Mathematics. — *On non-holonomic connexions.* By Prof. J. A. SCHOUTEN (Communicated by Prof. JAN DE VRIES).

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Introduction.

Non-holonomic parameters are well known in mechanics. In geometry they were first used by HESSENBERG [1]), by the author [2]), by CARTAN [3]) and by HLAVATY [4]). Using non-holonomic parameters, the equations of the general linear connexion get another form, given by HORAK [5]).

VRANCEANU [6]) has brought something essentially new. He has shown that in a V_n containing a non-V_m-building field of m-directions there exists a connexion for quantities belonging to the local R_m, and that this connexion can be deduced from the connexion of the V_n by the use of the coefficients of rotation of RICCI. HORAK [7]) has independently found this same connexion and in a paper that is to be published in a short time he will give especially mechanical applications.

Now we get a more general point of view starting from an A_n (X_n with a symmetrical linear connexion) containing a non-X_m-building field of m-directions. Then we can show that in the case of a general $(n\text{-}m)$-direction being given in every point (the case of „Einspannung" of WEYL) there is induced a connexion for all quantities belonging to the local E_m. So we get an A_n^m some of whose properties will be studied more in detail. Especially we will consider the properties of curvature and the generalised equations of GAUSS, which have the same form as in the case of an X_m in A_n, and also something will be said on the geodesics in A_n^m and A_n. Finally the "affine geometry" of an X_n^{n-1} in A_n will be treated. The first paragraph contains a short review on non-holonomic parameters in an A_n.

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[1]) Vektorielle Begründung der Differentialgeometrie, Math. Ann. **78** (18) 187—217.

[2]) Die direkte Analysis zur neueren Relativitätstheorie. Verh. Kon. Akad. v. Wet. Amsterdam **12** (18) 6.

[3]) Sur les variétés à connexion affine, Ann. de l'école normale (3) **40** (23) 325—412.

[4]) Sur le déplacement linéaire du point, Věstn. České Akademie (24) XIII 1—8.

[5]) Die Formeln für allgemeine lineare Uebertragung bei Benutzung von nichtholonomen Parametern, Nieuw Archief v. Wisk. **15** (27) 193—201.

[6]) Sur les espaces non holonomes, Comptes Rendus **183** (26) 825—854, Sur le calcul différentiel absolu pour les variétés non holonomes, Comptes Rendus **183** (26) 1083—1085.

[7]) (Czechisch) Sur une généralisation de la notion de variété, Publications de la Faculté des sciences de l'université Hasaryk, Brno.

§ 1. *Non-holonomic parameters in A_n.*

In every point of the A_n we introduce besides the measuring vectors $\overset{\nu}{e}$, $\underset{\lambda}{e}$, $\lambda, \mu, \nu = a_1, \ldots, a_n$, belonging to the variables x^ν, an arbitrary system $\overset{k}{e}$, $\underset{i}{e}$, $i, j, k = 1, \ldots, n$. Indicating the components with respect to the latter system with latin indices we have

$$v^k = v^\mu \overset{k}{e}_\mu \quad ; \quad w_i = w_\mu \underset{i}{e}^\mu \quad ; \quad \underset{i}{e}^k = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases} \quad ; \quad \overset{k}{e}_i = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases} \quad . \quad (1)$$

$$A_\lambda^\nu = \begin{cases} 1, \lambda = \nu \\ 0, \lambda \neq \nu \end{cases} \quad ; \quad A_\lambda^k = A_\lambda^\mu \overset{k}{e}_\mu \quad ; \quad A_i^\nu = A_\mu^\nu \underset{i}{e}^\mu \quad ; \quad A_i^k = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases} . \quad (2)$$

and

$$v^k = A_\mu^k v^\mu \quad ; \quad w_i = A_i^\mu w_\mu \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

In the expression

$$(dx)^k = A_\mu^k dx^\mu \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

the x^k have a signification by themselves if and only if

$$\partial_{[\omega} A_{\mu]}^k = 0 \quad ; \quad \left(\partial_\omega = \frac{\partial}{\partial x^\omega}\right) \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

