Mathematics. — *Comparison of the Simplified Theories of the Irrational Number.* By Prof. FRED. SCHUH. (Communicated by Prof. D. J. KORTEWEG).

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1. Introduction. In an earlier paper (Fusion of the existing Theories of the Irrational Number into a New Theory) I have shown that the different theories of the irrational number may be simplified so that they only consist of the proofs of the following properties:

a. *For any two real numbers* α *and* β *one and only one of the three relations* $\alpha = \beta$, $\alpha > \beta$, $\beta > \alpha$ *holds good.*

b. *If* $\alpha > \beta$ *and* $\beta > \gamma$, *we have* $\alpha > \gamma$.

c. *If* α *is a real number, there always exists a rational number* $> \alpha$.

d. *If* α *is a real number, there always exists a rational number* $< \alpha$.

e. *If* $\alpha > \beta$, *there always exists a rational number* c, *so that* $\alpha > c > \beta$.

f. *For the system of the real numbers the theorem of the upper boundary holds good.*

In this way the further discussion of addition and multiplication is made independent of any special theory of the irrational number and is based on the above mentioned properties.

The aim of this paper is to show how the different theories of the irrational number are connected and to prove the completeness of the system of the real numbers. We shall also give a simple proof of the possibility to extend the system of the rational numbers in such a way that the properties *a.—f.* hold good (cf. n°. 2—5); this proof, slightly altered, may serve at the same time to demonstrate the completeness of the system of the real numbers (cf. n°. 13).

2. Building of a system of real numbers. In several ways the system of the rational numbers can be extended to a system for which the properties *a.—f.* mentioned in n°. 1 hold good. We shall do it here in a way which differs from the usual theories of the irrational number and which has some resemblance to the theories of DEDEKIND and BAUDET.

By a **class** we understand *a set of rational numbers that is not empty, does not contain every rational number, has no greatest number and has the property that a rational number which is less than a number of the set, likewise belongs to the set.* From this it follows immediately *that a number of a class is less than a rational number that does not belong to the class* and *that a rational number which is greater than a* *rational number that does not belong to the class, does not belong to the class either.*

If one of the rational numbers that do not belong to the class is the smallest and if a is this smallest number, the class consists of the rational numbers $< a$. In this case we call the class **rational.**

We consider the classes as numbers (*real numbers*) and different classes as different numbers. By assuming *that a rational class must be identified with the smallest rational number that does not belong to it,* also the rational numbers are contained in the new system of numbers.

3. We call a class K **greater** *than a class* L, *if* K *contains a number* a *that does not belong to* L. In this case a number b of L is $< a$, hence a number of K, so that the relation $L > K$ is not satisfied. *Accordingly for two classes* K *and* L *one and only one of the three relations* $K = L$ (which means that the classes K and L are identical), $K > L$, $K < L$ *holds good.*

If the classes K and L are both rational, with k, resp. l, as smallest rational numbers that do not belong to them, and if $k > l$, l is a number of K that does not belong to L. Consequently $K > L$, *so that the new definition of greater applied to two rational numbers, leads to the same result as the old definition.*

4. *A class* K *is built of the rational numbers* $< K$.

For if a is a number of K, $a < K$, as a does not belong to the class formed by the rational numbers $< a$. If, inversely, a is a rational number $< K$, there exists a rational number b which belongs to K but not to the class of the rational numbers $< a$; accordingly $a \leqq b$, so that a belongs to K (because b belongs to it).

5. **Proofs of the properties mentioned in n⁰. 1.** *Proof of a.* The validity of a. is apparent from n⁰. 3.

Proof of b. If K, L and M are classes that satisfy the relations $K > L$ and $L > M$, there exists a rational number a which belongs to K but not to L and a rational number b which belongs to L but not to M. $b < a$ so that b belongs to K. As b does not belong to M we have $K > M$.

Proof of c. Let K be a class, a a rational number that does not belong to K, and b a rational number $> a$. In this case the number a belongs to the class of the rational numbers $< b$ but not to K, so that $b > K$.

