Mathematics. — *Linear Adjustment of a Set of Pairs of Numbers* (x_k, y_k). By Prof. M. J. VAN UVEN. (Communicated by Prof. A. A. NIJLAND).

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If, by some theoretical consideration, we assume a linear relation $\alpha x + \beta y + \gamma = 0$ between two variables x and y, the experiment however furnishing a set of n pairs $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ not agreeing completely with the same linear relation, the values x_k, y_k $(k = 1, \ldots n)$ obtained by experiment, and therefore subject to errors of observation, must be adjusted.

We have in general no right to consider one of the variables as free from error. So both values x_k and y_k must be "corrected".

Speaking geometrically we have the problem: To fit a straight line L as closely as possible through n points S_k having coordinates x_k, y_k $(k = 1, \ldots n)$ given by experiment.

As a rule no straight line exists, which passes exactly through all the given points S_k. Calling T_k for convenience sake the true position of S_k, the displacement $\delta_k = \overrightarrow{S_k T_k}$ is the correction of S_k, and $\overrightarrow{T_k S_k} = -\delta_k$ is the "error" of S_k. Generally both x_k and y_k must be corrected; that is to say: the displacements δ_k will not in general coincide with the direction of any coordinate.

We must also take into account the possibility that the x-coordinate has an uncertainty (mean error, weight) different from that of the y-coordinate. Nor is a certain dependence between the error $\triangle x$ of x and the error $\triangle y$ of y to be excluded; this dependence is revealed by the fact that the mean value of $\triangle x_k \, . \, \triangle y_k$ differs from zero.

We may consider the xy-plane as subject to elastic tensions, which perform a certain (negative) work by the displacement of a point, whence a certain potential energy is produced.

The plane not being isotropic, the elastic tension has its maximum and its minimum in two directions, perpendicular to each other, which need not coincide with the directions of the coordinate-axes.

We shall treat the given problem on the most general supposition: viz. that the plane is not isotropic, and that the main directions of elasticity form an angle (ω) with the coordinate-axes.

Each point S_k being shifted by a displacement δ_k to a point T_k, situated on a certain straight line L, there will be produced a certain total amount V of potential energy.

As the line L, best fitted through the points S_k, we shall consider that line L_0 for which the total potential energy V takes its minimum value.

The angle between the axes of elasticity and the coordinate-axes being ω, the coordinates x', y' of a point with respect to a system of coordinates having the same origin as the system x, y, and parallel to the axes of elasticity, are connected with x and y by

$$x' = x \cos \omega + y \sin \omega,$$
$$y' = -x \sin \omega + y \cos \omega.$$

The modulus of elasticity (elastic force in displacing a point over the distance 1) in the main direction x' being P, in the main direction y' Q, a displacement in the direction which makes the angle ζ' with the positive axis of x', generates an elastic force $\mathfrak{E}$ the components of which are

$$E_{x'} = -P \cos \zeta' \quad , \quad E_{y'} = -Q \sin \zeta' \quad . \quad . \quad . \quad . \quad (1)$$

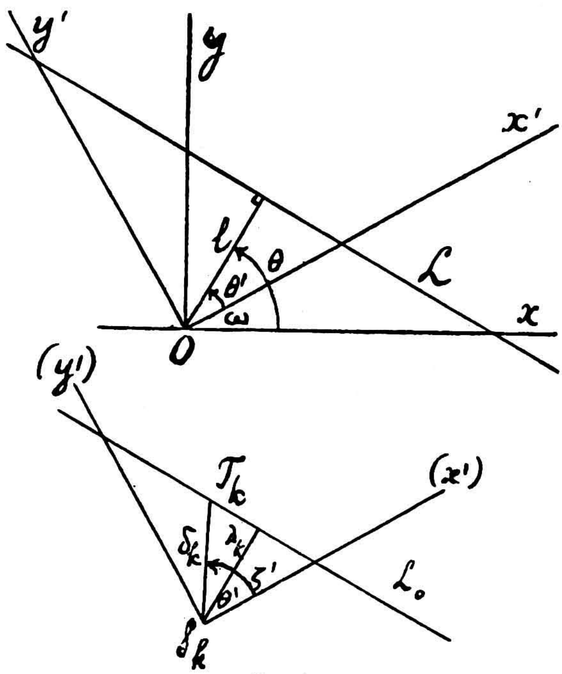

Fig. 1.

This force is balanced by the force attracting the point perpendicularly to L. This latter force having the components $+P \cos \zeta'$, $+Q \sin \zeta'$, the direction θ' of the normal of L is connected with the direction ζ' by the relation

$$\frac{E_{y'}}{E_{x'}} = \operatorname{tg} \theta' = \frac{Q \sin \zeta'}{P \cos \zeta'},$$

or

$$\frac{\sin \zeta'}{\cos \zeta'} = \frac{P \sin \theta'}{Q \cos \theta'},$$

thus

$$\cos \zeta' = \frac{Q \cos \theta'}{\sqrt{P^2 \sin^2 \theta' + Q^2 \cos^2 \theta'}}, \quad \sin \zeta' = \frac{P \sin \theta'}{\sqrt{P^2 \sin^2 \theta' + Q^2 \cos^2 \theta'}} . \quad (2)$$

The angle $\zeta' - \theta'$ between the directions of displacement and force is determined by

$$\cos (\zeta' - \theta') = \frac{Q \cos^2 \theta' + P \sin^2 \theta'}{\sqrt{P^2 \sin^2 \theta' + Q^2 \cos^2 \theta'}} . \quad . \quad . \quad . \quad (3)$$

So the component of the elastic force in displacing the point over the distance 1 in the direction ζ' (i.e. in the direction from S_k to T_k) amounts to

$$E_{\zeta'} = E_{x'} \cos \zeta' + E_{y'} \sin \zeta' = -(P \cos^2 \zeta' + Q \sin^2 \zeta') = -\frac{PQ^2 \cos^2 \theta' + QP^2 \sin^2 \theta'}{P^2 \sin^2 \theta' + Q^2 \cos^2 \theta'},$$

or

$$E_{\zeta'} = -PQ \cdot \frac{P \sin^2 \theta' + Q \cos^2 \theta'}{P^2 \sin^2 \theta' + Q^2 \cos^2 \theta'} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence the displacement δ_k (in the direction ζ') generates a force, the component of which in the direction ζ' (i. e. in the direction from S_k to T_k) is

$$E_{\zeta'} \cdot \delta_k .$$

By displacing S_k gradually to T_k on L, this component performs the work:

$$\int_0^{\delta_k} E_{\zeta'} \cdot \delta \cdot d\delta = \tfrac{1}{2} E_{\zeta'} \cdot \delta_k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The perpendicular distance λ_k from S_k to L being

$$\lambda_k = \delta_k \cos (\zeta' - \theta'),$$

we have

$$\delta_k = \frac{\lambda_k}{\cos (\zeta' - \theta')} = \lambda_k \cdot \frac{\sqrt{P^2 \sin^2 \theta' + Q^2 \cos^2 \theta'}}{P \sin^2 \theta' + Q \cos^2 \theta'} \quad . \quad . \quad . \quad (6)$$

From (4) and (6) follows that the work performed (5) amounts to

$$\tfrac{1}{2} E_{\zeta'} \cdot \delta_k^2 = -\frac{PQ}{2} \cdot \frac{P \sin^2 \theta' + Q \cos^2 \theta'}{P^2 \sin^2 \theta' + Q^2 \cos^2 \theta'} \times \frac{P^2 \sin^2 \theta' + Q^2 \cos^2 \theta'}{(P \sin^2 \theta' + Q \cos^2 \theta')^2} \times \lambda_k^2$$

$$= -\frac{PQ}{2} \cdot \frac{\lambda_k^2}{P \sin^2 \theta' + Q \cos^2 \theta'} .$$

