Physics — *A formula expressing the deflection of the plumb-line in the gravity anomalies and some formulae for the gravity-field and the gravity-potential outside the geoid.* By F. A. VENING MEINESZ.

(Communicated at the meeting of January 28, 1928).

§ 1. In the middle of the 19th century STOKES succeeded in deducing a formula, which expresses the distance between the geoid and some chosen spheroid in the gravity anomalies of the whole earth. These gravity anomalies have to be computed by taking the difference of the observed gravity, reduced to the geoid by free air reduction (reduction of FAYE), and the normal gravity value, corresponding to the spheroid, which has been chosen ; that is to say this normal value represents the gravity-field, which would exist on an outside potential surface of some theoretical earth of the same mass as the actual earth, for which this potential surface coincides with the spheroid. This condition determines the normal gravity-field completely, as a well-known potential theorem indicates. To the formula of the normal gravity may however be added any constant or any spherical harmonic of the first degree without changing the result given by the formula of STOKES. This is an advantage, which we will secure also in §§ 2 and 3 for some other formulae, as it renders harmless any error in the constant term of the formula for the normal gravity, which is not impossible so long as gravity is unknown for the greater part of the earth's surface.

The chosen spheroid has to fulfill the following conditions : The volume as well as the centre of gravity must coincide with the volume and the centre of gravity of the actual earth. The radius of curvature has not to deviate more from the earth's radius than in the ratio of the first order of the flattening and we assume further for the spheroid as well as for the geoid that the angle between the normal and the radius towards the earth's centre is of that same order. We suppose lastly that the distance between both surfaces is of the second order of the flattening in proportion to the earth's radius.

With these restrictions the spheroid is wholly arbitrary ; it may be an ellipsoid, but this is not necessary and it is not even necessary that it is a body of revolution. It may further be emphasized that the formula of STOKES, corresponding to the fixed relation between gravity-field and shape of the geoid, is independent of the distribution of the masses of the earth, which are the causes of both ; it is therefore independent of any assumption about the mass distribution.

It is clear that the existence of this fixed relation between the gravity-field and the shape of the geoid implies also a fixed relation between the gravity and the position of the normal on the geoid, that is to say between the gravity and the deflection of the plumb-line [1]). It is the object of this paper to deduce the corresponding equation, expressing the deflection of the plumb-line in the gravity-anomalies.

The easiest way to deduce such an equation is the differentiation of the distance between the geoid and the spheroid, given by the formula of STOKES, according to a direction tangent to the geoid; we find in this way the angle between the normal on the geoid and the normal on the spheroid, which may be considered as the deflection of the plumb-line with regard to the chosen spheroid. It is however necessary first to clear up two points concerning the equation of STOKES.

The first point is how far this formula gives also the local deviations of the geoid. This is questionable because of the fact, that in deducing it, a constant value R is substituted at a certain moment for the earth's radius. HELMERT for instance expresses doubt about it in "Die Theorien der höheren Geodäsie", and considers the equation as only valid for giving the general shape of the geoid. Would this be true, then the result for the deflection of the plumb-line got in this way, would be valueless; it is therefore necessary to prove, as we think it is possible to do, that this doubt is not founded. In this regard we may draw attention to a remarkable paper of POINCARÉ: "Les mesures de gravité et la géodésie", which is not generally known to geodesists and which appeared in the "Bulletin Astronomique" of 1901. POINCARÉ makes a study of the whole problem and, without knowing apparently about the work of STOKES, deduces the same formula and enlarges specially on the possibility of determining with this formula the local shape of the geoid with this equation. We will look into this question more closely in § 4.

The second point, which has to be examined, is the question concerning the effect of the masses outside the geoid. In deducing the formula of STOKES or one of the formulae of this paper, it is supposed that there are no masses outside the geoid, so that their validity is questionable so long as there are such masses. We will eliminate this difficulty by considering a regulated earth, for which these masses are removed. As the masses outside the geoid are fairly well known, the changing over from the actual earth to the regulated one presents no difficulty: we can compute the effect of these masses and find in this way the changes of the gravity anomaly and of the geoid, caused by the removal of these masses. The change of the gravity anomalies is about the same as the ordinary BOUGUER reduction. After having determined the shape of the regulated geoid by introducing these reduced anomalies in the formula of STOKES, the actual geoid can be derived by applying the difference between both geoids. It may be remarked — LAMBERT drew first attention to this point — that the change of the

[1]) See also of M. MARCEL BRILLOUIN: „Champ de grav. extérieur et densités internes", Comptes Rendus de l'Acad. d. Sciences, t. 180, and t. 184.

geoid will also bring along a shifting of the centre of gravity, so that the centre of gravity of the spheroid, which has to coincide with that of the regulated earth, will not quite coincide with that of the actual earth.