Proof of d. If K is a class and c a number of K, we have $c < K$ (see the property of n⁰. 4).

Proof of e. Let K and L be classes that satisfy $K > L$ and let a be a number of K that does not belong to L. As K has no greatest number, there is a rational number c that is $> a$ and belongs to K. This number c does not belong to L. As a belongs to the class of the rational numbers

$< c$, we have $c > L$. From the property of n^0. 4 it ensues further that $c < K$.

Proof of f. Let V be a non-empty set of real numbers all of which are smaller than the real number M. We form *the set B of the rational numbers that are smaller than a number of V.* According to *d.* this set is not empty. According to *c.* there is a rational number $> M$; according to *b.* this does not belong to B. If b is a number of B and v a number of V which is $> b$, according to e. there is a rational number c so that $b < c < v$; as c is a number of B, B has no greatest number. As moreover a rational number which is smaller than a number of B, evidently also belongs to B, *B is a class.* We shall now show *that B is the upper boundary of V.*

Let v be a number of V which is $> B$. Then there is a rational number c so that $B < c < v$. This number c belongs to the class B (as $c < v$) so that $B < c$ is in conflict with the property of n^0. 4. *Accordingly a number of V which is $> B$, does not exist.*

Let C be a real number $< B$. There is a rational number a so that $C < a < B$. According to the property of n^0. 4 a is a number of the class B so that a is smaller than a number of V. *This number of V is also greater than C.*

6. Similar systems. *A system of real numbers which contains all rational numbers and for which the properties a.—e. mentioned in n^0. 1 hold good,* we shall briefly call a **system.**

We shall call two systems S and S' **similar** if such a *(1, 1)-correspondence* can be established between the numbers of S and those of S' that the following two properties are valid:

1^0. *the rational numbers correspond to themselves;*

2^0. *if α and β are two numbers of S so that $\alpha > \beta$, and if α' and β' are the corresponding numbers of S', we have $\alpha' > \beta'$.*

We shall call a correspondence that has this property, a **representation** *of one system on the other.*

The similitude of systems is *commutative* and *transitive*. Every system is similar to itself as we can make every number of the system correspond to itself (**identical** *representation*).

7. *If S and S' are two similar systems, in the representation of S on S' the common numbers of the two systems correspond to themselves. We suppose that for two common numbers the relation "greater" is the same in the two systems.*

Let α be a number that belongs to the two systems and α' the number of S' which correspond to α (as number of S). As α and α' both belong to S', we have $\alpha' > \alpha$, $\alpha' < \alpha$ or $\alpha' = \alpha$. If $\alpha' > \alpha$ there is a rational number b so that $\alpha' > b > \alpha$. From $\alpha < b$ (numbers of S) follows $\alpha' < b$ (corresponding numbers of S') in contradiction to $\alpha' > b$. In the same way $\alpha' < \alpha$ leads to an absurdity so that $\alpha' = \alpha$ and, accordingly, α corresponds to itself.

Immediate consequences of this theorem are:

the only representation of a system on itself is the identical one;

two systems cannot be represented on each other in more than one way;

a system is not similar to a real part of itself.

8. Owing to the commutativity and the transitivity of similitude we can form a *set of systems any two of which are similar.* We can unite the corresponding numbers of these systems to one conception and consider this as a "number" of which the said corresponding numbers are the representatives.

For according to what was found in n^0. 7 no two of these corresponding numbers are different numbers of the same system so that a "number" in the new sense does not get two unequal numbers of an already existing system as representatives. As the relation "greater" remains unchanged by the representation, it does not matter from which system we derive the representatives if we have to judge which number is greater.

In the indicated way *similar systems may be united to one system.* By so doing the difference between similar systems disappears entirely as this difference only consists in the names that are given to the numbers or in the way in which they are indicated.

