Mathematics. — *On infinitesimal deformations of V_m in V_n.* By J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

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Let the points x^ν of a geodesic line in V_n be subjected to a transformation:

$$'x^\nu = x^\nu + \varepsilon v^\nu \,, \quad \ldots \ldots \ldots \quad (1)$$

where v^ν is a field of contravariant vectors defined along the line, and ε a small constant. Higher powers of ε may be neglected. Then we can deduce the conditions to which v^ν must conform in order that the transformed line may also be geodesic. A differential equation of the second order is found, which for $V_n = R_3$ is due to JACOBI, and in the general case to LEVI-CIVITA [1]).

An analogous question arises for a minimal-V_m in a V_n. This also leads to a differential equation of the second order, due for $V_n = R_3$, $m = 2$, to SCHWARZ, and for general V_n and $m = 2$ to CARTAN [2]).

We may now seek in general to find the equations expressing the change of the fundamental quantities of a V_m in V_n, when the points of this V_m are subjected to a displacement εv^ν. By fundamental quantities we understand the fundamental tensors and the different curvature quantities. After this we can easily find the differential equations for v^ν for the case that the displacement εv^ν does not change certain given properties of the V_m. It is only necessary to substitute the identities, characterizing this property, into the general equations.

In this paper we first deduce the conditions for a geodesic V_m and for a minimal-V_m, they are immediate generalisations of results found by LEVI-CIVITA and CARTAN; after this we deduce the equations for the bending [3]) of a V_m in V_n and find some interesting conclusions for the special case $V_n = R_n$. We conclude with the transformation of a V_{n-1} in V_n that leaves the principal directions of the second fundamental tensor invariant and with the equivoluminar transformation of a V_m in V_n.

§ 1. *The fundamental quantities of the V_m.*

We use two coordinate systems: $x^\nu, \lambda, \mu, \nu = a_1, \ldots, a_n$ in V_n and $y^c, a, b, c, d = 1, \ldots, m$ in V_m. According to a known property we can avoid the use of the coordinate system y. But it is useful for the present investigation, as we will accept that the deformation εv^ν takes it along

1) Sur l'écart géodésique, Math. Ann. **97** (26), 291—320.

2) Sur l'écart géodésique et quelques notions connexes, Rend. Acc. Lincei (6*a*) **5** (27) 609—613.

3) Under bending we understand a flexion without tearing or stretching.

with it. Hence the quantities of the V_n carry Greek, those of the V_m Latin indices. Under transformations of the x the components of the quantities of the V_m do not change, being fixed with respect to the y. The same holds for the components of the quantities of the V_n under a transformation of the y. Furthermore we have quantities with both Greek and Latin indices, their components being changed as the components of a quantity of V_m in as much as the Greek indices are concerned, and as those of a quantity of V_n in as much as the Latin indices are concerned [1]).

The most important of these quantities is:

$$B_a^{\nu} = \frac{\partial x^{\nu}}{\partial x^a} \quad \ldots \ldots \ldots \ldots \quad (2)$$

It can be shown very easily indeed that the B_a^{ν} behave like the components of a contravariant vector of the V_n under transformations of the x; and like components of a covariant vector under transformations of the y. With the aid of B_a^{ν} we deduce from the fundamental tensor $g_{\lambda\mu}$ of the V_n the fundamental tensor of the V_m:

$$g'_{ab} = B_{ab}^{\lambda\mu}\, g_{\lambda\mu} \;\; ; \;\; (B_{ab}^{\lambda\mu} = B_a^{\lambda}\, B_b^{\mu}) \quad \ldots \ldots \quad (3$$

and with the aid of g'_{ab} we form the quantity:

$$B_{\lambda}^{c} = g'^{cb}\, B_b^{\mu}\, g_{\lambda\mu} \quad \ldots \ldots \ldots \ldots \quad (4)$$

It follows from (2) and (4) that

$$B_{\mu}^{c}\, B_a^{\mu} = B_a^{c}, \quad \ldots \ldots \ldots \ldots \quad (5)$$

here we denote with B_a^c a set of m^2 numbers with value 1 when $a = c$, and 0 when $a \neq c$.

The quantities B_a^{ν} and B_{λ}^{c} allow us to define a unique correspondence between the quantities of the V_m and some of the quantites of the V_n and vice versa. We only need to show it for vectors. If

$$\overline{v}^{\nu} = B_b^{\nu}\, v^b \quad \ldots \ldots \ldots \ldots \quad (6^a)$$

corresponds to v^c, or, in covariant components:

$$\overline{v}_{\lambda} = g_{\lambda\nu}\, B_b^{\nu}\, g'^{bc}\, v_c = B_{\lambda}^{c}\, v_c\,, \quad \ldots \ldots \ldots \quad (6^b)$$

we have, on account of (5):

$$B_{\mu}^{c}\, \overline{v}^{\mu} = v^c \quad \ldots \ldots \ldots \ldots \quad (7^a)$$

$$B_a^{\mu}\, \overline{v}_{\mu} = v_a \quad \ldots \ldots \ldots \ldots \quad (7^b)$$