In order to lessen the difference between the geoids of the actual earth and of the regulated earth, of which we determine the shape with the formula of STOKES, we may partly compensate the taking away of the masses by adding at the same time corresponding masses inside the geoid. We may do this by applying the condensation method of HELMERT, who adds in the same vertical at a depth of 21 km the same quantity of mass as has been removed. Or we may do it according to the inversion method of RUDZKI, who compensates the removal of an outside mass at a distance l from the centre of the earth, by adding a mass, which is R/l times the former one, at a distance R^2/l from the earth's centre, R being the mean earth's radius. RUDZKI obtains in this way, that the geoid remains unchanged, so that the formula of STOKES gives at once the actual geoid. Or lastly we may follow the isostatic method, adding the same quantity of mass, as has been taken away, and distributing it inside the geoid according to one of the accepted methods. The reduction of the gravity anomalies, corresponding to this last method, is the ordinary isostatic reduction, if we extend this mass-regulation also to the oceanic part of the earth's crust, filling up the oceans till they have normal density with masses removed from the crust below. For the purpose, which we have in view here, this extension to the oceans would not of course be necessary.

Of the three compensation methods the isostatic one gives the greatest shift of the geoid, because the distance between the removed masses and the added masses is greatest. Still it seems to me that this method is preferable to the other ones. The greater shift of the geoid is no serious drawback, as it can be computed, and the method has the advantage that the field of gravity anomalies becomes more regular by the isostatic reduction than by any other: it removes as well as possible the effect of the local mass-irregularities in the crust. This advantage is worth mentioning because it makes each anomaly value representative for a greater area of the earth's surface, so that a certain limited number of anomalies will give a better image of the geoid.

When we adopt one of these methods for removing the outside masses in order to be able to apply the formulae on the regulated earth, we must take care of the following question. As has been remarked in the beginning, we have to reduce the observed gravity by free air reduction to the geoid, that is to say that we have now to reduce to the regulated geoid. We have therefore to apply an extra free air reduction for the distance between both geoids. If we take for instance the isostatic method of regulation we have the following series of reductions to the observed gravity: First the free air reduction to the actual geoid and then the reductions belonging to the regulation of the earth, including first the ordinary isostatic reduction, representing the change of gravity caused by the transport of masses and then the free air reduction from the original geoid to the regulated geoid. This last reduction is the same as the reduction of BOWIE.

Objection might be made that the masses outside the actual geoid have been removed and not the masses outside the regulated geoid. As a matter of fact we ought to remove the masses outside this last geoid. The difference may however be neglected if one of the compensation methods has been used, as in this case the distance between both geoids is small; it will in fact not exceed some twenty meters.

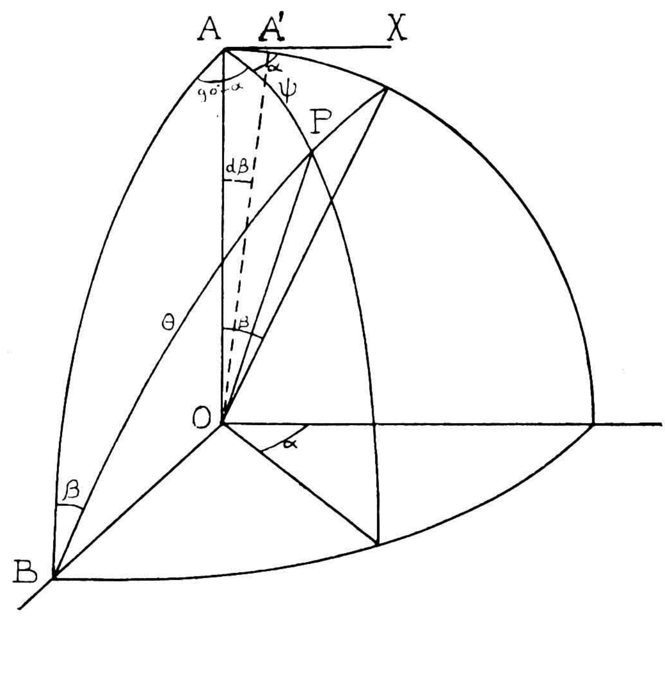
We will now proceed to the deduction of the formula for the deflection of the plumb-line, which we will do by differentiating the formula of STOKES according to a direction tangent to the geoid. This formula may be written:

