Mathematics. — *A Representation of the Congruence of* REYE. By Prof. JAN DE VRIES.

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1. The congruence van REYE is formed by the twisted cubics k^3 that pass through five fixed points B_k. Through a point M chosen at random there passes one bisecant b of a k^3; let the intersection P of b, with the fixed plane Φ be considered as the *image* of k^3. The curve k_0^3 through M has as images any point of the conic c^2 in which Φ is cut by the cone that projects k_0^3 out of M. The intersection C_k of MB_k is the image of each k^3 that has MB_k as bisecant.

2. The points P of a line l in Φ are the images of the curves k^3 that have the rays of a plane pencil (b) as bisecants. These curves form a surface Λ of which we may find the degree by examining the intersection of Λ with the plane $\beta_{123} \equiv B_1B_2B_3$. If D_{45} is the intersection of of B_4B_5 with the plane λ of (b), B_{45} the intersection of B_4B_5 with β_{123}, E_{45} the intersection of β_{123} with MD_{45}, the conic $B_1B_2B_3B_{45}E_{45}$ together with the line B_4B_5 forms a k^3, of which the bisecant $D_{45}E_{45}$ belongs to the plane pencil (b). The intersection of Λ with β_{123} consists, accordingly, of this conic and the lines B_1B_2, B_2B_3 and B_3B_1; the surface Λ is of the *fifth* degree and has *triple points* in B_k. The image of Λ^5 is the point range on l; as l contains two points of c^2, k_0^3 is a *nodal curve* of Λ^5.

The locus of the pairs of points that the curves k^3 of Λ have in common with the rays of (b), is a curve λ^4 with node M; together with a straight line m it forms the intersection of Λ with the plane λ of (b); m joins the other points of intersection of k_0^3 and the plane λ. Evidently Λ may also be considered as the locus of the curves k^3, that cut a bisecant m of k_0^3.

3. If l passes through the singular point C_1, Λ consists of the quadratic cone K_1^2, that has MB_1 and the four lines B_1B_k as generatrices and a surface Λ_1^3 with four conical points B_k.

The point range on $c_{12} \equiv C_1C_2$ is the image of the figure consisting of the cones K_1^2, K_2^2 and the plane β_{345}. For any conic through B_3, B_4, B_5 and the passage B_{12} of B_1B_2 forms a k^3 with B_1B_2 of which a bisecant $D_{12}E_{12}$ passes through M and cuts Φ in a point of c_{12}.

4. The curves k^3 that cut an arbitrary line g, form a surface Γ^5 with triple points B_k (§ 2). This contains, therefore, two curves k^3 that also cut MB_k outside B_k, and that, accordingly, have their images in C_k. As g does not generally cut the curve k_0^3 the image of Γ^5 can only meet the conic c^2 in the points C_k. Hence the image curve is a γ^5 with *double points* C_k; it has a *sixth* double point owing to the k^3 that cuts g twice.

Two image curves γ^5 have five points in common outside C_k; these are the images of the five k^3 that cut g and g'. The corresponding surface Γ^5 have the ten lines B_kB_l in common besides these five k^3.

If g rests on k_0^3, γ^5 consists of c^2 and a rational γ^3. If g is a bisecant of k_0^3, γ^5 is replaced by the c^2 which must be counted twice, and a line l.

The surfaces Λ^5 corresponding to two lines l and l', have a k^3 in common besides the ten lines $B_k B_l$ and the double curve k_0^3; this k^3 is represented in the point $l\,l'$.