Hence it is equally possible to deduce v from $\overline{v}$ as $\overline{v}$ from v. We use this property to write v^{ν} and v_{λ} in stead of $\overline{v}^{\nu}$ and $\overline{v}_{\lambda}$; we therefore consider *them as another kind of components* of the vector v. We have reached

[1]) Such quantities are already introduced, for the discussion of V_m in V_n, by E. BOMPIANI, Studi sugli spazi curvi, Atti Veneto **80** (20/21) 1113—1145, and more systematically by B. L. VAN DER WAERDEN, Differentialkovarianten von V_m in V_n, Abh. Math. Sem. Hamburg **5** (27) 153—160.

by proceeding in this way, that all vectors, and therefore all quantities, of the V_m can be considered as quantities of the V_n. On this property depends the mentioned possibility to discard the y totally.

We say that a vector v^ν, defined with respect to the V_n, *lies in the* V_m when $v^\nu = B^\nu_\lambda v^\lambda$. The geometrical meaning is clear: the direction of v^ν is tangent to V_m. It is obvious that a vector, not lying within the V_m, has no components with Latin indices. It is of course possible to form $B^c_\mu v^\mu$, but these are the components of the projection of v on V_m, not of v itself. In the same way we see that a quantity $P_{\lambda\mu\nu}$ has only then components $P_{ab\nu}$, if it "lies in the V_m with the indices λ and μ", that is to say, if $B^{\alpha\beta}_{\lambda\mu} P_{\alpha\beta\nu} = P_{\lambda\mu\nu}$.

A vectorfield v_λ, defined in V_n, has a covariant derivative

$$\nabla_\mu v_\lambda = \frac{\partial v_\lambda}{\partial x^\mu} - \Gamma^\nu_{\lambda\mu} v_\nu \quad ; \quad \Gamma^\nu_{\lambda\mu} = \left\{ \begin{matrix} \lambda\,\mu \\ \nu \end{matrix} \right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$\left\{ \begin{smallmatrix} \lambda\mu \\ \nu \end{smallmatrix} \right\}$ are the CHRISTOFFEL symbols belonging to the $g_{\lambda\mu}$. In the same way a vector field w_a, defined in V_m, has a covariant derivative, whose components with Latin indices are

$$\nabla'_b w_a = \frac{\partial w^a}{\partial y^b} - \Gamma'^c_{ab} w_c \quad ; \quad \Gamma'^c_{ab} = \left\{ \begin{matrix} a\,b \\ c \end{matrix} \right\}' . \quad . \quad . \quad . \quad . \quad (9)$$

and with Greek indices:

$$\nabla'_\mu w_\lambda = B^{ba}_{\lambda\mu} \frac{\partial w_a}{\partial y_b} - B^{ba}_{\mu\lambda} \Gamma'^c_{ab} w_c \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$\left\{ \begin{smallmatrix} ab \\ c \end{smallmatrix} \right\}'$ are the CHRISTOFFEL symbols belonging to the g'_{ab}. For a vector field u_λ of V_n, defined on V_m, the expression $\nabla_\mu u_\lambda$ has no meaning. But the expression

$$B^\alpha_\mu \nabla_\alpha u_\lambda = B^a_\mu B^\alpha_a \nabla_\alpha u_\lambda = B^a_\mu \frac{\partial u_\lambda}{\partial y^a} - B^\alpha_\mu \Gamma^\nu_{\lambda\alpha} u_\nu \quad . \quad . \quad . \quad (11)$$

has certainly a meaning, and represents another kind of covariant derivative. It can easily be proved, that for the case of u_λ lying in V_m:

$$\nabla'_b u_a = B^{\mu\lambda}_{ba} \nabla_\mu u_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In the same way we can build different kinds of derivatives for quantities of higher order of the V_n, defined on the V_m. One of the most frequently occurring quantities is $B^{\beta\alpha}_{\omega\mu} \nabla_\beta v_{\alpha\lambda}$, where $v_{\mu\lambda}$ is a field with index μ within V_m. For the *baλ*-component of this derivative we easily find

$$B^{\beta\alpha}_{ba} \nabla_\beta v_{\alpha\lambda} = \frac{\partial v_{a\lambda}}{\partial y_b} - \Gamma'^c_{ab} v_{c\lambda} - B^\mu_b \Gamma^\nu_{\lambda\mu} v_{a\nu} \,^{1)} \quad . \quad . \quad . \quad . \quad (13)$$

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[1]) The factors B could be avoided in most cases by the introduction of new differentiation symbols for the different derivatives, e.g. $\overset{1}{\nabla}$, $\overset{2}{\nabla}$. If however in this way we want to come to a systematic notation applyable to all cases, the sign ∇ must indicate in which way the factors B affect the indices. This makes the notation less clear and more complicated than the notation used here as well as in Chapter III and IV of "Der Ricci Kalkül", SPRINGER 1924.

Together with the fundamental tensor the following fundamental quantities are the most important [1]).