Denoting by [ ] the summation over k from 1 to n, we find for all the points S_k together:

$$\tfrac{1}{2} E_{\xi'} . [\delta_k^2] = -\frac{PQ}{2} \frac{[\lambda_k^2]}{P \sin^2 \theta' + Q \cos^2 \theta'} \quad . \quad . \quad . \quad . \quad (7)$$

So the potential energy produced is

$$V = -\tfrac{1}{2} E_{\xi'} . [\delta_k^2] = +\frac{PQ}{2} \cdot \frac{[\lambda_k^2]}{P \sin^2 \theta' + Q \cos^2 \theta'} \quad . \quad . \quad . \quad (8)$$

or, denoting the mean value $\frac{[\lambda_k^2]}{n}$ of λ_k^2 by $\overline{\lambda_k^2}$,

$$V = \frac{nPQ}{2} \cdot \frac{\overline{\lambda_k^2}}{P \sin^2 \theta' + Q \cos^2 \theta'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Introducing the perpendicular distance l from O to L, we find:

$$\lambda_k = x'_k \cos \theta' + y'_k \sin \theta' - l,$$

therefore

$$V = \frac{nPQ}{2} \cdot \frac{\overline{(x'_k \cos \theta' + y'_k \sin \theta' - l)^2}}{P \sin^2 \theta' + Q \cos^2 \theta'} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Passing to the original system of coordinates x, y, we have

$$\theta' = \theta - \omega,$$

$$x'_k \cos \theta' + y'_k \sin \theta' = x_k \cos \theta + y_k \sin \theta,$$

$$\begin{aligned} P \sin^2 \theta' + Q \cos^2 \theta' &= P \sin^2 (\theta - \omega) + Q \cos^2 (\theta - \omega) \\ &= (P \sin^2 \omega + Q \cos^2 \omega) \cos^2 \theta + \\ &\quad + 2(-P + Q) \sin \omega \cos \omega \cos \theta \sin \theta + \\ &\quad + (P \cos^2 \omega + Q \sin^2 \omega) \sin^2 \theta. \end{aligned}$$

Putting

$$\left. \begin{aligned} a &= P \sin^2 \omega + Q \cos^2 \omega, \\ b &= (-P + Q) \sin \omega \cos \omega, \\ c &= P \cos^2 \omega + Q \sin^2 \omega, \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

we find:

$$P \sin^2 \theta' + Q \cos^2 \theta' = a \cos^2 \theta + 2\, b \cos \theta \sin \theta + c \sin^2 \theta = f(\theta). \quad (12)$$

Hence the potential energy V is

$$V = \frac{nPQ}{2} \times \frac{\overline{\lambda_k^2}}{f(\theta)} = \frac{nPQ}{2} \times \frac{\overline{(x_k \cos \theta + y_k \sin \theta - l)^2}}{a \cos^2 \theta + 2\, b \cos \theta \sin \theta + c \sin^2 \theta} \quad (13)$$

The forces are in balance if V is a minimum, thus if

$$\frac{2V}{nPQ} = \varphi(\theta, l) = \frac{\overline{\lambda_k^2}}{f(\theta)} = \frac{\overline{(x_k \cos \theta + y_k \sin \theta - l)^2}}{a \cos^2 \theta + 2\, b \cos \theta \sin \theta + c \sin^2 \theta} \text{ minimum.} \quad (14)$$

Putting

$$\overline{x} = \frac{[x_k]}{n}, \quad \overline{y} = \frac{[y_k]}{n},$$

and

$$x_k = \bar{x} + u_k\,,\quad y_k = \bar{y} + v_k\,,$$

whence

$$\bar{u} = 0,\ \bar{v} = 0,$$

we find:

$$\overline{(x_k \cos\theta + y_k \sin\theta - l)^2} = \overline{\{(\bar{x}\cos\theta + \bar{y}\sin\theta - l) + (u_k\cos\theta + v_k\sin\theta)\}^2}$$
$$= (\bar{x}\cos\theta + \bar{y}\sin\theta - l)^2 + 2\,(\bar{x}\cos\theta + \bar{y}\sin\theta - l)(\bar{u}\cos\theta + \bar{v}\sin\theta) +$$
$$+ (\overline{uu}\cos^2\theta + 2\,\overline{uv}\cos\theta\sin\theta + \overline{vv}\sin^2\theta).$$

Taking into account $\bar{u}=0$, $\bar{v}=0$, and putting:

$$\overline{uu} = A,\ \overline{uv} = B,\ \overline{vv} = C, \quad . \ . \ . \ . \ . \ . \ . \quad (15)$$

we arrive at

$$\left.\begin{aligned}\overline{(x_k \cos\theta + y_k \sin\theta - l)^2} &= (\bar{x}\cos\theta + \bar{y}\sin\theta - l)^2 + \\ &+ (A\cos^2\theta + 2\,B\cos\theta\sin\theta + C\sin^2\theta).\end{aligned}\right\} \quad (16)$$

So the condition (14) runs:

$$\varphi(\theta,l) = \frac{(\bar{x}\cos\theta + \bar{y}\sin\theta - l)^2 + (A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta)}{a\cos^2\theta + 2\,b\cos\theta\sin\theta + c\sin^2\theta}\ \text{minimum} \quad (17)$$

The condition of minimum $\frac{\partial\varphi}{\partial l} = 0$ furnishes

$$\frac{\partial\varphi}{\partial l} = \frac{-2\,(\bar{x}\cos\theta + \bar{y}\sin\theta - l)}{f(\theta)} = 0,$$

or

$$\bar{x}\cos\theta + \bar{y}\sin\theta - l = 0; \quad . \ . \ . \ . \ . \ . \ . \quad (18)$$

that is to say: the line L required must pass through the "mean point" $(\bar{x}, \bar{y})$.

Th condition (17) asks now

$$\varphi(\theta) = \frac{A\cos^2\theta + 2\,B\cos\theta\sin\theta + C\sin^2\theta}{a\cos^2\theta + 2\,b\cos\theta\sin\theta + c\sin^2\theta}\ \text{minimum}, \ . \quad (19)$$

or, putting

$$A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta = F(\theta), \ . \ . \ . \quad (20)$$

$$\varphi(\theta) = \frac{F(\theta)}{f(\theta)}\ \text{minimum} \ . \ . \ . \ . \ . \ . \ . \quad (19')$$

From (19) (or (19')) we take the value of θ which minimizes $\varphi(\theta)$; denoting this solution by θ_0, and the value of l required, by l_0, we have, on account of (18),

$$l_0 = \bar{x}\cos\theta_0 + \bar{y}\sin\theta_0 \quad . \ . \ . \ . \ . \ . \ . \quad (21)$$

From

$$A = \overline{uu} = \frac{[u_k^2]}{n}\quad,\quad B = \overline{uv} = \frac{[u_k\,v_k]}{n}\quad,\quad C = \overline{vv} = \frac{[v_k^2]}{n}$$

follows

$$A > 0 \quad \text{and} \quad AC - B^2 = \frac{[u_k{}^2][v_k{}^2] - [u_k\, v_k]^2}{n^2} = \frac{[(u_k\, v_l - u_l\, v_k)^2]}{n^2} > 0.$$

Likewise, from (11) ensues:

$$a > 0 \quad \text{and} \quad ac - b^2 = PQ > 0.$$

So both the forms $F(\theta)$ and $f(\theta)$ are positive definite.