In the other case the x^k play the same part as the non-holonomic parameters in mechanics, only their differentials having a signification. We consider just this latter case. A symmetrical linear connexion being given by the parameters $\Gamma_{\lambda\mu}^\nu$ with respect to the system of measuring vectors $\left(\begin{smallmatrix} \nu \\ \lambda \end{smallmatrix}\right)$ we can fix this connexion also by parameters Λ_{ij}^k with respect to the system $\left(\begin{smallmatrix} k \\ i \end{smallmatrix}\right)$, so that

$$\begin{aligned} \nabla_j v^k &= A_{j\gamma}^{\beta k} \nabla_\beta v^\gamma = \partial_j v^k + \Lambda_{ij}^k v^i \\ \nabla_j w_i &= A_{ji}^{\beta\alpha} \nabla_\beta w_\alpha = \partial_j w_i - \Lambda_{ij}^k w_k \end{aligned} \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

writing ∂_i for $A_i^\mu \partial_\mu$. It is easily found that

$$\Lambda_{ij}^k = \Gamma_{ij}^k + A_\alpha^k \partial_j A_i^\alpha = \Gamma_{ij}^k - A_i^\alpha \partial_j A_\alpha^k \; . \; . \; . \; . \; . \quad (7)$$

hence

$$\Lambda_{[ij]}^k = - A_{ij}^{\alpha\beta} \partial_{[\beta} A_{\alpha]}^k \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

So $\Lambda_{[ij]}^k = 0$ is necessary and sufficient for the holonomity of the x^k, hence, notwithstanding the symmetry in λ and μ of the $\Gamma_{\lambda\mu}^\nu$, the parameters Λ_{ij}^k with respect to non-holonomic parameters are not symmetrical in i and j. Of course $\Lambda_{[ij]}^k$ is no affinor. Applicating (6) on $\underset{i}{e}^k$ and $\overset{k}{e}_i$ we get

$$\Lambda_{ij}^k = \nabla_j \underset{i}{e}^k = - \nabla_j \overset{k}{e}_i \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

If the A_n passes into a V_n, and if we choose the orthogonal system $i^\nu_i = \overset{i}{i}{}^\nu, \overset{k}{i}_\lambda$, as measuring vectors $e^\nu_i, \overset{k}{e}_\lambda$, then

$$\Lambda^k_{ij} = -\nabla_j \overset{k}{i}_i = +\nabla_j \overset{i}{i}_k = \gamma_{kij} = -\gamma_{ikj} \quad . \quad . \quad . \quad . \quad (10)$$

where the γ_{kij} are the coefficients of rotation of RICCI. We will write $\gamma^k_{ij} = -\gamma^i_{kj}$ instead of $\gamma_{ikj} = -\gamma_{kij}$ to render more conspicuous the signification of these coefficients as parameters of a connexion. Hence in orthogonal components the connexion is given by

$$\nabla_j v^k = \nabla_j v_k = \partial_j v^k + \gamma^k_{ij} v^i = \partial_j v_k - \gamma^i_{kj} v_i \, (= \partial_j v_k + \sum_i \gamma_{ikj} v_i). \quad (11)$$

By differentiating and alternating (6) we get

$$\nabla_{[l} \nabla_{j]} v^k = \{\partial_{[l} \Lambda^k_{|h|j]} + \Lambda^k_{p[l} \Lambda^p_{|h|j]} - \Lambda^p_{[jl]} \Lambda^k_{hp}\} v^h \quad . \quad . \quad . \quad (12)$$

hence for the quantity of curvature follows

$$R_{ljh}{}^{k} = -2\, \partial_{[l} \Lambda^k_{|h|j]} - 2\, \Lambda^k_{p[l} \Lambda^p_{|h|j]} + 2\, \Lambda^p_{[jl]} \Lambda^k_{hp} \quad . \quad . \quad . \quad (13)$$

an equation passing into the ordinary form for $\Lambda^k_{[ij]} = 0$, viz. for the case of holonomic parameters.

§ 2. X^m_n *in* X_n.

In every point of an X_n an m-direction be given. In the simplest case these m-directions are X_m-building, viz. it is possible to link them together in such a way that they build a system of ∞^{n-m} X_m. This simplest case will not be considered here. In general there will be a system of $\infty^{n-M} X_M$, $m < M \leqq n$, in such a manner that the given m-direction lies in every point in the local M-direction. We suppose that $M = n$ and give up all simplifications possible for $M < n$. An X_n equipped in this way with local m-directions will be called an X^m_n. An affinor of the local E_m, defined over X_n, will be called an "affinor of the X^m_n". Hence a field of contravariant vectors of the X^m_n is also a field of contravariant vectors of the X_n, but not vice versa, and a field of covariant vectors of the X_n (represented geometrically by two parallel E_{n-1} in every point) forms a field of covariant vectors of the X^m_n (represented geometrically by two parallel E_{m-1}) by intersection with the local E_m. All this is just the same as for an X_m in X_n.

