gamma$. From the property of N⁰. 11 it is further evident that $\alpha\beta + \alpha\gamma$ is the upper boundary of the set T of the numbers $a_1 b + a_2 c$ where a_1 and a_2 are rational numbers between 0 and α. As $a(b+c) = ab + ac$, any number of S is also a number of T. If $a_1 \leqq a_2$ we have $a_1 b + a_2 c \leqq a_2(b+c)$, so that to any number t of T there corresponds a number of S that is $\geqq t$. Hence S and T have the same upper boundary so that $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$.

If $\alpha = 0$, $\beta = 0$ or $\gamma = 0$, both $\alpha(\beta+\gamma)$ and $\alpha\beta + \alpha\gamma$ are resp. 0, $\alpha\gamma$ or $\alpha\beta$, so that in this case X holds good. We may therefore suppose α, β and γ all three $\neq 0$.

If α and β are positive, γ negative and $\beta + \gamma \geqq 0$, than $-\gamma$ is positive (cf. N⁰. 9). As X holds good for $\alpha > 0$, $\beta \geqq 0$, $\gamma > 0$, we have:

$$\alpha(\beta+\gamma) = \{\alpha(\beta+\gamma) - \alpha\gamma\} + \alpha\gamma = \{\alpha(\beta+\gamma) + \alpha(-\gamma)\} + \alpha\gamma =$$
$$= \alpha\{(\beta+\gamma) + (-\gamma)\} + \alpha\gamma = \alpha\beta + \alpha\gamma.$$

In connection with II (cf. N⁰. 7) this implies the validity of X in any case *where* $\alpha > 0$ *and* $\beta + \gamma \geqq 0$.

Taking N⁰. 9 into account, we derive from this the validity of X in the case *that* $\alpha > 0$ *and* $\beta + \gamma < 0$. For in this case $-(\beta+\gamma) > 0$, hence:

$$\alpha(\beta+\gamma) = -\alpha\{-(\beta+\gamma)\} = -\alpha\{-\beta + (-\gamma)\} =$$
$$= -\{\alpha(-\beta) + \alpha(-\gamma)\} = -\{-\alpha\beta + (-\alpha\gamma)\} = \alpha\beta + \alpha\gamma.$$

Consequently X always holds good *for* $\alpha > 0$. From this the validity *for* $\alpha < 0$ ensues thus:

$$\alpha(\beta+\gamma) = -(-\alpha)(\beta+\gamma) = -\{(-\alpha)\beta + (-\alpha)\gamma\} =$$
$$= -\{-\alpha\beta + (-\alpha\gamma)\} = \alpha\beta + \alpha\gamma.$$

Proof of XI. If α is a positive real number and if a and b are rational numbers satisfying $0 < a < \alpha$, $0 < b < 1$, we have $ab < a$, hence $ab < \alpha$. If, inversely, c is a positive rational number $< \alpha$ and a a rational number so that $c < a < \alpha$, we have $c = a \cdot \frac{c}{a}$ where $0 < a < \alpha$ and $0 < \frac{c}{a} < 1$. The set of the numbers ab (a and b rational, $0 < a < \alpha$,

$0 < b < 1$), that has $\alpha . 1$ as upper boundary, is, therefore, identical with the set of the positive rational numbers $< \alpha$, so that $\alpha . 1 = \alpha$.

$\alpha . 1 = \alpha$ also holds good for $\alpha = 0$ as in this case both members are 0 and the validity for $\alpha < 0$ follows from $\alpha . 1 = -\{(-\alpha) . 1\} = -(-\alpha) = \alpha$.

Proof of XII. If $\alpha > 0$ and $\beta > 0$ and if a and b are positive rational numbers resp. $< \alpha$ and $< \beta$, we have $ab \leqq \alpha\beta$ (according to the definition of upper boundary). Consequently from $ab > 0$ follows $\alpha\beta > 0$.

13. Division of real numbers. *The possibility and unambiguity of division except by 0,* rests on the fundamental properties VII, VIII, IX and XI of N^0. 12 and

XIII. *If α is a given real number different from zero, there is at least one real number ξ so that $\alpha\xi = 1$ (possibility of division in 1 except by* 0).

We shall first suppose $\alpha > 0$. We represent a rational number $> \alpha$ by a'. The set of the numbers $\frac{1}{a'}$ has an upper boundary (as all these numbers are $< \frac{1}{a}$ where a is a positive rational number $< \alpha$); let this upper boundary be ξ. If a' and p are rational numbers so that $a' > p > \alpha$, we have $\frac{1}{p} \leqq \xi$, hence $\frac{1}{a'} < \xi$. If, inversely, b is a positive rational number $< \xi$, there exists a rational number a' so that $a' > \alpha$ and $\frac{1}{a'} > b$; consequently $\frac{1}{b} > a'$, and therefore $\frac{1}{b} > \alpha$. *The set of the positive rational numbers $< \xi$ is, accordingly, identical with the set of the numbers of the form $\frac{1}{a'}$ where a' is rational and* $> \alpha$. Consequently $\alpha\xi$ is the upper boundary of the set of the numbers $a . \frac{1}{a'} = \frac{a}{a'}$ (a positive rational and $< \alpha$, a' rational and $> \alpha$). As $\frac{a}{a'}$ can assume any positive value < 1 (according to the property of N^0. 4), this upper boundary is 1 so that $\alpha\xi = 1$, hence $\xi = \frac{1}{\alpha}$.

If $\alpha < 0$ we have $\alpha . \left(-\frac{1}{-\alpha}\right) = (-\alpha) . \frac{1}{-\alpha} = 1$, so that XIII is also proved for negative values of α.

From XIII we find in the known way that $\frac{\beta}{\alpha} = \beta . \frac{1}{\alpha}$ $(\alpha \neq 0)$.