1st *The curvature affinor.*

$$H_{\lambda\mu}^{\cdot\cdot\nu} = B_{\lambda\mu}^{\alpha\beta} \nabla_\alpha B_\beta^\nu \quad \ldots \ldots \ldots \quad (14)$$

This quantity lies with its first two indices in the V_m. Hence it has also components with two Latin indices:

$$H_{ab}^{\cdot\cdot\nu} = B_{ab}^{\lambda\mu} H_{\lambda\mu}^{\cdot\cdot\nu} = B_{ab}^{\alpha\beta} \nabla_\alpha B_\beta^\nu \quad \ldots \ldots \quad (15)$$

The vanishing of $H_{ab}^{\cdot\cdot\nu}$ is necessary and sufficient for V_m being geodesic. For $m = n - 1$ $H_{ab}^{\cdot\cdot\nu}$ passes into $-h_{ab} n^\nu$, where h_{ab} is the second fundamental tensor and n^ν the unit vector normal to V_{n-1}.

2nd *The mean curvature vector.*

$$D^\nu = \frac{1}{m} g'^{ab} H_{ab}^{\cdot\cdot\nu} \quad \ldots \ldots \ldots \quad (16)$$

Its vanishing is necessary and sufficient for V_m being a minimal manifold. For $m = n - 1$ we have $-hn^\nu = -h_{\cdot a}^{a} n^\nu$ in stead of mD^ν.

§ 2. *The fundamental equations.*

Under a deformation εv^ν these quantities are changed in the following way:

I. $\delta g'_{\lambda\mu} = 2\,\varepsilon\, B_{(\lambda}^{\alpha} C_{\mu)}^{\beta} \nabla_\alpha v_\beta$ II. $dg'_{ab} = 2\,\varepsilon\, B_{(ab)}^{\alpha\beta} \nabla_\alpha v_\beta$

$$\text{III.}\quad \delta H_{\lambda\mu}^{\cdot\cdot\nu} = 2\,\varepsilon\, H_{\cdot(\lambda}^{\alpha\cdot\nu} C_{\mu)}^{\beta} \nabla_\alpha v_\beta - 2\,\varepsilon\, H_{\cdot(\lambda}^{\beta\cdot\nu} B_{\mu)}^{\alpha} \nabla_\alpha v_\beta - \varepsilon\, H_{\lambda\mu}^{\cdot\cdot\beta} g'^{\nu\alpha} \nabla_\alpha v^\beta -$$
$$- \varepsilon\, B_{\lambda\mu}^{\alpha\beta} C_\delta^\nu K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\delta} v^\gamma + \varepsilon\, C_\delta^\nu B_{\lambda\mu}^{\alpha\beta} \nabla_\alpha B_\beta^\gamma \nabla_\gamma v^\delta .$$

$$\text{IV.}\quad dH_{ab}^{\cdot\cdot\nu} = -\,\varepsilon\, H_{ab}^{\cdot\cdot\beta} g'^{\nu\alpha} \nabla_\alpha v_\beta - \varepsilon\, B_{ab}^{\alpha\beta} C_\delta^\nu K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\delta} v^\gamma +$$
$$+ \varepsilon\, C_\alpha^\nu B_{ab}^{\beta\gamma} \nabla_\beta B_\gamma^\delta \nabla_\delta v^\alpha - \varepsilon\, H_{ab}^{\cdot\cdot\alpha} \Gamma_{\alpha\beta}^{\nu} v^\beta .$$

$$\text{V.}\quad \delta D^\nu = -\,\varepsilon\, D^\beta g'^{\nu\alpha} \nabla_\alpha v_\beta - \frac{1}{m}\,\varepsilon\, C_\delta^\nu g'^{\alpha\beta} K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\delta} v^\gamma +$$
$$+ \frac{1}{m}\,\varepsilon\, C_\alpha^\nu g'^{\beta\gamma} \nabla_\beta B_\gamma^\delta \nabla_\delta v^\alpha - \frac{1}{m}\,\varepsilon\, H^{\alpha\beta\nu} \nabla_\alpha v_\beta .$$

$$\text{VI.}\quad dK'_{abcd} = -\,4\,\varepsilon\, B_{[a[c}^{\alpha\beta} H_{b]d]}^{\cdot\cdot\delta} K_{\gamma\alpha\beta\delta} v^\gamma + 4\,\varepsilon\, H_{[a[c}^{\cdot\cdot\alpha} B_{b]d]}^{\beta\delta} \nabla_\beta B_\delta^\gamma \nabla_\gamma v_\alpha$$
$$+ \varepsilon\, B_{abcd}^{\omega\mu\lambda\nu} \{K_{\alpha\mu\lambda\nu} \nabla_\omega v^\alpha + K_{\omega\alpha\lambda\nu} \nabla_\mu v^\alpha + K_{\omega\mu\alpha\nu} \nabla_\lambda v^\alpha + K_{\omega\mu\lambda\alpha} \nabla_\nu v^\alpha \}$$
$$+ \varepsilon\, v^\varepsilon B_{abcd}^{\alpha\beta\gamma\delta} \nabla_\varepsilon K_{\alpha\beta\gamma\delta}.$$