We now introduce in every point of the X_n an $(n-m)$-direction, having no direction in common with the local m-direction. An X_n equipped in this way will be called *rigged* [1]). The essential difference between covariant vectors of the X_n and of the X^m_n disappears in a rigged X^m_n, because here every covariant vector of the X^m_n is in one to one correspondence with the covariant vector

[1]) WEYLS expression "eingespannt" being untranslatable and the $(n-m)$-direction reminding of a hoisted sail on a ship (the local m-direction), the word "rigged" was suggested.

of the X_n, whose E_{n-1} are determined by the E_{m-1} of the former vector and the $(n-m)$-direction of the rigging. Analytically we express this as follows. Besides the system $\binom{\nu}{\lambda}$ belonging to the x^ν we introduce in every point a system $\binom{k}{i}$, in such a manner that the first m contravariant measuring vectors $\underset{a}{e}^\nu$, $a, b, c, d = 1, \ldots, m$, lie arbitrarily in the local m-direction, and that the other ones $\underset{e}{e}^\nu$, $e, f, g, h = m+1, \ldots, n$, lie also arbitrarily in the local $(n-m)$-direction. With respect to this system the quantities of the X_n^m only have components with indices from 1 to m. Writing

$$B_\lambda^\nu = \overset{a}{e}_\lambda \underset{a}{e}^\nu \quad ; \quad C_\lambda^\nu = A_\lambda^\nu - B_\lambda^\nu = \overset{e}{e}_\lambda \underset{e}{e}^\nu \quad . \quad . \quad . \quad . \quad . \quad (14)$$

we have

$$B_a^c = \begin{cases} 1, & a = c \\ 0, & a \neq c \end{cases} \quad , \quad B_e^f = 0 \quad , \quad B_a^f = 0, \quad B_e^c = 0 \quad . \quad . \quad (15)$$

and the X_n^m-component of a vector of X_n is given by the equation

$$v'^\nu = B_\mu^\nu v^\mu \quad ; \quad w'_\lambda = B_\lambda^\mu w_\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

or, with regard to the system $\binom{k}{i}$:

$$v'^k = B_\mu^k v^\mu \quad ; \quad w'_i = B_i^\mu w_\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If we write $(dy)^c$ for the $\binom{k}{i}$-components of a translation dx^ν lying in X_n^m:

$$dx^\nu = B_a^\nu (dy)^a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

then the y^c are non-holonomic parameters and for B_a^ν follows

$$\boxed{B_a^\nu = \frac{\partial x^\nu}{(\partial y)^a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

§ 3. *The connexion induced in a rigged X_n^m in A_n.*

By introducing the $\Gamma_{\lambda\mu}^\nu$ the X_n becomes an A_n. We are going to prove that the connexion of A_n induces a connexion in an X_n^m in A^n provided that this X_n^m is rigged. This connexion is defined as follows:

The covariant differential quotient of a quantity in X_n^m is the X_n^m-component of the covariant differential quotient in A_n.

Thus, indicating the covariant differential quotient in X_n^m by ∇' we have for vectors:

$$\left.\begin{aligned}\nabla'_\mu v^\nu &= B_{\mu\gamma}^{\beta\nu}\, \partial_\beta\, v^\gamma + B_{\mu\gamma}^{\beta\nu}\, \Gamma_{\lambda\beta}^{\gamma}\, v^\lambda \\ \nabla'_\mu w_\lambda &= B_{\mu\lambda}^{\beta\alpha}\, \partial_\beta w_\alpha - B_{\mu\lambda}^{\beta\alpha}\, \Gamma_{\alpha\beta}^{\nu}\, w_\nu\end{aligned}\right. \quad \ldots\ldots \quad (20)$$

or, with regard to the system $\binom{k}{i}$:

$$\left.\begin{aligned}\nabla'_b\, v^c &= B_{b\nu}^{\mu c}\, \nabla_\mu\, v^\nu = \partial_b\, v^c - B_a^\nu\, v^a\, \partial_b\, B_\nu^c + v^a\, B_{b\nu a}^{\mu c \lambda}\, \Gamma_{\lambda\mu}^{\nu} \\ \nabla'_b\, w_a &= B_{ba}^{\mu\lambda}\, \nabla_\mu\, w_\lambda = \partial_b\, w_a - w_c\, B_\lambda^c\, \partial_b\, B_a^\lambda - w_c\, B_{ba\nu}^{\mu\lambda c}\, \Gamma_{\lambda\mu}^{\nu}\end{aligned}\right\} . \quad (21)$$

