**Mathematics.** — *The Definition of Euclidean Measure in the Projective Plane.* By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE).

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Euclidean metrics in the plane are fixed by choosing 2 conjugate complex points $I_1$ and $I_2$, the isotropic points.

The projective transformations that leave these points invariant, form the group of similitude. The definition of *angular measure* rests immediately on the projective invariant par excellence, the *anharmonic ratio*. For, if $A$, $B$ and $C$ are three points of the plane, according to the formula of LAGUERRE:

$$\angle BAC = \frac{1}{2i} \log A\,(I_1\, I_2\, B\, C),$$

where $A\,(I_1 I_2 BC)$ is the anharmonic ratio of the rays $AI_1$, $AI_2$, $AB$ and $AC$.

The definition of *distance* is less simple. It is often given as the limit of the expression which holds good in non-euclidean geometry, where angle and distance are entirely dualistic and the latter is likewise defined by the aid of an anharmonic ratio.

In what follows we shall give a direct definition of distance that is entirely based on anharmonic ratios.

We must notice that by choosing $I_1$ and $I_2$ only *ratios* of distances will be defined. For the definition of ratios of distances it is further sufficient to restrict ourselves to the comparison of segments that have a common extremity as after introducing the straight line at infinity, we can already ascertain the equality of parallel line segments.

*If $\overline{AB}$ and $\overline{AC}$ represent the lengths of the segments* we give the following definition:

$$\overline{AB}^2 : \overline{AC}^2 = I_1\,(I_2\, A\, B\, C)\,.\,I_2\,(I_1\, A\, B\, C).$$

*The square of the ratio of the distances is the product of two anharmonic ratios.*

In order to prove that this definition agrees with the usual definition of length we choose the homogeneous coordinates of $I_1$ and $I_2$ resp. $(1, i, 0)$ and $(1, -i, 0)$, and the Cartesian coordinates of $A, B$ and $C$ resp. $(a_1, a_2)$, $(b_1, b_2)$, $(c_1, c_2)$. The four lines $I_1\, I_2$, $I_1\, A$, $I_1\, B$, $I_1\, C$ cut the $Y$-axis at four points of which the $Y$-coordinates are resp.:

$$\infty, \quad (a_2 - ia_1), \quad (b_2 - ib_1), \quad (c_2 - ic_1),$$

so that

$$I_1\,(I_2\,A\,B\,C) = \frac{(b_2 - ib_1) - (a_2 - ia_1)}{(c_2 - ic_1) - (a_2 - ia_1)} = \frac{(b_2 - a_2) - i\,(b_1 - a_1)}{(c_2 - a_2) - i\,(c_1 - a_1)},$$

and for $I_2\,(I_1\,ABC)$ the conjugate complex value is found.

We find, therefore, in fact:

$$I_1\,(I_2\,A\,B\,C)\,.\,I_2\,(I_1\,A\,B\,C) = \frac{(b_1 - a_1)^2 + (b_2 - a_2)^2}{(c_1 - a_1)^2 + (c_2 - a_2)^2} = \frac{\overline{AB}^2}{\overline{AC}^2}.$$

The sides and the angles of a triangle $ABC$ are now defined in a similar way, viz. as functions of the anharmonic ratios of the four rays drawn out of each of the five points $I_1, I_2, A, B, C$ to each of the other four. These anharmonic ratios are the most simple projective invariants of the quintuple. They are for the rest equal to the anharmonic ratios of any four of the points of the conic which can be passed through the quintuple, which, speaking metrically, is the circumscribed circle about the triangle $A, B, C$. Not every function of the anharmonic ratios is an invariant of the triangle; it is necessary that $I_1$ and $I_2$ appear symmetrically.

The definitions of angle and distance agree still more if we write the anharmonic ratio that appears in the angular measure, likewise as the product of two anharmonic ratios. So e.g.:

$$A\,(I_1\,I_2\,B\,C) = B\,(I_2\,I_1\,C\,A)\,.\,C\,(I_2\,I_1\,A\,B).$$

The given definitions may be used to prove simple theorems of Euclidean geometry by the aid of projectivity.