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[1]) Compare e.g. Chapter V of "Der Ricci Kalkül" (further on referred to as R.K.).

$$\text{VII. } dK'_{bc} = \varepsilon\,(-B^{\beta}_{c} H_{b}^{\cdot\,\alpha\delta} + m\,B^{\alpha\beta}_{bc} D^{\delta} - B^{\alpha}_{b} H_{c}^{\cdot\,\beta\delta} + g'^{\alpha\beta} H_{bc}^{\cdot\cdot\,\delta})\,v^{\gamma} K_{\gamma\alpha\beta\delta} +$$
$$+ 4\,\varepsilon\,g'^{ad} H_{[a[c}^{\cdot\,\cdot\,\alpha} B^{\beta\delta}_{b]d]} \nabla_{\beta} B^{\gamma}_{\delta} \nabla_{\gamma} v_{\alpha}$$
$$+ 2\,\varepsilon\,B^{\mu\lambda}_{bc} g'^{\omega\nu} K_{\alpha(\lambda\mu)\nu} \nabla_{\omega} v^{\alpha} + 2\,\varepsilon\,B^{\mu\lambda}_{bc} g'^{\omega\nu} K_{\alpha\omega\nu(\lambda} \nabla_{\mu)} v^{\alpha} -$$
$$-2\,\varepsilon\,K'_{abcd}\, g'^{a\lambda} g'^{d\mu} \nabla_{(\mu} v_{\lambda)} + \varepsilon\,v^{\varepsilon} g'^{\alpha\delta} B^{\beta\gamma}_{bc} \nabla_{\varepsilon} K_{\alpha\beta\gamma\delta}.$$

$$\text{VIII. } dK' = -2\,\varepsilon\,H^{\beta\alpha\delta} v^{\gamma} K_{\gamma\alpha\beta\delta} + 2\,m\,\varepsilon\,g'^{\alpha\beta} D^{\delta} v^{\gamma} K_{\gamma\alpha\beta\delta} +$$
$$+ 2\,\varepsilon\,H^{\beta\gamma\alpha} \nabla_{\beta} B^{\delta}_{\gamma} \nabla_{\delta} v_{\alpha} - 2\,m\,\varepsilon\,D^{\alpha} g'^{\beta\gamma} \nabla_{\beta} B^{\delta}_{\gamma} \nabla_{\delta} v_{\alpha}$$
$$+ 4\,\varepsilon\,K_{\alpha\mu\lambda\nu}\, g'^{\omega\nu} g'^{\mu\lambda} \nabla_{\omega} v^{\alpha} - 4\,\varepsilon\,K'^{\alpha\beta} \nabla_{\alpha} v_{\beta} + \varepsilon\,v^{\varepsilon} g'^{\alpha\delta} g'^{\beta\gamma} \nabla_{\varepsilon} K_{\alpha\beta\gamma\delta}.$$

We obtain (I) starting from (1) and (2). From (I) equations (II) are deduced. For $m = n$ (II) passes into the well known equation for the variation of the fundamental tensor of the V_n under an infinitesimal transformation [1]). We obtain (III) and (IV) from (I) and (14); and (V) from (11). (VI—VIII) are deduced from (IV) and GAUSS' equation. For $m = n$ the quantities $H_{\mu\lambda}^{\cdot\cdot\,\nu}$ and D^{ν} vanish, and (VI) passes into the equation expressing the change of the curvature quantity under an infinitesimal transformation [2]). For $m = n - 1$ we have in stead of III, IV and V:

$$\text{III}'.\ \delta h_{\lambda\mu} = +2\,\varepsilon\,h^{\alpha}_{\cdot\,(\mu} C^{\beta}_{\lambda)} \nabla_{\alpha} v_{\beta} - 2\,\varepsilon\,h^{\beta}_{\cdot\,(\mu} B^{\alpha}_{\lambda)} \nabla_{\alpha} v_{\beta} +$$
$$+ \varepsilon\,B^{\alpha\beta}_{\lambda\mu} K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta} n_{\delta} v^{\gamma} - \varepsilon\,n_{\alpha} B^{\beta\gamma}_{\lambda\mu} \nabla_{\beta} B^{\delta}_{\gamma} \nabla_{\delta} v^{\alpha}.$$

$$\text{IV}'.\ dh_{ab} = \varepsilon\,B^{\alpha\beta}_{ab} K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta} n_{\delta} v^{\gamma} - \varepsilon\,n_{\alpha} B^{\beta\delta}_{ab} \nabla_{\beta} B^{\gamma}_{\delta} \nabla_{\gamma} v^{\alpha}.$$

$$\text{V}'.\ dh_{a}^{\cdot\,a} = \varepsilon\,K_{\alpha\beta} n^{\alpha} v^{\beta} - \varepsilon\,n_{\alpha} g'^{\beta\gamma} \nabla_{\beta} B^{\delta}_{\gamma} \nabla_{\delta} v^{\alpha} - 2\,\varepsilon\,h^{\alpha\beta} \nabla_{\alpha} v_{\beta}.$$