From this equation follows for the parameters Λ'^{c}_{ab} of the induced connexion

$$\boxed{\Lambda'^{c}_{ab} = B_{ab\nu}^{\lambda\mu c}\, \Gamma_{\lambda\mu}^{\nu} + B_\nu^c\, \partial_b\, B_a^\nu} \quad \ldots\ldots \quad (22)$$

Quite as in an A_n the alternating part

$$\Lambda'^{c}_{[ab]} = B_\nu^c\, \partial_{[b}\, B_{a]}^\nu \quad \ldots\ldots \quad (23)$$

is no affinor and depends on the choice of the systems $\binom{c}{a}$.

A rigged X_n^m thus equipped with a connexion will be called an A_n^m. Applying (21) to $\underset{a}{e^c}$ and $\overset{c}{e_a}$ we get

$$\Lambda'^{c}_{ab} = \nabla'_b\, \underset{a}{e^c} = -\nabla'_b\, \overset{c}{e_a} . \quad \ldots\ldots \quad (24)$$

but also, in consequence of the choice of the measuring vectors

$$\Lambda'^{c}_{ab} = \nabla_b\, \underset{a}{e^c} = -\nabla_b\, \overset{c}{e_a} = \Lambda^{c}_{ab} \quad \ldots\ldots \quad (25)$$

Starting from a V_n instead of from an A_n and choosing for $\binom{k}{i}$ an orthogonal system, we get easily

$$\Lambda'^{c}_{ab} = \gamma_{ab}^{c}\, (= -\gamma_{cab}) \quad \ldots\ldots \quad (26)$$

Hence the connexion induced in a V_n^m ($= X_n^m$ in V_n) is obtained in a very simple manner by using the coefficients of rotation of RICCI with respect to a suitable chosen system of m congruences of curves [1]).

[1]) VRANCEANU has found the connexion, induced in V_n^m, just in this way.

§ 4. *Properties of curvature of an* A_n^m *in* A_n.

We define the first and the second *affinor of curvature* in the same way as in an A_m in A_n:

$$\left.\begin{aligned} H_{ba}^{\cdot\cdot\nu} &= B_{ba}^{\mu\lambda} \nabla_\mu B_\lambda^{\nu'} = -(\nabla'_b \overset{e}{e}_a) \underset{e}{e}^\nu = \Lambda_{ab}^e \underset{e}{e}^\nu \\ L_b^{\cdot c}{}_\lambda &= B_{b\nu}^{\mu c} \nabla_\mu B_\lambda^\nu = -(\nabla_b \underset{e}{e}^c) \overset{e}{e}_\lambda = -\Lambda_{eb}^c \overset{e}{e}_\lambda \end{aligned}\right\} \quad . \quad . \quad . \quad (27)$$

It strikes that, just as in an A_m in A_n, $H_{ba}^{\cdot\cdot\nu}$ lies with the index ν in the local $(n-m)$-direction and $L_b^{\cdot c}{}_\lambda$ contains with the index λ the local m-direction:

$$B_\alpha^\nu H_{ba}^{\cdot\cdot\alpha} = 0 \quad ; \quad B_\lambda^\alpha L_b^{\cdot c}{}_\alpha = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

But here $H_{ba}^{\cdot\cdot\nu}$ is no longer symmetrical in a and b, because

$$H_{[ba]}^{\cdot\cdot\nu} = -B_{ba}^{\mu\lambda} (\nabla_{[\mu} \overset{e}{e}_{\lambda]}) \underset{e}{e}^\nu = \Lambda_{[ab]}^e \underset{e}{e}^\nu \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and this expression vanishes if and only if all vectors $\overset{c}{e}_\lambda$ are X_{n-1}-building viz. if the field of m-directions is X_m-building, the case which we have excluded expressely.