$$N = \frac{R}{4\pi\gamma}\int^{\sigma} S_T \,\Delta_0\, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1^A)$$

in which: N = distance between geoid and spheroid,
R = mean earth's radius,
$d\sigma$ = surface-element of a sphere with radius 1,
Δ_0 = gravity anomaly corresponding to the element $d\sigma$,
S_T = function of the angle ψ between the radius of the point A, where N is computed, and the radius of the point P coinciding with $d\sigma$:

$$S_T = cosec\,\tfrac{1}{2}\psi + 1 - 6\,sin\,\tfrac{1}{2}\psi - 5\cos\psi - 3\cos\psi\; lg\,[sin\,\tfrac{1}{2}\psi\,(1 + sin\,\tfrac{1}{2}\psi)] \quad (1^B)$$

We choose in A an X direction tangent to the geoid and introduce an azimuthal coordinate α, representing the angle between OAX and OAP



Besides the system of coordinates ψ and α we introduce a second system θ and β, θ being the angle between the radii of P and B (OB perpendicular to OAX), and β the angle between AOB and POB.

Let ϑ be the deflection of the plumb-line in A with regard to the spheroid, and let ϑ_x be its component in the X direction. Neglecting quantities of higher order, we have then :

$$\vartheta_x = \frac{\partial N}{\partial x} = \frac{1}{R}\frac{\partial N}{\partial \beta}$$

in which last formula N is expressed in the second system of coordinates θ and β :

$$N = \frac{R}{4\pi\gamma}\int^{\sigma} S_T(\theta, \beta) \times \Delta_0(\theta, \beta)\, d\sigma.$$

For differentiating N according to β we have to determine the value of $N + \frac{\partial N}{\partial \beta}\, d\beta$ in a point A' at a distance $Rd\beta$ from A. The field Δ_0 remains at the same place, but the field S_T shifts with the centre A towards A', that is to say it rotates an angle $d\beta$. If the system of coordinates is kept unmoved, we find :

$$N + \frac{\partial N}{\partial \beta}\, d\beta = \frac{R}{4\pi\gamma}\int^{\sigma} S_T[\theta, (\beta - d\beta)] \times \Delta_0(\theta, \beta)\, d\sigma$$

but we may also shift the system of coordinates in the same way as the field S_T by rotating it an angle $d\beta$, and we find then :

$$N + \frac{\partial N}{\partial \beta}\, d\beta = \frac{R}{4\pi\gamma}\int^{\rho} S_T(\theta, \beta) \times \Delta_0[\theta, (\beta + d\beta)]\, d\sigma$$

The two expressions give the formulae :

$$\vartheta_x = \frac{1}{R}\frac{\partial N}{\partial \beta} = -\frac{1}{4\pi\gamma}\int^{\sigma} \frac{\partial S_T}{\partial \beta}\Delta_0\, d\sigma \quad . \; . \; . \; . \; . \quad (2^A)$$

$$\vartheta_x = \frac{1}{R}\frac{\partial N}{\partial \beta} = +\frac{1}{4\pi\gamma}\int^{\sigma} S_T\,\frac{\partial \Delta_0}{\partial \beta}\, d\sigma \quad . \; . \; . \; . \; . \quad (3^A)$$

which are both worth while examining : The first expresses ϑ_x in the gravity anomaly itself and the second expresses ϑ_x in the horizontal gradient of the anomaly.