If we decompose in this case v^{ν} into a component w^{ν} in the V_{n-1} and another, ψn^{ν}, normal to the V_{n-1} (n^{ν} being unit vector) we find

$$B^{\alpha}_{\mu} \nabla_{\alpha} v_{\lambda} = \nabla'_{\mu} w_{\lambda} + \psi\,h_{\mu\lambda} - h^{\alpha}_{\mu} w_{\alpha} n_{\lambda} + n_{\lambda} \nabla'_{\mu} \psi \quad . \quad . \quad (17)$$

$$n^{\alpha} B^{\beta\gamma}_{\omega\mu} \nabla_{\beta} B^{\delta}_{\gamma} \nabla_{\delta} v_{\alpha} = -h_{\omega}^{\cdot\,\alpha} \nabla'_{\mu} w_{\alpha} - \nabla'_{\omega} h^{\alpha}_{\mu} w_{\alpha} - \psi\,h_{\omega}^{\cdot\,\alpha} h_{\mu\alpha} + \nabla'_{\omega} \nabla'_{\mu} \psi. \quad (18)$$

The equation of KILLING $\nabla_{(\mu} v_{\lambda)} = 0$ [3]) is characteristic for the rigid motions in V_n. It can indeed be shown without difficulty that in this case all diffferentials vanish.

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[1]) R.K., p. 209.
[2]) R.K., p. 208.
[3]) R.K., p. 212.

§ 3. *Geodesic V_m.*

Necessary and sufficient condition that a geodesic V_m remains geodesic under a deformation εv^ν, is, after (IV), that

$$C^\nu_\alpha B^{\beta\gamma}_{ab} \nabla_\beta B^\delta_\gamma \nabla_\delta v^\alpha - B^{\alpha\beta}_{ab} C^\nu_\delta K_{\gamma\alpha\beta}{}^{\delta} v^\gamma = 0 \quad . \quad . \quad . \quad , \quad (19)$$

For $m = n - 1$ this equation passes into

$$n_\alpha B^{\beta\gamma}_{ab} \nabla_\beta B^\delta_\gamma \nabla_\delta v^\alpha - B^{\alpha\beta}_{ab} K_{\alpha\gamma\beta\delta} v^\gamma n^\delta = 0 \quad . \quad . \quad . \quad . \quad (20)$$

and for $m = 1$ into:

$$\frac{\delta^2}{ds^2} v^\delta - i^\alpha i^\beta K_{\gamma\alpha\beta}{}^{\delta} v^\gamma = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

the equation of LEVI-CIVITA.

§ 4. *Minimal-V_m.*

Necessary and sufficient condition that the minimal property is not changed is

$$C^\nu_\alpha g'^{\beta\gamma} \nabla_\beta B^\delta_\gamma \nabla_\delta v^\alpha - C^\nu_\delta g'^{\alpha\beta} K_{\gamma\alpha\beta}{}^{\delta} v^\gamma - H^{\alpha\beta\nu} \nabla_\alpha v_\beta = 0. \quad . \quad (22)$$

For $m = 2$ and $v^\nu \perp V_2$ this equation is equivalent to CARTAN's equation.

For $m = n - 1$ equation (22) passes into

$$n^\alpha g'^{\beta\gamma} \nabla_\beta B^\delta_\gamma \nabla_\delta v_\alpha - K_{\alpha\beta} n^\alpha v^\beta + h^{\alpha\beta} \nabla_\alpha v_\beta = 0 \quad . \quad . \quad . \quad (23)$$

If we take $v^\nu = \psi n^\nu$ and the unit vector $n^\nu \perp V_{n-1}$, we get

$$\nabla'^a \nabla'_a \psi - \psi K_{\alpha\beta} n^\alpha n^\beta + \psi h^{\alpha\beta} h_{\alpha\beta} = 0 \quad . \quad . \quad . \quad . \quad (24)$$

and, if in this case $V_n = R_3$, $m = 2$, we obtain the equation of SCHWARZ

$$\nabla'^a \nabla'_a \psi - 2 K'_0 \psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

in wich K'_0 is the curvature of the V_2.