It follows from

$$\left.\begin{aligned} H_{[ba]}^{\cdot\cdot\nu} &= B_{[ba]}^{\mu\lambda} (\partial_\mu B_\lambda^\nu + \Gamma_{\alpha\mu}^\nu B_\lambda^\alpha - \Gamma_{\lambda\mu}^\alpha B_\alpha^\nu) = B_{ba}^{\mu\lambda} \partial_{[\mu} B_{\lambda]}^\nu = \\ &= \partial_{[b} B_{a]}^\nu - B_\alpha^\nu \partial_{[b} B_{a]}^\alpha = C_\alpha^\nu \partial_{[b} B_{a]}^\alpha \end{aligned}\right\} \quad (30)$$

that the field of m-directions is X_m-building if and only if $C_\alpha^\nu \partial_{[b} B_{a]}^\alpha$ vanishes.

The ordinary method of obtaining the quantity of curvature is here useless because in an A_n^m *it is generally impossible to construct a parallelogram*, this impossibility being exactly characteristic for a non X_m-building field of m-directions. In fact, if on the one side a translation $\underset{1}{dy^c}$ is followed by a translation $\underset{2}{dy^c}$, and on the other side $\underset{2}{dy^c}$ by $\underset{1}{dy^c}$, then by using (23) and (30) we find for the closing vector the equation

$$2 \underset{1}{d} y^b \underset{2}{d} y^a \partial_{[b} B_{a]}^\nu = 2 \underset{1}{d} y^a \underset{2}{d} y^b (H_{[ba]}^{\cdot\cdot\nu} + \Lambda_{[ab]}^c B_c^\nu) \quad . \quad . \quad . \quad (31)$$

giving the decomposition into one component in the A_n^m and one in the local $(n-m)$-direction. The latter one only vanishes when the field of m-directions is X_m-building, the other one depends on the choice of the systems $\begin{pmatrix} c \\ a \end{pmatrix}$.

So we choose another way and start with $\nabla'_{[i} \nabla'_{j]} v^k$ which certainly is an affinor. We get

$$\left.\begin{aligned} \nabla'_{[d} \nabla'_{b]} v^c &= H_{[db]}^{\cdot\cdot\alpha} B_\beta^c \nabla_\alpha v^\beta + \{H_{[db]}^{\cdot\cdot\alpha} (B_a^\beta \partial_\alpha B_\beta^c - B_{\gamma a}^{c\beta} \Gamma_{\beta\alpha}^\gamma) + \\ &+ \partial_{[d} \Lambda_{|a|b]}^c + \Lambda_{p[d}^c \Lambda_{|a|b]}^p - \Lambda_{[bd]}^p \Lambda_{ap}^c\} v^a \quad ; \quad p = 1, \ldots, m. \end{aligned}\right\} \quad . \quad (32)$$

The expression corresponding with the right hand side of (13)[1]) is here no longer an affinor, but the expression

$$\left.\begin{aligned} R'^{\;\cdot\cdot\;c}_{dba} = -2\, H_{[db]}^{\;\cdot\cdot\;\alpha} (B^{\beta}_{a}\, \partial_{\alpha}\, B^{c}_{\beta} - B^{c\beta}_{\gamma a}\, \Gamma^{\gamma}_{\beta\alpha}) - 2\partial_{[d}\, \Lambda^{c}_{|a|\, b]} - 2\, \Lambda^{c}_{p[d}\, \Lambda^{p}_{|a|\, b]} + \\ + 2\, \Lambda^{p}_{[bd]}\, \Lambda^{c}_{ap} \quad ; \quad p = 1, \ldots, m, \end{aligned}\right\} . \quad (33)$$

which we call the *quantity of curvature* of the A^m_n, is. Using the parameters belonging to the systems $\binom{k}{i}$ and the equations (25) and (29), we may write for the first term of $R'^{\;\cdot\cdot\;c}_{dba}$

$$-2\, H_{[db]}^{\;\cdot\cdot\;\alpha}\, B^{\beta}_{a}\, \nabla_{\alpha}\, \overset{c}{e}_{\beta} = -2\, H_{[bd]}^{\;\cdot\cdot\;e}\, \Lambda^{c}_{ae} = 2\, \Lambda^{e}_{[bd]}\, \Lambda^{c}_{ae} \quad . \quad . \quad (34)$$

and this expression can be added to the last term of (33) so that finally

$$\left.\begin{aligned} R'^{\;\cdot\cdot\;c}_{dba} = -2\, \partial_{[d}\, \Lambda^{c}_{|a|\, b]} - 2\, \Lambda^{c}_{p[d}\, \Lambda^{p}_{|a|\, b]} + 2\Lambda^{j}_{[bd]}\, \Lambda^{c}_{aj} \\ p = 1, \ldots, m, \\ j = 1, \ldots, n. \end{aligned}\right\} \quad . \quad . \quad (35)$$

