Physics. — *On the mean values of straight and curved chords of geometrical bodies.* By P. CLAUSING. (Communicated by Dr. G. HOLST).

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§ 1. In working out a diffusion-problem connected with the adsorption-phenomenon of the molecules of very rare gases on a glass-wall we found a simple relation between the area of the cross section of an infinite long cylindrical tube, the circumference of this section and the mean distance traversed by a molecule between two collisions with the wall of the tube. It was assumed that the molecules did not collide with each other and that they left the wall according to the "cosine-law". This law corresponds to a random distribution of moving points in space and implies that the number of molecules which passes in unit time through an element of surface in a certain direction, is proportional to cos ϑ, if ϑ is the angle between the perpendicular to the element and the said direction.

On closer inspection of the derivation of this relation it turned out to be a particular case of a much more general theorem, which can be stated in the following way.

The mean chord of a "body" with "volume" V_m and "area" S_m in a m-dimensional space is given by

$$\bar{\varrho}_m = \lambda_m \cdot \frac{V_m}{S_m} \quad \ldots \ldots \ldots \ldots \quad (1)$$

with

$$\lambda_m = \frac{1 \cdot 3 \cdot 5 \ldots\ldots (m-1)}{2 \cdot 4 \ldots\ldots (m-2)} \cdot \pi$$

for even m, and

$$\lambda_m = \frac{2 \cdot 4 \cdot 6 \ldots\ldots (m-1)}{1 \cdot 3 \cdot 5 \ldots\ldots (m-2)} \cdot 2$$

for odd m[1]).

The formula (1) also holds for the mean length of the arcs which are

[1]) We find e.g. for a m-dimensional sphere

$$\lim_{m=\infty} \bar{\varrho}_m = 0.$$

cut off from the great circles of a m-dimensional spherical space by a body in this space [1]).

A proof of the theorem for $m=3$ and non-curved spaces does not offer difficulties. The defininiton of $\bar{\varrho}$ is

$$\bar{\varrho} = \frac{\int\limits_S \int\limits_{\frac{\Omega}{2}} \varrho \cos\vartheta \, d\Omega \, dS}{\int\limits_S \int\limits_{\frac{\Omega}{2}} \cos\vartheta \, d\Omega \, dS}, \qquad (2)$$

where ϑ is the angle between the chord ϱ and the normal to the element of surface dS; $d\Omega$ is an element of the complete solid angle Ω. Considering that $d\Omega = d\vartheta \sin\vartheta \, d\varphi$ and that $\varrho \cos\vartheta \, dS$ represents an element of volume dV with base dS and length ϱ, extending in the direction ϑ, we find

$$\bar{\varrho} = \frac{\int\limits_S \int\limits_{\frac{\Omega}{2}} dV \, d\Omega}{S \int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{2}} \sin\vartheta \cos\vartheta \, d\vartheta} = \frac{2V \cdot \frac{4\pi}{2}}{S \cdot 2\pi \cdot \frac{1}{2}} = \frac{4V}{S}, \qquad (3)$$

in agreement with (1). The factor 2 in the numerator in front of V is inserted because in the integration over the total surface S each element dV is counted twice. A similar proof holds for more-dimensional spaces.

§ 2. The second proof of (3), which emerged from our considerations about the said adsorption, is much simpler.

For this purpose we write $\bar{\varrho} = 4V/S$ in the form

$$S \cdot \frac{1}{4} n u \cdot \frac{\bar{\varrho}}{u} = V n. \qquad (4)$$

[1]) We find for the area of a segment from a m-dimensional sphere, interpreted as a $(m-1)$-dimensional volume, generally

$$\lim_{m=\infty} \bar{\varrho}_{m-1} = 0,$$

but

$$\bar{\varrho}_{m-1} = \pi R,$$

for each value of $m > 1$, if the segment coincides with half a sphere. R is the radius of the sphere.

n is the number of molecules in unit volume and u their velocity. We assume namely that all molecules move equally fast and do not interact with each other. In the dynamical theory of gases it is shown that $\nu = (nu)/4$ molecules strike the unit of surface in unit time, independent of the distribution of the velocities of the molecules and therefore also if all the molecules have the same velocity. Thus $S \, . \, (nu)/4$ is the number of molecules which leaves the surface in unit time. Considering that ϱ/u represents the mean time between two collisions of a molecule with the surface, it is easily seen that both members of the equation (4) give the number of molecules present in the volume. Simplification of the identical relation (4) gives the theorem (3).

For more-dimensional spaces the proof remains unchanged. Only the formula $\nu = nu/4$ must be replaced by $\nu = nu/\lambda_m$, which can be easily demonstrated.

These considerations are not only much simpler than those of the preceding paragraph but may also be generalised in an easy way. It is namely evident that nothing essential in this proof changes if we consider a number of molecules moving in a spherical surface only. V becomes the area inside a certain closed curve on the sphere, S the total length of this curve and the factor 4 is to be replaced by π, in agreement with formula (1) for a 2-dimensional (be it curved) space.

In this case however the formula is limited in the way that no great circles of the sphere (the orbits of the molecules) may be situated inside the curve in question. This is a consequence of equation (4), which expresses that the molecules present inside V must come from the wall S. Moving about inside V must be impossible.

§ 3. Finally we point out that our theorem still holds in all linear and spherical spaces if we suppose the sets of straight lines and great circles (the tracks of the molecules) replaced by sets of curves of a different shape, of course with the limitation mentioned at the end of § 2.

Hence, with the aid of (1), we find for the mean length of a helix inside a sphere

$$\bar{\varrho}_3 = \frac{4 \, . \, \frac{4}{3} \pi R^3}{4 \pi R^2} = \frac{4}{3} R$$

In calculating this mean value we must in the first place consider that an equal number of helices goes out in every direction (in a little elementary solid angle), but in the second place that the planes of osculation of these helices are distributed in a regular way about this direction. If the curve is not movable in itself as the helix we must divide the curve in elements of equal length and take the mean value of ϱ for all the possible positions of the curve. Proceeding in this way we apply the only method which agrees with a really random distribution of curved lines in a space.

We were not yet able to give an analogical proof as in § 1 of the theorem in its full generality, without the time as a parameter, as in § 2.

In the accessible literature [1]) we have only found some very special cases of the general theorem.

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[1]) M. W. CROFTON: Phil. Trans. Roy. Soc. Lond., **158**, blz. 181 (1868).
E. CZUBER: Geometrische Wahrscheinlichkeiten, blz. 213 (Leipzig, 1884); Wahrscheinlichkeitsrechnung, erster Band, blz. 115 (Leipzig, 1914).
R. DELTHEIL: Probabilités géométriques, blz. 84 (Paris, 1926).