§ 5. *Bending.*

A V_m is bended when its metric is not changed under the deformation. Hence the necessary and sufficient condition is $dg'^{ab} = 0$, or, with respect to (II)

$$B^{\alpha\beta}_{ab} \nabla_{(\alpha} v_{\beta)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

For $m = n - 1$ we get from (17)

$$\nabla'_{(a} w_{b)} = -\psi h_{ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If $V_n = R_n$ and the rank of h_{ab} is larger than 1, we can obtain from this differential equation an equation of the second order with $\nabla_{[a} w_{b]}$ as dependent variable, which does no longer contain the function ψ. If we write $\nabla'_{[a} w_{b]} = f_{ab}$, the integrability conditions of (27) are:

$$\tfrac{1}{2} K'_{cab}{}^{d} w_d - \nabla'_{[c} f_{a]b} = -(\nabla'_{[c} \psi) h_{a]b}, \quad . \quad . \quad . \quad . \quad (28)$$

For the deduction of the right member we have used CODAZZI's equation

$$\nabla'_{[c} h_{a]b} = 0 \quad \ldots \ldots \ldots \quad (29)$$

From (28) we obtain, making use of GAUSS' equation:

$$h_{b[c} h_{a]}^{\cdot d} w_d - \nabla'_{[c} f_{a]b} = -(\nabla'_{[c} \psi) h_{a]b} \quad \ldots \ldots \quad (30)$$

Transvection with h^{ab} gives

$$p_c^{\cdot a} h_a^{\cdot d} w_d + h^{ab} \nabla'_a f_{cb} = p_c^{\cdot a} \nabla'_a \psi \quad \ldots \ldots \quad (31)$$

in which we have used the abbreviation

$$h^{ab} h_{cb} - h^{db} h_{db} A_c^a = p_c^{\cdot a} \quad \ldots \ldots \ldots \quad (32)$$

The rank of p_{ab} is always n when the rank of h_{ab} is not equal to 1. Hence there exists an inverse tensor P_{ab} of p_{ab} and the introduction of P_{ab} into equation (31) gives the simpler formula

$$h_c^{\cdot a} w_a + P_c^{\cdot a} h^{db} \nabla'_d f_{ab} = \nabla'_c \psi \quad \ldots \ldots \quad (33)$$

from which we obtain by differentiation and alternation

$$\boxed{h_{[c}^{\cdot a} f_{d]a} = \nabla'_{[c} u_{d]}} \quad \ldots \ldots \ldots \quad (A)$$

in which we have used the abbreviation:

$$P_c^{\cdot a} h^{db} \nabla'_d f_{ab} = u_c \quad \ldots \ldots \ldots \quad (34)$$

If (33) is substituted into (17), we get, making use of (27) and (34)

$$B_\mu^\alpha \nabla_\alpha v_\lambda = f_{\mu\lambda} + u_\mu n_\lambda \quad \ldots \ldots \quad (35)$$

The integrability conditions of this equation are, so far as they are not a consequence of (A):

$$\boxed{\nabla'_a f_{bc} = 2\, h_{a[b} u_{c]}} \quad \ldots \ldots \ldots \quad (B)$$

It can easily be shown that the integrability conditions of (A) are a consequence of (B). Those of (B) are

$$\boxed{h_{[a[c} h_{b]}^{\cdot \alpha} f_{d]\alpha} = h_{[a[c} \nabla'_{b]} u_{d]}} \quad \ldots \ldots \quad (C)$$

and the integrability conditions of this equation are a consequence of (B). Hence the system (A), (B), (C) is complete [1]). If we compute for this case the right side of equation (VI), we see that (C) expresses the fact that $dK'_{abcd} = 0$.

Now we will first investigate under which conditions the bending is improper, that is to say, is only a pure motion of the R_n. Necessary and sufficient condition for this is that we can find a vector field in the R_n being equal to v_λ in every point of the V_m and being choosen in other points in such a way, that

$$\nabla_\mu v_\lambda = \nabla_{[\mu} v_{\lambda]} = F_{\mu\lambda} \quad \ldots \ldots \ldots \quad (36)$$

[1]) The equations (A), (B) and (C) are equivalent with a system deduced by SBRANA: Sulla deformazione infinitesima delle ipersuperficie, Ann. di Mat. (3) **15** ('08) 329—348.

is a constant bivector in R_n; that is to say that there exists in V_m a vector field p_λ, such that the bivector

$$F_{\mu\lambda} = f_{\mu\lambda} + 2 p_{[\mu} i_{\lambda]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

is constant in R_n. This is however then and only then the case, when the system

$$\boxed{h_c^{\cdot a} f_{da} = \nabla_c p_d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A_0)$$

$$\boxed{\nabla'_a f_{bc} = 2 h_{a[b} p_{c]}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_0)$$

admits a solution. It can be shown without difficulty that (A_0) and (B_0) also form a complete system, the equation corresponding tot (C) being here a consequence of (A_0). If a solution of (A_0, B_0) is found, we have for the corresponding motion

$$B^\alpha_\mu \nabla_\alpha v_\lambda = f_{\mu\lambda} + p_\mu i_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Hence a solution of (A, B, C) is then and only then not a proper bending, if this solution also satisfies (A_0) for $u = p$.