Hence the present minimumproblem: $\frac{F(\theta)}{f(\theta)}$ min. is formally equivalent to determining the direction of minimum curvature at an elliptic point of a surface.

Putting:

$$\left.\begin{aligned} \operatorname{tg}\theta &= z, \\ A + 2\,Bz + Cz^2 &= G(z), \\ a + 2\,bz + cz^2 &= g(z), \end{aligned}\right\} \quad \dots\dots\dots \quad (22)$$

we find

$$\varphi(\theta) = \psi(z) = \frac{A + 2\,Bz + Cz^2}{a + 2\,bz + cz^2} = \frac{G(z)}{g(z)} \quad \dots\dots \quad (23)$$

Now we have

$$\frac{d\psi}{dz} = \frac{g\frac{dG}{dz} - G\frac{dg}{dz}}{g^2} = \frac{1}{g^2}\begin{vmatrix} g & \frac{dg}{dz} \\ G & \frac{dG}{dz} \end{vmatrix} = \frac{2}{g^2}\begin{vmatrix} a + 2\,bz + cz^2, & b + cz \\ A + 2\,Bz + Cz^2, & B + Cz \end{vmatrix}$$

$$= \frac{2}{g^2}\begin{vmatrix} a + bz, & b + cz \\ A + Bz, & B + Cz \end{vmatrix}.$$

So the value z_0 which minimizes $\psi(z)$, is one of the roots of the equation

$$\begin{vmatrix} a + bz, & b + cz \\ A + Bz, & B + Cz \end{vmatrix} = 0, \quad \dots\dots\dots \quad (24)$$

or

$$(aB - bA) - (cA - aC)\,z + (bC - cB)\,z^2 = 0, \quad \dots \quad (24')$$

for which we, putting

$$\left.\begin{aligned} bC - cB &= \alpha \\ cA - aC &= \beta \\ aB - bA &= \gamma \end{aligned}\right\} \quad \dots\dots\dots \quad (25)$$

can write

$$\alpha z^2 - \beta z + \gamma = 0. \quad \dots\dots\dots \quad (24'')$$

The corresponding value of ψ is

$$\psi(z_0) = \psi_0 = \frac{G(z_0)}{g(z_0)} = \frac{G_0}{g_0} = \frac{(A + Bz_0) + (B + Cz_0)\,z_0}{(a + bz_0) + (b + cz_0)\,z_0},$$

or, by (24),

$$\psi_0 = \frac{A + Bz_0}{a + bz_0} = \frac{B + Cz_0}{b + cz_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Hence

$$z_0 = -\frac{a\psi_0 - A}{b\psi_0 - B} = -\frac{b\psi_0 - B}{c\psi_0 - C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

so that ψ_0 is a root of the equation

$$\begin{vmatrix} a\psi - A & , & b\psi - B \\ b\psi - B & , & c\psi - C \end{vmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

or

$$(ac - b^2)\,\psi^2 - (aC - 2\,bB + cA)\,\psi + (AC - B^2) = 0, \quad . \quad . \quad (27')$$

or, putting for abbreviation's sake

$$ac - b^2 = s \quad , \quad AC - B^2 = S \quad , \quad aC - 2\,bB + cA = \sigma, \quad . \quad (28^a)$$

$$s\,.\,\psi^2 - \sigma\,.\,\psi + S = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27'')$$

From $s > 0$, $S > 0$, $a > 0$, $A > 0$ follows

$$b < \sqrt{ac} \quad , \quad B < \sqrt{AC} \quad , \quad \text{thus } bB < \sqrt{ac\,AC},$$

therefore

$$\sigma = aC - 2\,bB + cA = (\sqrt{aC} - \sqrt{cA})^2 + 2\,(\sqrt{ac\,AC} - bB) > 0.$$

The solution of (27''):

$$\psi = \frac{\sigma \pm \sqrt{\sigma^2 - 4\,sS}}{2s}$$

requires an investigation of the eliminant of $g(z)$ and $G(z)$:

$$\left.\begin{aligned} R = \sigma^2 - 4\,sS &= (aC - 2\,bB + cA)^2 - 4\,(ac - b^2)\,(AC - B^2) \\ &= (cA - aC)^2 - 4\,(bC - cB)\,(aB - bA) \\ &= \beta^2 - 4\,\alpha\gamma. \end{aligned}\right\} \quad (28^b)$$

$$\begin{aligned} R = \sigma^2 - 4\,sS &= \{(\sqrt{aC} - \sqrt{cA})^2 + 2\,(\sqrt{ac\,AC} - bB)\}^2 - 4\,(ac - b^2)\,(AC - B^2) \\ &= (\sqrt{aC} - \sqrt{cA})^4 + 4\,(\sqrt{aC} - \sqrt{cA})^2\,(\sqrt{ac\,AC} - bB) + \\ &\quad + 4\,(ac\,AC - 2bB\sqrt{ac\,AC} + b^2B^2 - ac\,AC + b^2\,AC + B^2\,ac - b^2\,B^2) \\ &= (\sqrt{aC} - \sqrt{cA})^4 + 4\,(\sqrt{aC} - \sqrt{cA})^2\,(\sqrt{ac\,AC} - bB) + \\ &\qquad + 4\,(b\sqrt{AC} - B\sqrt{ac})^2 > 0. \end{aligned}$$

Hence the roots of (27'') are real.

Obviously the minimum value ψ_0 of ψ is the inferior root of (27''), thus:

$$\psi_0 = \frac{\sigma - \sqrt{R}}{2s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The corresponding value z_0 of z is that root of (24″) which answers to ψ_0; it follows also from (26):

$$z_0 = \frac{-ac\psi_0 + cA}{cb\,\psi_0 - cB} = \frac{-b^2\psi_0 + bB}{bc\,\psi_0 - bC} = \frac{-(ac-b^2)\,\psi_0 + (cA - bB)}{bC - cB}$$

$$= \frac{-2\,s\,\psi_0 + 2\,(cA - bB)}{2\,a} = \frac{-\sigma + \sqrt{R} + 2\,(cA - bB)}{2\,a} =$$

$$= \frac{-cA + 2\,bB - aC + 2\,cA - 2\,bB + \sqrt{R}}{2\,a},$$

or

$$z_0 = \frac{\beta + \sqrt{R}}{2\,a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Hence the value of z required is, with the definitions for α, β and γ given in (25), the root of (24″) having the positive root of R.

The angle θ_0 required is therefore determined by

$$\operatorname{tg}\theta_0 = \frac{\beta + \sqrt{R}}{2\,a} = \frac{(cA - aC) + \sqrt{(cA - aC)^2 - 4\,(bC - cB)\,(aB - bA)}}{2\,(bC - cB)}. \quad (30')$$

So the angle θ_0 is known, all but 180°; we must choose that value of θ_0, which, being substituted in (21), makes l_0 positive.

In the formula (30′) for θ_0 the coefficients a, b, c, being functions of P, Q and ω, are given a priori, thus independent of the result of the experiment, which furnishes the pairs of numbers (x_k, y_k) $(k = 1, \ldots n)$. On the contrary the magnitudes A, B, C are really dependent of this result; so are the magnitudes $\bar{x}, \bar{y}$ appearing in the formula (21) for l_0. Repeating the proof, we can expect other values of $A, B, C, \bar{x}, \bar{y}$.

