Mathematics. — *A Group of Null Systems.* By Prof. JAN DE VRIES.

(Communicated at the meeting of Januari 28, 1928).

§ 1. In a null system (α, β, γ) a point has α nullplanes, a plane has β nullpoints and a line is γ times a nullray. In a paper "On Bilinear Null Systems" (These Proceedings **15**, 879) I have proved the existence of nullsystems $(1, 1, \gamma)$ for any value of γ and I have derived their properties. In a paper "Null Systems that are Defined by Two Linear Congruences of Rays" (These Proceedings **21**, 309) I have considered nullsystems $(1, pq, p+q)$ that are defined by two congruences of rays $[1, p]$ and $[1, q]$. Now I shall consider a group of nullsystems that are characterised by the symbol $(1, 2n, n+2)$.

§ 2. We shall consider as given the congruence $[k^3]$ of twisted cubics that pass through 5 points B_k and the congruence of rays $[1, n]$ formed by the lines t resting on the twisted curve α^n and on the line a which cuts α^n in $(n-1)$ points A_k.

Through a point M there pass one k^3 and one ray t; let r be the straight line that touches k^3 at M; we shall associate the plane rt to M as *nullplane* μ.

The points of contact R of the k^3 that touch a plane μ, lie on a conic ϱ [1]); its points of intersection with the n rays t in μ are the *nullpoints* of μ. Hence $\beta = 2n$.

The rays t that cut a line l, form a scroll of the degree $(n+1)$: the points of contact R of the tangents r that rest on l, lie on a cubic surface through l. Besides l the two surfaces have a curve of the order $(3n+2)$ in common that is formed by the nullpoints M of the planes μ through l. This curve cuts l in $(n+2)$ points M; hence $\gamma = n+2$ [2]).

§ 3. A point M for which r coincides with t, has a pencil of nullplanes and is, therefore, *singular* for the nullsystem. I shall indicate such a point by S; let the axis of the pencil (σ) of the nullplanes be indicated by s. The lines s form a scroll; this is the intersection of the congruence $[1, n]$ and the complex of the tangents r to the curves k^3.

[1]) The plane $B_1B_2B_3$ contains a pencil (k^2) of conics each of which forms a composite k^3 together with B_4B_5. Two of these k^2 touch the plane μ.

[2]) For $n=1$ we find a null system (1, 2, 3); I have treated its properties in these Proceedings **26**, 124.

As this is a complex of the degree *six* [1], the lines s form a scroll of the degree $6(n+1)$. The *curve* λ^{3n+2} (§ 2) corresponding to l has, therefore, $6(n+1)$ points S in common with the *curve* (S) of the singular points.

In order to determine the order of (S) we shall consider the congruence $[r]$ of the tangents r that have their points of contact R in a plane φ. The points R of which the lines r meet in a point P, lie on a twisted curve of the order 7. A plane π contains two lines r; their points of contact are the points R in φ of the conics ϱ^2 in π (§ 2). The congruence $[7,2]$ of the lines r has $(2n+7)$ rays s in common with the $[1,n]$ of the rays t; accordingly the locus of the *singular points* is a curve of the *order* $(2n+7)$.

It contains the 5 base points B_k and the 10 points D each of which is the intersection of a plane $B_l B_m B_n$ and the line $B_p B_q$. Through each of these 15 points there passes one k^3 that touches a ray t. The plane $B_1 B_2 B_3$ contains four points D; they may be indicated by (12,345), (13,245), (23,145), and (45,123). Besides these four points and the three base points this plane contains $2n$ points S; they lie in pairs on the n rays t; for each of these is touched by two conics k^2 that are component parts of composite k^3.

§ 4. The nullpoints M of the planes μ that pass through a point P, lie on a surface $(P)^{n+3}$, for any ray through P is a nullray for $(n+2)$ of its points.

The surfaces $(P)^{n+3}$ and $(Q)^{n+3}$ have in common: the curve λ^{3n+2} defined by PQ, the curve $(S)^{2n+7}$, the curve α^n and the line a. Any point of α^n carries a pencil (t), hence a pencil (μ); any point of a is the vertex of a cone $(t)^n$, hence nullpoint of ∞^1 planes μ each of which contains n rays t. Accordingly a is an n-fold line and α^n is a single curve on (P). In fact $(n+3)^2 = (3n+2) + (2n+7) + n + n^2$.

The cubic surface of the conics ϱ^2 in planes through l (§ 2) has $3n$ points in common with α^n; each of these singular points has a nullplane through l. Analogously a contains three singular points that have a nullplane through l. Hence λ^{3n+2} has $3n$ *points* in common with α^n and it contains three *n-fold points* on a.

A surface $(O)^{n+3}$ has in common with λ^{3n+2}: the $2n$ nullpoints of the plane OPQ, the $6(n+1)$ points S that lie on λ (§ 3), the $3n$ points on α^n and the three n-fold points of λ. In fact $(n+3)(3n+2) = 2n + 6(n+1) + 3n + 3n^2$.

The 10 planes $B_k\ B_l\ B_m$ are *singular nullplanes*; their nullpoints lie on the n lines t in that plane; these are *singular nullrays*. Also the 10 lines $B_k\ B_l$ are *singular nullrays*, for they may be considered as lines r.

[1]) Each of the rays through a point P is cut in two points Q by a k^3. The locus (Q) is a surface of the 4th degree with conical point P. Any plane through P contains 6 tangents of (Q) that meet in P. The locus of the points of contact is a twisted curve of the order 7.

§ 5. If we replace the congruence [1,3] that has the curve α^3 and one of its chords a as directrices, by the [1,3] of the *bisecants* of α^3, we find another nullsystem (1, 6, 5) [1]. Analogous considerations lead to a curve $(S)^{13}$ that cuts the curve λ^{11} in 24 points. The curve α^3 is double on the surface $(P)^6$ and λ^{11} has 9 double points on α^3.

[1]) Also the planes of osculation of the curves k^3 form with their points of contact a nullsystem (1, 6,5). Cf. STURM, Die Lehre von den geometrischen Verwandtschaften, IV, 469.