Now the following theorem holds and can be proved easily by writing out the components with respect to the principal directions of h_{ab}:

Given the equation $h_{[a[c} k_{b]d]} = 0$, *in which* h_{ab} *is real and symmetrical and* k_{ab} *arbitrary. Then if* h_{ab} *has the rank 2,* k_{ab} *lies totally in the* R_2 *of* h_{ab}, *and if* h_{ab} *has a rank* > 2, k_{ab} *vanishes.*

Hence we deduce from (C) that a V_m in R_n admits only then proper bendings, if the rank of h_{ab} is 2 or less, a well-known property, first published by KILLING [1]). If the rank of h_{ab} is 2, we have the only case that the ∞^{n-3} directions of h_{ab} form, at each point, a *plane* R_{n-3} lying totally in the V_{n-1} and with the same tangent-R_{n-1} at each point. This was proved bij BOMPIANI [2]). Hence the V_{n-1} is thus built up by ∞^2 of such R_{n-3}. According to (B) $f_{\lambda\mu}$ is constant in each of these R_{n-3}. If the above mentioned theorem is applied to (C), we find that $h_b^{\cdot\alpha} f_{d\alpha} - \nabla'_b n_d$ lies totally in the R_2 of $h_{\lambda\mu}$; and this shows that also u_a in each of the R_{n-3} of V_{n-1} is constant. We have besides, from (IV′) and (35):

$$d\, h_{ab} = - h_a^{\cdot\alpha} f_{b\alpha} + \nabla'_a u_b \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Hence if y^1 and y^2 are chosen in such a way that the R_{n-3} become

[1]) Die nichteuklidischen Raumformen in analytischer Behandlung, Leipz., 1885, p. 236 a.f.

[2]) Forma geometrica delle condizione per la deformabilità delle ipersuperficie, Rend. Acc. Lincei (5) **23**. 1 (14) 126—131. The first part is an immediate consequence of CODAZZI's equation, if written in orthogonal components with respect to the principal direction of h_{ab}, the second part follows from the geometrical meaning of $B^\alpha_\mu \nabla_\alpha n_\lambda$. Comp. STRUIK, Grundzüge der mehrdimensionalen Differentialgeometrie, SPRINGER 1922, p. 140. CARTAN, La déformation des hypersurfaces dans l'espace euclidéen réel à n dimensions, Bull. Soc. Math. de France **44** (16) 65—99, has a complete classification of all possible cases where a V_{n-1} is bended in a R_n.

the intersections of the systems of V_{n-2} belonging to y^1 and y^2, we see from this last equation that dh_{ab} always vanishes except for $n \leqq 2$ and $b \leqq 2$ simultaniously. Hence the plane R_{n-3} remain plane under bending. [1]).

Starting with a definite solution v_λ of (A, B, C) we can always obtain that $f_{\mu\lambda}$ and u_λ vanish at some point P, without essential change of the solution. We have only to determine the corresponding proper motion v'^λ, giving at P the same value for $f_{\mu\lambda}$ and u_λ. Then $v_\lambda - v'_\lambda$ is the desired solution not differing essentially from v_λ. This will be called the *reduction* of the bending with respect to P.

Let us give besides a short treatment of the case $m = 2$, in which case $f_{\mu\lambda}$ passes into $\varphi I_{\mu\lambda}$; $I_{\mu\lambda}$ being the unit bivector of the V_2. The function φ is WEINGARTEN's "Verschiebungsfunktion" [2]). Then equations (A, B, C) become:

$$\boxed{\varphi\, h_{[c}^{\cdot\, a}\, I_{d]\,a} = \nabla'_{[c}\, u_{d]}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (A_1)$$

$$\boxed{\nabla'_a\, \varphi = {}^1/_2\, h_{ab}\, I^{bc}\, u_c} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (B_1)$$

$$\boxed{0 = h_{[a\,[c}\, \nabla'_{b]}\, u_{d]}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (C_1)$$

From (A_1) and (B_1) we obtain:

$$\nabla'_a\, H^{ab} \nabla'_b\, \varphi = -{}^1/_8\, \varphi\, h^{a}_{\cdot\, a} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (40)$$

in which H^{ab} is reciprocal to h_{ab}.

This equation comes in stead of (A) and is equivalent to the characteristic equation of WEINGARTEN [3]). It can easily be shown that every value of φ deduced from a solution of the characteristic equation satisfies (C_1) identically.

A remarkable case of proper bending is that in which the $(n-1)$-direction of each element of the V_{n-1} remains unchanged. The necessary and sufficient condition for this is that not only dg'_{ab}, but also $\delta g'_{\lambda\mu}$ vanishes. This occurs then and only then if, as we see from (I) and (II) $B_\mu^\alpha \nabla_\alpha v_\lambda$ lies totally in V_{n-1} and is at the same time alternating. Then the equations (A, B, C) pass into

$$\boxed{h_{[c}^{\cdot\, a}\, f_{d]\,a} = 0} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (A_2)$$

$$\boxed{\nabla'_a\, f_{bc} = 0} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (B_2)$$

$$\boxed{h_{[a\,[c}\, h_{b]}^{\cdot\,\alpha}\, f_{d]\,\alpha} = 0} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (C_2)$$

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[1]) BOMPIANI, Forma geometrica delle condizioni per la deformabilità delle superficie, Rend. Linc. **33** (14) 126—131.