The values of θ and l obtained in (30′) and (21) by taking the average, furnish L_0 as the „apparent line" (apparently true line), on which the points S_k ought to lie. According to usage, we shall call the deviations of the points S_k from this line: the apparent errors (residuals) of the position of S_k. T_k being the point on L_0 corresponding to S_k, $\overrightarrow{T_k S_k} = -\delta_k$ is the apparent error of S_k, as to amount and as to direction.

Denoting the coordinates of T_k by X_k, Y_k, we have

$$X_k \cos\theta_0 + Y_k \sin\theta_0 - l_0 = 0.$$

The residuals of the coordinates of S_k being

$$x_k - X_k = \xi_k\,,\; y_k - Y_k = \eta_k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and the coordinates of S_k satisfying

$$x_k \cos\theta_0 + y_k \sin\theta_0 - l_0 = \lambda_k,$$

we find:

$$\xi_k \cos\theta_0 + \eta_k \sin\theta_0 = \lambda_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Passing to the system of coordinates of the main directions of elasticity (x', y'), the residuals ξ', η' of the coordinates of this system are connected to ξ, η by

$$\left.\begin{aligned}\xi &= \xi' \cos\omega - \eta' \sin\omega,\\ \eta &= \xi' \sin\omega + \eta' \cos\omega.\end{aligned}\right\} \quad \ldots \quad (33)$$

As the moduli of elasticity P and Q (cf. (8)) play the part of "weights", P and Q are inversely proportional to the squares of the mean errors of x' and y', hence to the mean squares $\overline{\xi'^2}$ and $\overline{\eta'^2}$ of ξ' and η' resp.; moreover $\overline{\xi'\eta'} = 0$, ξ' and η' being measured along the main directions of elasticity.

So we have

$$P \, . \, \overline{\xi'^2} = Q \, . \, \overline{\eta'^2}, \quad \overline{\xi'\eta'} = 0, \quad \ldots \quad (34)$$

whence

$$\overline{\xi'^2} = \varrho\, Q, \quad \overline{\eta'^2} = \varrho\, P, \quad \overline{\xi'\eta'} = 0; \quad \ldots \quad (34')$$

where ϱ is a factor of proportionality to be determined afterwards.

From (33) follows

$$\overline{\xi^2} = \overline{\xi'^2} \cos^2\omega - 2\,\overline{\xi'\eta'} \sin\omega\cos\omega + \overline{\eta'^2} \sin^2\omega = \varrho\,(Q\cos^2\omega + P\sin^2\omega),$$

$$\overline{\xi\eta} = \overline{\xi'^2} \sin\omega\cos\omega + \overline{\xi'\eta'}\,(\cos^2\omega - \sin^2\omega) - \overline{\eta'^2} \sin\omega\cos\omega = \varrho(Q-P)\sin\omega\cos\omega,$$

$$\overline{\eta^2} = \overline{\xi'^2} \sin^2\omega + 2\,\overline{\xi'\eta'} \sin\omega\cos\omega + \overline{\eta'^2} \cos^2\omega = \varrho\,(Q\sin^2\omega + P\cos^2\omega),$$

hence, on account of (11):

$$\overline{\xi^2} = \varrho\, a, \quad \overline{\xi\eta} = \varrho\, b, \quad \overline{\eta^2} = \varrho\, c. \quad \ldots \quad (35)$$

By taking the mean squares of both sides of (32) we find

$$\overline{\xi^2} \cos^2\theta_0 + 2\,\overline{\xi\eta} \cos\theta_0 \sin\theta_0 + \overline{\eta^2} \sin^2\theta_0 = \overline{\lambda^2},$$

thus, by (35),

$$\varrho\,(a\cos^2\theta_0 + 2\,b\cos\theta_0\sin\theta_0 + c\sin^2\theta_0) = \overline{\lambda^2},$$

whence, on account of (12) and of (14),

$$\varrho\, f(\theta_0) = (\overline{\lambda^2})_0 = \varphi_0 \, . \, f(\theta_0);$$

therefore

$$\varrho = \varphi_0 = \varphi\,(\theta_0) = \psi_0 = \psi\,(z_0) \quad \ldots \quad (36)$$

and

$$\overline{\xi^2} = \varphi_0\, a, \quad \overline{\xi\eta} = \varphi_0\, b, \quad \overline{\eta^2} = \varphi_0\, c, \quad \ldots \quad (37)$$

$\varphi_0 = \psi_0$ being determined by (29).

In this way the mean squares and the mean product of the residuals are found.

Denoting the true errors of x_k, y_k, u_k, v_k, $\bar{x}$, $\bar{y}$, etc. by $\triangle x_k$, $\triangle y_k$, etc., the formula

$$u_k = x_k - \bar{x} = x_k - \frac{x_1 + x_2 + \ldots + x_k + \ldots + x_n}{n} = \frac{-x_1 - x_2 - \ldots + (n-1)x_k - \ldots - x_n}{n}$$

gives

$$\triangle u_k = \frac{-\triangle x_1 - \triangle x_2 - \ldots + (n-1)\triangle x_k - \ldots - \triangle x_n}{n} \quad \ldots \quad (38^a)$$

Likewise

$$\triangle v_k = \frac{-\triangle y_1 - \triangle y_2 - .. + (n-1)\triangle y_k - .. - \triangle y_n}{n} \quad . \quad . \quad (38^b)$$

The errors of $A = \frac{[uu]}{n}$, $B = \frac{[uv]}{n}$, $C = \frac{[vv]}{n}$ are:

$$\triangle A = \frac{2}{n}[u\triangle u], \ \triangle B = \frac{1}{n}\{[v\triangle u] + [u\triangle v]\}, \ \triangle C = \frac{2}{n}[v\triangle v].$$

In determining the mean error of a function of A, B and C, we have to do with the mean values of $\triangle A^2$, $\triangle A \triangle B$, $\triangle A \triangle C$, $\triangle B^2$, $\triangle B \triangle C$, $\triangle C^2$, thus with the mean values of the squares and products built up out of $[u\triangle u]$, $[v\triangle u]$, $[u\triangle v]$, $[v\triangle v]$.

In computing the mean value of $[u\triangle u]^2$ for instance, we meet with two sums, viz.: $[u_k u_k . \triangle u_k \triangle u_k]$ and $[[u_k u_l . \triangle u_k \triangle u_l]]$, where $[[\ ]]$ designates a summation of the $n(n-1)$-terms wherein $l \neq k$.

The errors $\triangle u_k$, $\triangle v_k$ being supposed independent of the very u_k, v_k, the mean value of $[u_k u_k . \triangle u_k \triangle u_k]$, represented by $M([u_k u_k . \triangle u_k \triangle u_k])$, may be written:

$$M([u_k u_k . \triangle u_k \triangle u_k]) = M([u_k u_k]) \times M(\triangle u_k \triangle u_k);$$

likewise

$$M([[u_k u_l . \triangle u_k \triangle u_l]]) = M([[u_k u_l]]) \times M(\triangle u_k \triangle u_l).$$

Now

$$[u_k u_k] = nA, \ [u_k]^2 = [u_k u_k] + [[u_k u_l]] = 0, \text{ thus } [[u_k u_l]] = -nA.$$

likewise

$$[u_k v_k] = nB, \ [u_k][v_k] = [u_k v_k] + [[u_k v_l]] = 0, \text{ thus } [[u_k v_l]] = -nB,$$

$$[v_k v_k] = nC, \ [v_k]^2 = [v_k v_k] + [[v_k v_l]] = 0, \text{ thus } [[v_k v_l]] = -nC.$$