9. Systems for which the theorem of the upper boundary is valid. We prove:

two systems S and S' for which the theorem of the upper boundary is valid, are similar.

Let α be a number of S and A the set of the rational numbers $< \alpha$. In S the number α is the upper boundary of A; for in the first place A has no number $> \alpha$; if γ is a number of S that is $< \alpha$, there is a rational number a so that $\gamma < a < \alpha$; hence a is a number of A that is $> \gamma$. The set A has also an upper boundary in S'; we shall call this α'. We shall make this number α' correspond to α.

Let β be another number of S and *suppose* $\alpha < \beta$. In the same way from β we can derive a number β' of S', which we shall make correspond to β. Now we can choose the rational numbers p and q such that $\alpha < p < q < \beta$. As β' is the upper boundary in S' of the set B of the rational numbers $< \beta$, and as q belongs to B, we have $q \leqq \beta'$. As $p > \alpha$ a rational number $> p$ is also $> \alpha$, hence not a number of A; accordingly A does not contain any number $> p$, so that $p \geqq \alpha'$; for $p < \alpha'$ is contradictory to the fact that α' is the upper boundary of A in S'. From $q \leqq \beta'$, $p \geqq \alpha'$ and $p < q$ we may further conclude *that* $\alpha' < \beta'$. Consequently to unequal numbers of S there correspond unequal numbers of S'.

A number a of A is $\leqq \alpha'$; however, $a = \alpha'$ is excluded as A has no greatest number; a number of A is, therefore, $< \alpha'$. If, inversely, a is a rational number $< \alpha'$, A contains such a number a_1 that $a_1 > a$ (as α' is the upper boundary of A in S'); this number a_1 satisfies the relation $a < a_1 < \alpha$, so that $a < \alpha$ and a is a number of A. The set A is, accordingly, the same as that of the rational numbers $< \alpha'$. Hence to any number α' of S' there corresponds a number α of S, viz. the upper boundary in S of the set of the rational numbers $< \alpha'$.

In the indicated way we get a (1,1)-correspondence between the numbers of S and those of S' for which the property 2^0 of n^0. 6 holds good. If α is a rational number, α is the upper boundary in S' of the set of the rational numbers $< \alpha$; hence the correspondence has also the property 1^0 of n^0. 6, *so that this correspondence is a representation of S on S'.*

10. *If we extend the system of the rational numbers to a system S for which the properties a.—f. of n^0. 1 hold good, and if we omit irrational numbers and add irrational numbers so that a system S' arises for which the said properties likewise hold good, the systems S and S' can be represented on each other* (according to the theorem of n^0. 9). According to the theorem of n^0. 7 *the omitted numbers correspond to the added ones.* This shows that the transition from S to S' only consists in this *that numbers are omitted and afterwards added again under another name*, for instance that the irrational numbers according to the theory of CANTOR are omitted and the irrational numbers according to the theory of DEDEKIND are added.

11. Completeness of a system of numbers for which the theorem of the upper boundary holds good. We have:

to a system S for which the theorem of the upper boundary holds good, not a single number can be added if we want the properties a.—e. of n^0. 1 to remain valid after the addition and also the relation "greater" to remain unchanged for two numbers of S.

Suppose that S may be extended to the system S' (with conservation of the properties a.—e. of n^0. 1). Let α' be a number of S' that does not belong to S and A the set of the rational numbers $< \alpha'$. This set has an upper boundary α in S; this number α also belongs to S'. As α' does not belong to S, we have $\alpha < \alpha'$ or $\alpha > \alpha'$.

If $\alpha < \alpha'$ and if p is a rational number so that $\alpha < p < \alpha'$, p is a number of A and hence $p > \alpha$ is contradictory to the fact that α is the upper boundary of A in S. If $\alpha > \alpha'$ and if p is a rational number so that $\alpha > p > \alpha'$, it follows from $p < \alpha$ that there is a number a of A greater than p; from $a > p$ and $p > \alpha'$ follows $a > \alpha'$, in contradiction to the definition of A. Hence in both cases we arrive at an absurdity.