From (32) and (33) follows

$$\nabla'_{[d}\nabla'_{b]}\, v^c = H_{[db]}^{\;\cdot\cdot\;\beta}\, B^{c}_{\gamma}\, \nabla_{\beta}\, v^{\gamma} - {}^1/_2\, R'^{\;\cdot\cdot\;c}_{dba}\, v^a \quad . \quad . \quad . \quad . \quad (36)$$

If the field of m-directions is X_m-building, then $H_{[ba]}^{\;\cdot\cdot\;\nu}$ vanishes, (36) takes the ordinary form and (33) regains the same form as (13).

If the A_n passes into a V_n, the quantity of curvature passes into

$$\left.\begin{aligned} K'^{\;\cdot\cdot\;c}_{dba} = -2\, \partial_{[d}\, \gamma^{c}_{|a|b]} - 2\, \gamma^{c}_{p[d}\, \gamma^{p}_{|a|b]} + 2\, \gamma^{j}_{[bd]}\, \gamma^{c}_{aj} \\ p = 1, \ldots, m, \\ j = 1, \ldots, n. \end{aligned}\right\} . \quad (37)$$

§ 5. *The generalised equation of* GAUSS.

From the definition of the induced connexion it is easily deduced for a field v^c of the A^m_n:

$$\nabla'_{[d}\nabla'_{b]}\, v^c = H_{[db]}^{\;\cdot\cdot\;\beta}\, B^{c}_{\gamma}\, \nabla_{\beta}\, v^{\gamma} + L_{[d}^{\;\cdot\;c}{}_{|\alpha|}\, H_{b]a}^{\;\cdot\cdot\;\alpha}\, v^a - {}^1/_2\, B^{\delta\beta c}_{db\gamma}\, R_{\delta\beta\alpha}^{\;\cdot\cdot\;\gamma}\, v^{\alpha} . \quad (38)$$

from which follows

$$\boxed{B^{\delta\beta\alpha c}_{dba\gamma}\, R_{\delta\beta\alpha}^{\;\cdot\cdot\;\gamma} = R'^{\;\cdot\cdot\;c}_{dba} + 2\, H_{[b|a|}^{\;\cdot\cdot\;\alpha}\, L_{d]}^{\;\cdot\;c}{}_{\alpha}} \quad . \quad . \quad . \quad : \quad (39)$$

This is the generalised equation of GAUSS for an A^m_n in A_n and we see that it has the same form as the equation for an A_m in A^*_n.

§ 6. *Geodesics in A^m_n and in A_n.*

A geodesic in A^m_n is a curve, generated by the pseudoparallel

[1]) This expression has been found by VRANCEANU but the affinor $K'^{\;\cdot\cdot\;c}_{dba}$ from (37) does not occur in his papers.

displacement of a contravariant vector in its own direction. t being a parameter on a geodesic, $\frac{dy^b}{dt} \nabla'_b \frac{dy^c}{dt}$ must have the direction of dy^c :

$$\frac{d^2y^c}{dt^2} + \Lambda_{ab}^{\ c} \frac{dy^a dy^b}{dt\ dt} = \alpha \frac{dy^c}{dt} \quad . \; . \; . \; . \; . \; . \; . \quad (40)$$

Hence a geodesic in A_n^m is also a geodesic in A_n if and only if the vector

$$\frac{dx^\mu}{dt} \nabla_\mu \frac{dx^\nu}{dt} - \frac{dx^\mu}{dt} \nabla'_\mu \frac{dx^\nu}{dt} = \frac{dx^\mu}{dt} \left(\nabla_\mu \frac{dx^\alpha}{dt} \right) C_\alpha^\nu = $$
$$= \frac{dx^\mu}{dt} \frac{dx^\lambda}{dt} \nabla_\mu B_\lambda^\nu = \frac{dy^a}{dt} \frac{dy^b}{dt} H_{ba}^{\cdot\cdot\nu} \quad . \; . \; . \quad (41)$$