Moreover:

$$\left.\begin{aligned} \triangle u_1 &= \frac{(n-1)\triangle x_1 - \triangle x_2 - \triangle x_3 - .. - \triangle x_n}{n}, \\ \triangle u_2 &= \frac{-\triangle x_1 + (n-1)\triangle x_2 - \triangle x_3 - .. - \triangle x_n}{n} \end{aligned}\right\} \quad . \quad . \quad (38c)$$

whence

$$\triangle u_1 \triangle u_1 = \frac{1}{n^2}\{+(n-1)^2 \triangle x_1^2 + \triangle x_2^2 + \triangle x_3^2 + .. + \triangle x_n^2 + [[p_{kl}\triangle x_k \triangle x_l]]\}$$

$$\triangle u_1 \triangle u_2 = \frac{1}{n^2}\{-(n-1)\triangle x_1^2 - (n-1)\triangle x_2^2 + \triangle x_3^2 + .. + \triangle x_n^2 + [[q_{kl}\triangle x_k \triangle x_l]]\}.$$

Now we have

$$M(\triangle x_1^2) = M(\triangle x_2^2) = .. = M(\triangle x_n^2) = M(\triangle x^2), \ M(\triangle x_k \triangle x_l) = 0 \text{ (for } l \neq k\text{)},$$

therefore

$$M(\triangle u_1 \triangle u_1) = \frac{1}{n^2}\{(n-1)^2 M(\triangle x_1^2) + (n-1) M(\triangle x^2)\} = \frac{n-1}{n} M(\triangle x^2),$$

$$M(\triangle u_1 \triangle u_2) = \frac{1}{n^2}\{-(n-1) M(\triangle x_1^2) - (n-1) M(\triangle x_2^2) + (n-2) M(\triangle x^2)\} =$$

$$= -\frac{1}{n} M(\triangle x^2).$$

Hence:

$$M(\triangle u_k \triangle u_k) = \frac{n-1}{n} M(\triangle x^2), \quad M(\triangle u_k \triangle u_l) = -\frac{1}{n} M(\triangle x^2).$$

Likewise:

$$M(\triangle u_k \triangle v_k) = \frac{n-1}{n} M(\triangle x \triangle y), \quad M(\triangle u_k \triangle v_l) = -\frac{1}{n} M(\triangle x \triangle y),$$

$$M(\triangle v_k \triangle v_k) = \frac{n-1}{n} M(\triangle y^2), \quad M(\triangle v_k \triangle v_l) = -\frac{1}{n} M(\triangle y^2).$$

So we arrive at

$$M([u \triangle u]^2) = M([u_k u_k \,.\, \triangle u_k \triangle u_k] + [[u_k u_l \,.\, \triangle u_k \triangle u_l]]) =$$
$$= M([u_k u_k]) \,.\, M(\triangle u_k \triangle u_k) + M([[u_k u_l]]) \,.\, M(\triangle u_k \triangle u_l)$$
$$= nM(A) \,.\, \frac{n-1}{n} M(\triangle x^2) - nM(A) \times -\frac{1}{n} M(\triangle x^2) =$$
$$= nM(A) \,.\, M(\triangle x^2).$$

Likewise:

$$M([u \triangle u][u \triangle v]) = nM(A) \,.\, M(\triangle x \triangle y),$$
$$M([u \triangle v]^2) = nM(A) \,.\, M(\triangle y^2),$$
$$M([u \triangle u][v \triangle u]) = n\, M(B) \,.\, M(\triangle x^2),$$
$$M([u \triangle u][v \triangle v]) = M([u \triangle v][v \triangle u]) = n\, M(B) \,.\, M(\triangle x \triangle y),$$
$$M([u \triangle v][v \triangle v]) = n\, M(B) \,.\, M(\triangle y^2),$$
$$M([v \triangle u]^2) = n\, M(C) \,.\, M(\triangle x^2),$$
$$M([v \triangle u][v \triangle v]) = n\, M(C) \,.\, M(\triangle x \triangle y),$$
$$M([v \triangle v]^2) = n\, M(C) \,.\, M(\triangle y^2).$$

By means of these formulae we find

$$M(\triangle A^2) = \frac{4}{n^2} M([u \triangle u]^2) = \frac{4}{n} M(A) \,.\, M(\triangle x^2),$$

$$M(\triangle A \,.\, \triangle B) = \frac{2}{n^2} M([u \triangle u]\{[v \triangle u] + [u \triangle v]\}) =$$
$$= \frac{2}{n} \{M(B) \,.\, M(\triangle x^2) + M(A) \,.\, M(\triangle x \triangle y)\},$$

$$M(\triangle A \triangle C) = \frac{4}{n^2} M([u \triangle u][v \triangle v]) = \frac{4}{n} M(B) \,.\, M(\triangle x \triangle y),$$

$$M(\triangle B^2) = \frac{1}{n^2} M(\{[v \triangle u] + [u \triangle v]\}^2) =$$
$$= \frac{1}{n} \{M(C) \,.\, M(\triangle x^2) + 2\, M(B) \,.\, M(\triangle x \triangle y) + M(A) \,.\, M(\triangle y^2)\},$$

$$M(\triangle B \triangle C) = \frac{2}{n^2} M(\{[v \triangle u) + [u \triangle v]\}[v \triangle v]) =$$
$$= \frac{2}{n} \{M(C) \,.\, M(\triangle x \triangle y) + M(B) \,.\, M(\triangle y^2)\},$$

$$M(\triangle C^2) = \frac{4}{n^2} M([v \triangle v]^2) = \frac{4}{n} M(C) \,.\, M(\triangle y^2).$$

As the present problem of adjustment deals with two unknowns (θ and l), so that there is only question of adjustment if $n > 2$, the formula for the square of the mean true error runs:

$$\mathsf{E}^2(x) = M(\triangle x^2) = \frac{[\xi^2]}{n-2} = \frac{n\overline{\xi^2}}{n-2} = \frac{n}{n-2}\varphi_0 a.$$

Likewise:

$$M(\triangle x \triangle y) = \frac{[\xi\eta]}{n-2} = \frac{n\overline{\xi\eta}}{n-2} = \frac{n}{n-2}\varphi_0 b, \quad \ldots \quad (39)$$

$$\mathsf{E}^2(y) = M(\triangle y^2) = \frac{[\eta^2]}{n-2} = \frac{n\overline{\eta^2}}{n-2} = \frac{n}{n-2}\varphi_0 c.$$

Replacing moreover the values $M(A)$, $M(B)$, $M(C)$, essentially unknown, by the values A, B, C actually found, we obtain

$$\left.\begin{aligned}
M(\triangle A^2) &= \frac{4\varphi_0}{n-2} \cdot aA,\\
M(\triangle A \triangle B) &= \frac{2\varphi_0}{n-2} \cdot (aB + bA),\\
M(\triangle A \triangle C) &= \frac{4\varphi_0}{n-2} \cdot bB,\\
M(\triangle B^2) &= \frac{\varphi_0}{n-2} \cdot (aC + 2\,bB + cA),\\
M(\triangle B \triangle C) &= \frac{2\varphi_0}{n-2} \cdot (bC + cB),\\
M(\triangle C^2) &= \frac{4\varphi_0}{n-2} \cdot cC.
\end{aligned}\right\} \quad \ldots \quad (40)\,^{1)}$$

As $\theta_0 = \text{arc tg}\, z_0$, by (24''), is a function of α, β and γ, the mean error of θ_0 (and of z_0) will be built up out of the means

$$M(\triangle \alpha^2) \quad , \quad M(\triangle \alpha \triangle \beta), \ldots \text{ etc.}$$

We have

$$\triangle\alpha = b\triangle C - c\triangle B, \quad \triangle\beta = c\triangle A - a\triangle C, \quad \triangle\gamma = a\triangle B - b\triangle A,$$

thus:

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[1]) The formulae (40) differ from the formulae $M(\triangle A^2) = \frac{2A^2}{n-1}$, $M(\triangle A \triangle B) = \frac{2AB}{n-1}$, $M(\triangle A \triangle C) = \frac{2B^2}{n-1}$, $M(\triangle B^2) = \frac{B^2 + AC}{n-1}$, $M(\triangle B \triangle C) = \frac{2BC}{n-1}$, $M(\triangle C^2) = \frac{2C^2}{n-1}$, valid in the case of linear correlation between x and y. In the case of correlation between x and y the deviations u_k and v_k of x_k and y_k from $\overline{x}$ and $\overline{y}$ are considered as accidental errors. In the present case however the coordinates x_k and y_k are yet considered as free from error, provided the point S_k be situated on L_0.

$$M(\triangle\alpha^2) = b^2\,M(\triangle C^2) - 2bc\,M(\triangle B\,\triangle C) + c^2\,M(\triangle B^2)$$

$$= \frac{\varphi_0}{n-2}\,\{b^2 \,.\, 4\,cC - 2\,bc \,.\, 2\,(bC + cB) + c^2\,(aC + 2\,bB + cA)\} =$$

$$= \frac{\varphi_0}{n-2}\,c^2\,(aC - 2\,bB + cA),$$

$$M(\triangle\alpha\,\triangle\beta) = bcM(\triangle A\triangle C) - c^2M(\triangle A\triangle B) - abM(\triangle C^2) + acM(\triangle B\triangle C)$$

$$= \frac{\varphi_0}{n-2}\,\{bc\,.\,4\,bB - c^2\,.\,2(aB+bA) - ab\,.\,4\,cC + ac\,.\,2\,(bC+cB)\} =$$

$$= \frac{-2\varphi_0}{n-2}\,bc\,(aC - 2\,bB + cA),$$

$$M(\triangle\alpha\,\triangle\gamma) = abM(\triangle B\,\triangle C) - b^2M(\triangle A\,\triangle C) - acM(\triangle B^2) + bcM(\triangle A\,\triangle B)$$

$$= \frac{\varphi_0}{n-2}\{ab.2(bC+cB) - b^2.4bB - ac(aC+2bB+cA) + bc.2(aB+bA)\} =$$

$$= \frac{\varphi_0}{n-2}\,(2\,b^2 - ac)\,(aC - 2\,bB + cA),$$

$$M(\triangle\beta^2) = c^2M(\triangle A^2) - 2\,acM(\triangle A\,\triangle C) + a^2M(\triangle C^2)$$

$$= \frac{\varphi_0}{n-2}\,\{c^2.\,4aA - 2ac\,.\,4b\,B + a^2.\,4cC\} = \frac{4\varphi_0}{n-2}\,ac\,(aC - 2bB + cA),$$

$$M(\triangle\beta\,\triangle\gamma) = acM(\triangle A\,\triangle B) - bcM(\triangle A^2) - a^2M(\triangle B\,\triangle C) + abM(\triangle A\,\triangle C)$$

$$= \frac{\varphi_0}{n-2}\,\{ac\,.\,2(aB+bA) - bc\,.4aA - a^2.2(bC + cB) + ab\,.4bB\} =$$

$$= \frac{-2\varphi_0}{n-2}\,ab\,(aC - 2\,bB + cA),$$

$$M(\triangle\gamma^2) = a^2\,M(\triangle B^2) - 2\,ab\,M(\triangle A\,\triangle B) + b^2\,M(\triangle A^2)$$

$$= \frac{\varphi_0}{n-2}\,\{a^2\,(aC + 2\,bB + cA) - 2\,ab\,.\,2\,(aB + bA) + b^2.\,4\,aA\} =$$

$$= \frac{\varphi_0}{n-2}\,a^2\,(aC - 2\,bB + cA),$$

or, on account of (28^a),

$$\left.\begin{aligned}
&M(\triangle\alpha^2) = \frac{\varphi_0\sigma}{n-2}\,.\,c^2, \qquad M(\triangle\alpha\triangle\beta) = \frac{-2\varphi_0\,\sigma}{n-2}\,.\,bc,\\
&\qquad\qquad\qquad\qquad M(\triangle\alpha\,\triangle\gamma) = \frac{\varphi_0\sigma}{n-2}\,.\,(2\,b^2 - ac),\\
&M(\triangle\beta^2) = \frac{4\,\varphi_0\sigma}{n-2}\,.\,ac, \qquad M(\triangle\beta\triangle\gamma) = \frac{-2\varphi_0\,\sigma}{n-2}\,.\,ab,\\
&\qquad\qquad\qquad\qquad M(\triangle\gamma^2) = \frac{\varphi_0\sigma}{n-2}\,.\,a^2.
\end{aligned}\right\} \quad (41)$$

Taking z_0 as central value, we derive from (24''):

$$2\,a z_0 \triangle z_0 - \beta \triangle z_0 + z_0^2 \triangle a - z_0 \triangle \beta + \triangle \gamma = 0$$

or

$$-(2\,a z_0 - \beta) \triangle z_0 = z_0^2 \triangle a - z_0 \triangle \beta + \triangle \gamma \quad . \quad . \quad . \quad . \quad (42)$$

whence

$$(2\,a z_0 - \beta)^2 M(\triangle z_0^2) = z_0^4 M(\triangle a^2) - 2\,z_0^3 M(\triangle a \triangle \beta) + \\ + 2\,z_0^2 M(\triangle a \triangle \gamma) + z_0^2 M(\triangle \beta^2) - 2\,z_0 M(\triangle \beta \triangle \gamma) + M(\triangle \gamma^2)$$

Now

$$(2\,a z_0 - \beta)^2 = 4\,a\,(a z_0^2 - \beta z_0 + \gamma) + (\beta^2 - 4\,a\gamma),$$

or, z_0 being a root of (24''), and by (28^b),

$$(2\,a z_0 - \beta)^2 = R.$$

Substituting for $M(\triangle a^2)$ etc. the values found in (41), we obtain

$$R\,.\,M(\triangle z_0^2) = \frac{\varphi_0 \sigma}{n-2} \{z_0^4\,.\,c^2 - 2\,z_0^3\,.-2\,bc + 2\,z_0^2\,.\,(2\,b^2 - ac) + \\ + z_0^2\,.\,4\,ac - 2\,z_0\,.-2\,ab + a^2\}$$

$$= \frac{\varphi_0 \sigma}{n-2} (c\,z_0^2 + 2\,b z_0 + a)^2,$$

or, by (22) and (23),

$$M(\triangle z_0^2) = \frac{\varphi_0 \sigma}{(n-2)\,R} \{\,g(z_0)\,\}^2 = \frac{\sigma \varphi_0 g_0^2}{(n-2)\,R} = \frac{\sigma g_0 G_0}{(n-2)\,R}.$$

We have further for the true error of θ_0:

$$\triangle\,\theta_0 = \frac{\triangle z_0}{1 + z_0^2},$$

thus

$$M(\triangle \theta_0^2) = \frac{M(\triangle z_0^2)}{(1+z_0^2)^2} = \frac{\sigma \varphi_0}{(n-2)\,R} \left(\frac{g_0}{1+z_0^2}\right)^2 = \\ = \frac{\sigma \varphi_0}{(n-2) R} \left(\frac{a + 2\,b \operatorname{tg} \theta_0 + c \operatorname{tg}^2 \theta_0}{\sec^2 \theta_0}\right)^2 =$$

$$= \frac{\sigma \varphi_0}{(n-2)\,R} (a \cos^2 \theta_0 + 2\,b \cos \theta_0 \sin \theta_0 + c \sin^2 \theta_0)^2 = \\ = \frac{\sigma \varphi_0}{(n-2)\,R} \{\,f(\theta_0)\,\}^2 = \frac{\sigma f_0 F_0}{(n-2)\,R}.$$