5. The k^3 that cut a given conic s^2, form a surface Σ^{10}; for β_{123} contains two conics k^2 through B_1, B_2, B_3 and B_{45} that meet s^2 so that $B_4 B_5$ is a double line of Σ and the intersection of Σ with β_{123} consists of these two k^2 and the double lines $B_1 B_2$, $B_2 B_3$, $B_3 B_1$. As B_k is a *sextuple* point on Σ^{10}, there are four k^3 of Σ that have the line MB_k as bisecant. Hence the image curve σ of Σ has five *quadruple* points C_k; as it cannot cut c^2 outside C_k, the image of Σ is a σ^{10}. As there exists a (1,1) correspondence between the point ranges an c^2 and σ^{10}. σ^{10} is rational and has, therefore, six more double points. Accordingly there are six curves k^3 that cut s^2 *twice*.

If s^2 meets the curve k_0^3, σ^{10} is replaced by c^2 and a σ^8 with triple points C_k. If s^2 rests on k_0^3 in two points, the c^2, which must be counted double, is supplemented by a rational σ^6 with double points in C_k and five double points outside c^2 owing to the five k^3 besides k_0^3 that cut the conic s^2 twice.

If s^2 cuts the curve k_0^3 three times, it replaces three of the six k^3 that meet s^2 twice; the remaining three are represented in the double points of the rational σ^4, that together with the c_{22}, which must be counted three times, forms the image of the system Σ.

6. The k^3 that touch a given plane ψ, form a surface Ψ^{10}; its intersection with β_{123} consists of the two conics k^2 that touch ψ and the three double lines $B_1 B_2$, $B_2 B_3$, $B_3 B_1$. The base points B_k are again *sextuple* so that the image curve has *quadruple* points in C_k and is a ψ^{10}.

As β_{123} contains two points of contact of figures k^3 that belong to Ψ, the locus of the points of contact of the k^3 touching ψ is a *conic* ψ^2. The plane ψ has also a rational ψ^6 in common with Ψ^{10}; this has ten double points in the intersections of the lines $B_k B_l$.

The image curve ψ^{10} is rational and has, therefore, six double points outside c^2; these are the images of *six* k^3 that *osculate* the plane ψ. The curves ψ^2 and ψ^6 touch each other in the six points where ψ is osculated by curves k^3.

The surface Ψ^{10} is formed by the curves k^3 that rest on the conic ψ^2; hence it belongs to the surfaces Σ^{10} discussed in § 5. As any plane contains a rational ψ^6 besides a conic ψ^2, Ψ^{10} may also be considered as the locus of the k^3 resting on a ψ^6.

7. An arbitrary conic φ^2 in the plane φ is the image of a system Φ of curves k^3 that have the generatrices of a quadratic cone as bisecants. The image curve γ^5 of the system of the k^3 that cut a line g (§ 4), has ten points in common with φ^2; accordingly the system Φ lies on a surface Φ^{10}.

The line c_{kl} (§ 3) cuts φ^2 in the images of two composite k^3. Consequently Φ^{10} has two conics and three double lines in common with β_{123}; this proves that the surface has sextuple points in the base points B_k. The four points of intersection of φ^2 with c^2 show that the curve k_0^3 is *quadruple* on Φ^{10}.

8. Let Ω be the system of the k^3 that have the generatrices of a quadratic cone ω^2 with vertex O as bisecants; they form an Ω^{10} with *sextuple* points B_k and a *quadruple* curve (the k^3 that passes through O). This surface has ten curves k^3 in common with the surface Λ^5 of the k^3 that rest on a bisecant of k_0^3 (§ 2) and are represented by the points of a line l. This image curve of the system Ω is, therefore, a curve ω^{10}.

As ω^2 contains two generatrices resting on $B_1 B_2$, there lie two conics through B_3, B_4, B_5, B_{12} on Ω^{10}; hence c_{12} has two points besides C_1 and C_2 in common with ω^{10} and B_1 and B_2 are quadruple points. Consequently the image curve γ^5 (C_k^2) has ten points besides C_k in common with ω^{10}; they are the images of the ten k^3 that rest on the line g (§ 4).

The curve ω^{10} has a *sixth* quadruple point; it is the image of the k^3 that passes through O; the singular points of the ω^{10}, as naturally rational, are, therefore, represented by six quadruple points.