has the direction of dx^ν . In consequence the geodesics in A_n^m are always geodesics in A_n if and only if $H_{(ba)}^{\cdot\cdot\nu}$ vanishes. Thus the alternating part $H_{[ba]}^{\cdot\cdot\nu}$ which is of such a fundamental importance for the non-holonomity of A_n^m, has nothing to do with this question concerning the geodesics. To the case of a geodesic A_m in A_n corresponds the case of an A_n^m with $H_{(ba)}^{\cdot\cdot\nu} = 0$, all geodesics being also geodesics of A_n. If the A_n passes into a V_n , there exist also shortest curves in V_n^m. But it is immediately clear that shortest curves and geodesics are not identical here. In fact, through a point of V_n^m only ∞^{m-1} geodesics pass but generally ∞^{n-1} shortest curves, because every point of the V_n can be connected with every other point by a curve lying wholly in V_n^m. As an example we take the linear complex in R_3 belonging to a system of forces. The field of the 2-directions belonging to every point is not V_2-building and may be given by the equation

$$p_\lambda = a_\lambda + r^\alpha f_{\alpha\lambda} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (42)$$

a_λ being a constant vector, $f_{\lambda\mu}$ a constant bivector and r^ν the radius-vector. Writing p for the length of p_λ and i_λ for the unit vector belonging to p_λ we have

$$B_{\mu\lambda}^{\beta\alpha} \nabla_\beta i_\alpha = \frac{1}{p} B_{\mu\lambda}^{\beta\alpha} \nabla_\beta p_\alpha = \frac{1}{p} B_{\mu\lambda}^{\beta\alpha} f_{\beta\alpha} = \frac{1}{p} f'_{\mu\lambda} \quad . \; . \; . \quad (43)$$

and

$$H_{\mu\lambda}^{\cdot\cdot\nu} = -\frac{1}{p} f'_{\mu\lambda} i^\nu \quad . \; . \; . \; . \; . \; . \; . \; . \quad (44)$$

$f'_{\mu\lambda}$ being the V_3^2-component of $f_{\mu\lambda}$. The straight lines of the complex are geodesics as well in R_3 as in V_3^2. Obviously two arbitrary points in R_3 can not be connected by a geodesic of V_3^2 but always by a curve lying wholly in V_3^2. The quantity of curvature of V_3^2 is

$$K'_{dbac} = -2 H_{[b|a|}^{\cdot\cdot e} H_{d]ce} = -\frac{2}{p^2} f'_{[b|a|} f'_{d]c} \quad . \; . \; . \; . \quad (45)$$

§ 7. *Affine geometry of an* X_n^{n-1} *in* A_n.

We will prove that an X_n^{n-1} in A_n determines an affine-normal direction in the same way as an X_{n-1} in A_n does, if the following two conditions are fulfilled.

1. The connexion in A_n leaves invariant each volume. (In E_n this condition is always fulfilled).

2. t_λ being a covariant vector having in every point the $(n-1)$-direction of the X_n^{n-1}, the affinor $h_{ba} = B_{ba}^{\mu\lambda} \nabla_\mu t_\lambda$ has the rank $n-1$.

If the connexion A_n leaves invariant every volume, there exists a constant n-vectorfield $P_{\lambda_1 \dots \lambda_n}$. Every other constant n-vectorfield can be obtained by multiplying $P_{\lambda_1 \dots \lambda_n}$ with a *constant* scalar. Now if h_{ba} has the rank $n-1$, t_λ can be chosen in a unique way, so that

$$t_{[\mu_1} t_{[\lambda_1} k_{\mu_2 \lambda_2} \dots k_{\mu_n] \lambda_n]} = P_{\mu_1 \dots \mu_n} P_{\lambda_1 \dots \lambda_n} \quad . \quad . \quad . \quad . \quad (46)$$

If the constant n-vectorfield be changed, t_λ only takes a *constant* scalar factor. The affine-normal vector can now be defined by means of the equations

$$\left. \begin{array}{l} t_\mu n^\mu = 1 \\ B_a^\mu (\nabla_\mu t_\lambda) n^\lambda = 0 \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

h_{ba} having the rank $n-1$, n^ν is determined but for a *constant* scalar factor. Thus the affine-normal direction is found.

By use of the direction of n^ν just found, the X_n^{n-1} can be rigged, and an affine geometry can be obtained, as indicated in the former paragraphs.

Instead of h_{ba} also $k_{ba} = h_{(ba)}$ or $f_{ba} = h_{[ba]}$ can be used to construct the affine-normal direction.