[2]) Comp. e.g. BIANCHI-LUKAT, Vorlesungen über Differentialgeometrie, Leipzig, 1899, p. 289 a.f.

[3]) E.g. BIANCHI-LUKAT, l.c. p. 292, equation (7*).

It follows from (C_2) that, except for a scalar factor, f_{ab} is equal to the unit bivector in the R_2 of h_{ab}. It follows from (B_2), that this scalar factor is a constant and that the R_2 of h_{ab} is geodesically parallel (in V_{n-1}) at all points of V_{n-1}. If (A_2) is written out in orthogonal components with respect to the principal axes of h_{ab}, it appears that this equation is then and only then satisfied if the V_{n-1} is minimal. If we reduce the bending with respect to some point P, we see that there is essentially only one solution. Hence we have obtained the theorem, obtained by DARBOUX for the case of a V_2 in R_3 [1]):

Necessary and sufficient condition that a V_{n-1} in R_n, for which h_{ab} has the rank 2, can be subjected to a proper infinitesimal bending, with preservation of the (n—1)-direction of each element, is, that the V_{n-1} be a minimal-V_{n-1} and that the R_2 of h_{ab} be geodesically parallel in V_{n-1} at all points of V_{n-1}. If one such a bending is given, then any other can be obtained from it by the adjunction of a proper motion.

§ 7. *Infinitesimal deformations normal to V_{n-1} that keep the principal directions of h_{ab} invariant.*

Suppose $v^\nu \perp V_{n-1}$. Then we have from (IV′) and (18):

$$d\, h_{ab} = \varepsilon\, \psi\, B_{ab}^{\alpha_{|}\beta}\, K_{\gamma\alpha_{|}\beta}^{\cdot\;\cdot\;\cdot\;\delta}\, n_\delta\, n^\gamma + \varepsilon\, \psi\, h_a^{\cdot c}\, h_{bc} - \varepsilon \nabla'_a \nabla'_b\, \psi \quad . \quad . \qquad (41)$$

The tensor $h_a^{\cdot c}\, h_{bc}$ has the same principal directions as h_{ab}. Hence the necessary and sufficient condition that the principal directions of h_{ab} remain invariant, is:

$$\underset{a}{i^\alpha}\, \underset{b}{i^{\beta}}\, (\nabla'_\alpha \nabla'_{|\beta}\, \psi - \psi\, K_{\gamma\alpha_{|}\beta}^{\cdot\;\cdot\;\cdot\;\delta}\, n_\delta\, n^\gamma) = 0; \quad a, b = 1, \ldots, n-1, \; a \neq b, \quad (42)$$

in which the i are unit vectors in the principal directions of h_{ab}. Such a transformation exists when we pass from one of the V_{n-1} of an n-uple orthogonal system to a neighbouring V_{n-1}. It can be shown indeed, that in this case one of the conditions for the existence of such a system is given by equation (42) [2]).

§ 8. *Infinitesimal transformations that keep invariant the the m-dimensional volume.*

The volume of the parallepiped with sides $dy^1, dy^2, \ldots, dy^m$ is, according to a well-known formula:

$$d\tau = dy^1 \ldots dy^m \sqrt{g'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

[1]) Leçons sur la théorie générale des surfaces, Paris, 1914, I. p. 383.

[2]) For literature compare SCHOUTEN and STRUIK, On n-uple orthogonal systems of V_{n-1} in V_n, These Proceedings 22, (1919), p. 594–605, 680—695.

A transformation is equivoluminar, when $d\tau$ remains invariant, or in consequence of (43) when $d\sqrt{g'}=0$. But

$$d\sqrt{g'} = {}^1/_2\, g'^{-1/_2}\, dg' = {}^1/_2 \sqrt{g'}\, g'^{ab}\, dg'_{ab} = \varepsilon \sqrt{g'} g'^{\mu\lambda} \nabla_\mu v_\lambda \quad . \quad (44)$$

This formula passes for $m = n - 1$ into

$$d\sqrt{g'} = \varepsilon\, \psi \sqrt{g'}\, (\nabla'^{a}\, w_a + \psi\, h^{a}_{\cdot a}) \quad . \quad . \quad . \quad . \quad . \quad (45)$$

on account of (17). This shows that an infinitesimal transformation of a V_{n-1} in V_n perpendicular to this V_{n-1} is then and only then equivoluminar if the V_{n-1} is minimal. This theorem is due to BOMPIANI.[1] [2]).

[1]) Studi sugli spazi curvi, Atti del R.I. Veneto **80.** 2 (20/21) 1113—1145, p. 1141.

[2]) In a recent dissertation at the Massachusetts Institute of Technology, with title "Infinitesimal Deformation of Surfaces in Riemannian Space", W. F. Cheney investigates the bending of V_2 in V_n, especially for the cases $n = 3$, $n = 4$. In these cases he comes to equations, which for the case $V_n = R_n$ correspond to the equations (A_1, B_1) of our paper. An abstract of this dissertation is published in "Abstracts of Publications of the Massachusetts Institute of Technology", Vol. I (1928).