So the square of the mean true error $\mathsf{E}(\theta_0)$ of θ_0 is

$$\mathsf{E}^2(\theta_0) = \frac{1}{n-2} \cdot \frac{\sigma}{R} \cdot f_0 F_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

The mean error of l_0 is to be derived from

$$\triangle\,l_0 = \triangle\,\bar{x}\,.\cos \theta_0 + \triangle \bar{y}\,.\sin \theta_0 + (-\bar{x} \sin \theta_0 + \bar{y} \cos \theta_0) \triangle\,\theta_0 . \;.$$

Here we have (see (42))

$$\triangle\,\theta_0 = \frac{\triangle z_0}{1+z_0{}^2} = -\,\frac{z_0{}^2\,\triangle\,\alpha - z_0\,\triangle\,\beta + \triangle\,\gamma}{(1+z_0{}^2)\,(2\,\alpha z_0 - \beta)} =$$

$$= -\,\frac{z_0{}^2\,(b\triangle C - c\triangle B) - z_0(c\triangle A - a\triangle C) + (a\triangle B - b\triangle A)}{(1+z_0{}^2)\,(2\alpha z_0 - \beta)}$$

$$= \lambda\,\triangle\,A + \mu\,\triangle\,B + \nu\,\triangle\,C.$$

So we obtain:

$$\left.\begin{aligned} M\,(\triangle l_0{}^2) = \cos^2\theta_0\,.\,M\,(\triangle\,\bar{x}^2) + 2\cos\theta_0\sin\theta_0\,.\,M\,(\triangle\bar{x}\triangle\bar{y}) + \\ + \sin^2\theta_0\,.\,M(\triangle\bar{y}^2) + 2\,(-\bar{x}\sin\theta_0 + \bar{y}\cos\theta_0)\,\{\cos\theta_0\,.\,M(\triangle\bar{x}\triangle\theta_0) + \\ + \sin\theta_0\,.\,M\,(\triangle\bar{y}\,\triangle\,\theta_0)\,\} + (-\,\bar{x}\sin\theta_0 + \bar{y}\cos\theta_0)^2\,M\,(\triangle\theta_0{}^2). \end{aligned}\right\} \quad (44)$$

Now from

$$\bar{x} = \frac{[x]}{n},\quad \bar{y} = \frac{[y]}{n}$$

ensues

$$M\,(\triangle\bar{x}^2) = \frac{1}{n}\,M(\triangle x^2),\ M(\triangle\bar{x}\triangle\bar{y}) = \frac{1}{n}\,M(\triangle x\triangle y),\quad M(\triangle\bar{y}^2) = \frac{1}{n}\,M\,(\triangle y^2),$$

hence, by (39),

$$M\,(\triangle\bar{x}^2) = \frac{\varphi_0 a}{n-2},\ M\,(\triangle\bar{x}\triangle\bar{y}) = \frac{\varphi_0 b}{n-2},\ M\,(\triangle\bar{y}^2) = \frac{\varphi_0 c}{n-2} \qquad (45)$$

Moreover:

$$M\,(\triangle\bar{x}\triangle\theta_0) = \lambda M\,(\triangle\bar{x}\triangle A) + \mu M\,(\triangle\bar{x}\triangle B) + \nu M\,(\triangle\bar{x}\triangle C);$$

likewise:

$$M\,(\triangle\bar{y}\triangle\theta_0) = \lambda M\,(\triangle\bar{y}\triangle A) + \mu M\,(\triangle\bar{y}\triangle B) + \nu M\,(\triangle\bar{y}\triangle C).$$

Now

$$M\,(\triangle\bar{x}\triangle A) = \frac{2}{n^2}\,M\,([\triangle x]\,[u\triangle u]) = \frac{2}{n^2}M\,([u_k\triangle u_k\triangle x_k] + [[u_k\triangle u_k\triangle x_l]])$$

$$= \frac{2}{n^2}\,\{\,M\,([u_k])\,M\,(\triangle u_k\triangle x_k) + M\,[[u_k]]\,M\,(\triangle u_k\triangle x_l)\,\}.$$

As however $[u_k] = 0$, also $M\,([u_k]) = 0$; therefore:

$$M\,(\triangle\bar{x}\,.\,\triangle A) = 0;$$

likewise

$$M\,(\triangle\bar{x}\,.\,\triangle B) = 0,\quad M\,(\triangle\bar{x}\,.\,\triangle C) = 0,\quad M\,(\triangle\bar{y}\,.\,\triangle\,A) = 0,$$

$$M\,(\triangle\bar{y}\,.\,\triangle B) = 0,\ M\,(\triangle\bar{y}\,.\,\triangle C) = 0,$$

whence

$$M\,(\triangle\bar{x}\,.\,\triangle\theta_0) = 0,\ M\,(\triangle\bar{y}\,.\,\triangle\theta_0) = 0. \quad .\ .\ .\ .\ . \quad (46)$$

In virtue of (44), (45), (46) and (43), we find at last for $M\,(\triangle l_0^2) = \mathrm{E}^2\,(l_0)$:

$$\mathrm{E}^2\,(l_0) = \frac{\varphi_0}{n-2}\,\{a\cos^2\theta_0 + 2\,b\cos\theta_0\sin\theta_0 + c\sin^2\theta\} +$$

$$+ (-\bar{x}\sin\theta_0 + \bar{y}\cos\theta_0)^2 \cdot \frac{1}{n-2}\cdot\frac{\sigma}{R} f_0\,F_0$$

$$= \frac{\varphi_0 f_0}{n-2} + (\bar{x}^2 + \bar{y}^2 - l_0^2)\cdot\frac{1}{n-2}\cdot\frac{\sigma}{R}\cdot f_0\,F_0,$$

or

$$\mathrm{E}^2\,(l_0) = \frac{F_0}{n-2}\,\{1 + (\bar{x}^2 + \bar{y}^2 - l_0^2)\cdot\frac{\sigma}{R} f_0\} \quad . \quad . \quad . \quad . \quad (47)$$

Here $\bar{x}^2 + \bar{y}^2 - l_0^2$ represents the square of the projection p of the radius vector of the "mean point" $(\bar{x}, \bar{y})$ on the line L_0.

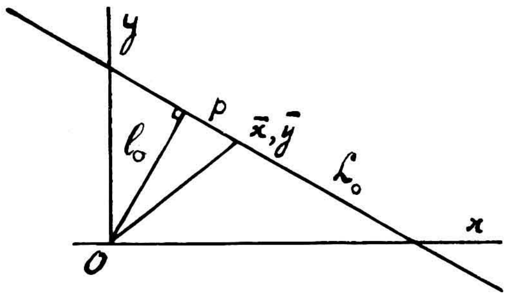

Fig. 2.