12. The theorem of n^0. 11 shows *that we cannot omit any number from the system S considered there, if we want the properties a.—f. of n^0. 1 to remain valid.* For the application of the theorem of n^0. 11 to the system that arises from S through the omission of numbers would lead to a contradiction.

13. Theorem of completeness. We understand by this:

for a system that is not liable to extension if we want the properties a.—e. of n^0. 1 to remain valid, the theorem of the upper boundary holds good.

We mean, of course, such an extension *that the relation "greater" for the numbers of the original system remains unchanged.*

We shall extend the system S by forming classes. In deviation from n^0. 2 we shall now understand by a **class** *a set of numbers of S which is not empty, does not contain every number of S, has no greatest number and has the property that a number of S which is smaller than a number of the set, likewise belongs to the set.*

If among the numbers of S that do not belong to the class there is a smallest a, the class consists of the numbers of S which are $< a$.

We shall consider the classes formed in this way as *numbers of a new system S'* where we identify the class of the numbers of S which are $< a$ (a is a number of S) with a. In this way the numbers of S are also contained in S'.

If K and L are two classes, we give the same definition of $K > L$ as in n^0. 3. In the same way as there it appears *that the definition of greater which is valid in S', applied to two numbers of S, leads to the same result as the definition which holds good in S.* Accordingly for the system S' *the property a. of n^0. 1* is valid.

The validity of the *property b. of n^0. 1* for the system S' is proved as in n^0. 5.

If K is a class, b a number of S that does not belong to K, and c a rational number $> b$, we have $c > K$ (*property c. of n^0. 1*). If a is a number of K and d a rational number $< a$, we have $d < K$ (*property d. of n^0. 1*).

Let K and L be two classes so that $K > L$ and let a be a number of K that does not belong to L. The class K contains a number $b > a$. If c is a rational number so that $b > c > a$, we have $K > c > L$ (*property e. of n^0. 1*).

Let V be a non-empty set of numbers of S' all of which are smaller than the number M of S'. We shall now build *the set B of the numbers of S that are smaller than a number of V.* Just as in n^0. 5 it appears *that B is a class* so that B is a number of S'. If v is a number of V which is $> B$, there is a rational number c (hence a number of S) so that $B < c < v$; according to $c < v$ c is a number of B and from $B < c$ it follows that c is not a number of B; consequently $v > B$ is impossible.

If C is a number of S' which is $< B$ and if a is a rational number so that $C < a < B$, a is a number of B, hence a, and also C, is smaller than a number of V. This shows *that B is the upper boundary of V in S'*, so that for the system S' also the *property f. of n⁰. 1* (theorem of the upper boundary) holds good.

As for the system S' the properties *a.—e.* of n⁰. 1 are valid, *S' is a system as defined in n⁰. 6.* It is further evident from the suppositions of the theorem of completeness, *that the systems S and S' are identical.* As the theorem of the upper boundary holds good for S', this is also valid for S.

14. The theorem of n⁰. 13 together with that of n⁰. 11 (of which it is the converse) shows, *that it amounts to the same if we say of a system that the theorem of the upper boundary holds good for it or that a further extension of the system is impossible.* We can also say *that the condition that is necessary and sufficient for the impossibility of a further extension of the system, is the validity for that system of the theorem of the upper boundary.* It follows further from the theorem of n⁰. 9 *that two systems which are not liable to extension* (with conservation of the properties *a.—e.* of n⁰. 1) *are similar.*

15. A system S can be completed in the way indicated in n⁰. 13 to a system for which the theorem of the upper boundary holds good. From this in connection with the above it is evident that:

a system S is a part of the system of the real numbers and a real part or not according as an extension of S (with conservation of the properties a.—e. of n⁰. 1) is possible or impossible, hence according as the theorem of the upper boundary holds not good for S or holds good.

This means *that S is similar to a part of any system for which the theorem of the upper boundary holds good.*