Hence

$$\mathrm{E}^2\,(l_0) = \frac{F_0}{n-2}\,\{1 + p^2\frac{\sigma}{R} f_0\} \quad . \quad . \quad . \quad . \quad . \quad (47')$$

Moreover we have, in virtue of (46),

$$\left.\begin{aligned} M(\triangle\, l_0\, \triangle\, \theta_0) &= (-\bar{x}\sin\theta_0 + \bar{y}\cos\theta_0)\, M\,(\triangle\,\theta_0^2) = \\ &= \frac{1}{n-2}\cdot\frac{\sigma}{R}\cdot f_0\,F_0\,.\,(-\bar{x}_0\sin\theta_0 + \bar{y}\cos\theta_0) \end{aligned}\right\} \quad . \quad (48)$$

If there is occasion to consider the different "observed points" $S_k\,(x_k, y_k)$ unequally certain, we can assign to each point S_k a weight g_k. Then we operate with $u'_k = u_k\sqrt{g_k}$, $v'_k = v_k\sqrt{g_k}$ and with $A' = \frac{[g_k\,u_k\,u_k]}{n} = \overline{guu}$, $B' = \frac{[g_k\,u_k\,v_k]}{n} = \overline{guv}$, $C' = \frac{[g_k\,v_k\,v_k]}{n} = \overline{gvv}$ in the same manner as we have operated above with u_k, v_k, A, B and C.

Summarising the results obtained above, we can say that the most probable line L_0

$$x\cos\theta_0 + y\sin\theta_0 - l_0 = 0$$

is determined by (30′) and (21), thus by

$$\mathrm{tg}\,\theta_0 = \frac{\beta + \sqrt{R}}{2a}\quad,\quad l_0 = \bar{x}\cos\theta_0 + \bar{y}\sin\theta_0 > 0,$$

the mean errors of θ_0 and l_0 being determined by (43) and (47′), thus by

$$\mathsf{E}^2(\theta_0) = \frac{1}{n-2} \cdot \frac{\sigma}{R} \cdot f_0 F_0 \quad , \quad \mathsf{E}^2(l_0) = \frac{F_0}{n-2} \left\{ 1 + p^2 \cdot \frac{\sigma}{R} \cdot f_0 \right\},$$

where

$$\alpha = bC - cB \; , \; \beta = cA - aC \; , \; \gamma = aB - bA \; , \; \sigma = aC - 2\,bB + cA,$$
$$R = \beta^2 - 4\,\alpha\gamma = (aC - 2\,bB + cA)^2 - 4\,(ac - b^2)\,(AC - B^2),$$
$$f_0 = a\cos^2\theta_0 + 2\,b\cos\theta_0 \sin\theta_0 + c\sin^2\theta_0 \; , \; F_0 = A\cos^2\theta_0 + 2\,B\cos\theta_0\sin\theta_0 + C\sin^2\theta_0.$$
$$p^2 = \overline{x}^2 + \overline{y}^2 - l_0^2.$$

Particular cases.

I. The axes of coordinates coincide with the main axes of elasticity. Here we have $\omega = 0$, thus $a = Q$, $b = 0$, $c = P$, whence

$$\alpha = -PB \quad , \quad \beta = PA - QC \quad , \quad \gamma = QB \quad , \quad \sigma = PA + QC,$$
$$R = (PA - QC)^2 + 4\,PQB^2 = (PA + QC)^2 - 4\,PQ\,(AC - B^2),$$
$$f_0 = Q\cos^2\theta_0 + P\sin^2\theta_0.$$

$M(\triangle x \triangle y) = 0$, so the errors of x and y are independent of each other.

II. The plane is isotropic, or, what comes to the same thing: x and y are equally uncertain.

Then we have $P = Q = 1$, so that $a = c = 1$, $b = 0$, whence

$$\alpha = -B \quad , \quad \beta = A - C \quad , \quad \gamma = B \quad . \quad \sigma = A + C,$$
$$R = (A - C)^2 + 4\,B^2 = (A + C)^2 - 4\,(AC - B^2),$$
$$f_0 = 1.$$

Therefore

$$\operatorname{tg}\theta_0 = \frac{A - C + \sqrt{(A - C)^2 + 4\,B^2}}{-2\,B} = \frac{C - A - \sqrt{(C - A)^2 + 4\,B^2}}{2\,B};$$

$$\operatorname{tg} 2\,\theta_0 = \frac{2\,B}{A - C} \; , \; \cos 2\,\theta_0 = \frac{C - A}{\sqrt{R}} \; , \; \sin 2\,\theta_0 = \frac{-2B}{\sqrt{R}};$$

$$\mathsf{E}^2(\theta_0) = \frac{1}{n-2} \cdot \frac{(A + C)F_0}{(A + C)^2 - 4\,(AC - B^2)},$$

$$\mathsf{E}^2(l_0) = \frac{F_0}{n-2} \left\{ 1 + p^2 \cdot \frac{A + C}{\sqrt{(A + C)^2 - 4\,(AC - B^2)}} \right\};$$

$$F_0 = \frac{A+C}{2} + \frac{A-C}{2}\cos 2\theta_0 + B\sin 2\theta_0 = \frac{A + C - \sqrt{(A+C)^2 - 4(AC - B^2)}}{2}. \text{ [1]}$$

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[1]) The formula for $\operatorname{tg}\theta_0$ (and $\operatorname{tg} 2\theta_0$) is the same as in the case of linear correlation between the variables x_k, y_k. The expression for the mean error of θ_0 however is different (see the footnote on page 1032).

III. x is perfectly certain: $\frac{P}{Q}=\infty$, or $P=1$, $Q=0$.

So we have $a=b=0$, $c=1$, whence

$$\alpha=-B,\ \beta=A,\ \gamma=0,\ \sigma=A,\ R=A^2;\ f_0=\sin^2\theta_0.$$

$\operatorname{tg}\theta_0=\frac{2A}{-2B}=-\frac{A}{B}$, or, what comes to the same thing: the direction tangent of the line L_0 is $\frac{B}{A}=\frac{\overline{uv}}{\overline{uu}}$.

$$f_0=\frac{1}{1+\frac{B^2}{A^2}}=\frac{A^2}{A^2+B^2},\quad F_0=\frac{A+2B\operatorname{tg}\theta_0+C\operatorname{tg}^2\theta_0}{1+\operatorname{tg}^2\theta_0}=\frac{A-2A+\frac{A^2C}{B^2}}{1+\frac{A^2}{B^2}}=$$

$$=\frac{A(AC-B^2)}{A^2+B^2}.$$

$$\mathsf{E}^2(\theta_0)=\frac{1}{n-2}\cdot\frac{1}{A}\frac{A^3(AC-B^2)}{(A^2+B^2)^2}=\frac{A^2(AC-B^2)}{(n-2)(A^2+B^2)^2},\ \text{or}$$

$$\mathsf{E}(\theta_0)=\frac{1}{\sqrt{n-2}}\cdot\frac{A\sqrt{AC-B^2}}{A^2+B^2};$$

$$\mathsf{E}^2(l_0)=\frac{A(AC-B^2)}{(n-2)(A^2+B^2)}\left\{1+p^2\cdot\frac{A}{A^2+B^2}\right\}.\ ^{1)}$$

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1) Here we have the same case as that of n observational equations $mx_k+h=y_k$, where m and h are the unknowns, x_k the known (and perfectly exact) coefficients, y_k the observations. The solution of the normal equations $[xx]\,m+[x\,.\,1]\,h=[xy]$, $[x\,.\,1]\,m+$ $+[1\times1]\,h=[1\,.\,y]$ $([1\times1]=n)$ furnishes, after introducing $u_k=x_k-\overline{x}$, $v_k=y_k-\overline{y}$, the same expression for m, viz. $\frac{\overline{uv}}{\overline{uu}}$. We also find for $\theta=\operatorname{arc}\operatorname{tg}m\pm\frac{\pi}{2}$ and $l=h\sin\theta=$ $=\frac{\pm h}{\sqrt{1+m^2}}$ the same mean errors as above.