We will go back to the original system of coordinates, and express both formulae in ψ and a. The spheric triangle APB gives:

$$\left.\begin{aligned} \frac{\partial \psi}{\partial \beta} &= \cos a \\ \frac{\partial a}{\partial \beta} &= -\sin a \, \operatorname{cotg} \psi \end{aligned}\right\} \quad \ldots \ldots \quad (4^A)$$

and therefore:

$$\frac{\partial}{\partial \beta} = \cos a \frac{\partial}{\partial \psi} - \sin a \, \operatorname{cotg} \psi \frac{\partial}{\partial a} \quad \ldots \ldots \quad (4^B)$$

so that we get:

$$\vartheta_x = -\frac{1}{4\pi\gamma} \int^{\sigma} \cos a \frac{\partial S_T}{\partial \psi} \Delta_g \, d\sigma \quad \ldots \ldots \quad (2^B)$$

$$\vartheta_x = +\frac{1}{4\pi\gamma} \int^{\sigma} S_T \left[\cos a \frac{\partial \Delta_0}{\partial \psi} - \sin a \, \operatorname{cotg} \psi \frac{\partial \Delta_0}{\partial a} \right] d\sigma \quad \ldots \quad (3^B)$$

We may introduce

$$d\sigma = \sin \psi \, da \, d\psi$$

and multiply with $\varrho'' = \operatorname{cosec} 1''$, in order to express ϑ_x in seconds. The second formula gives then:

$$\vartheta''_x = \frac{\varrho''}{4\pi\gamma} \int_0^{2\pi} \cos a \, da \int_0^{\pi} \sin \psi \, S_T \frac{\partial \Delta_0}{\partial \psi} d\psi - \frac{\varrho''}{4\pi\gamma} \int_0^{2\pi} \sin a \, da \int_0^{\pi} \cos \psi \, S_T \frac{\partial \Delta_0}{\partial a} d\psi \quad (3^C)$$

which we will not further examine in this paper.

The first formula gives:

$$\vartheta''_x = \frac{1}{2\pi} \int_0^{2\pi} \cos a \, da \int_0^{\pi} Q \, \Delta_0 \, d\psi \quad \ldots \ldots \quad (2^C)$$

with

$$Q = -\frac{\varrho''}{2\gamma} \sin \psi \frac{\partial S_T}{\partial \psi}$$

that is to say in introducing the expression for S_T:

$$\left.\begin{aligned} Q = \frac{\varrho''}{2\gamma} \cos^2 \tfrac{1}{2} \psi \Big[& \operatorname{cosec} \tfrac{1}{2} \psi + 12 \sin \tfrac{1}{2} \psi - 32 \sin^2 \tfrac{1}{2} \psi + \\ & + \frac{3}{1 + \sin \tfrac{1}{2} \psi} - 12 \sin^2 \tfrac{1}{2} \psi \, lg \left\{ \sin \tfrac{1}{2} \psi \left(1 + \sin \tfrac{1}{2} \psi\right) \right\} \Big] \end{aligned}\right\} \quad (2^D)$$

The following table gives the values of Q for intervals of 10° for ψ (Δ_0 and γ are supposed to be expressed in 0.001 cm) :

ψ	Q	ψ	Q	ψ	Q
1°	+12.35	70°	+0.03	140°	—0.25
10°	+ 1.59	80°	—0.15	150°	—0.16
20°	+ 1.02	90°	—0.29	160°	—0.08
30°	+ 0.79	100°	—0.38	170°	—0.02
40°	+ 0.61	110°	—0.41	180°	—0.00
50°	+ 0.43	120°	—0.40		
60°	+ 0.22	130°	—0.34		

For increasing ψ, Q first decreases rather quickly, afterwards more slowly, is zero for ψ more than 70°, gets to a minimum —0.41 for ψ about 110°, and increases then towards zero, which value is reached for $\psi = 180°$.

For small ψ we can neglect in the formula for Q all but the first term, and we get by introducing in stead of ψ the linear distance $AP = r$, so that $r = 2R \sin \frac{1}{2} \psi$:

$$Q = \frac{1338}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2^E)$$

in which r is expressed in km.

By substituting in (2C) the surface element df in stead of the expression in $d\psi$ and da, we find for the effect of the anomalies in the neighbouring region :

$$\vartheta''_x = \frac{\varrho''}{2\pi\gamma} \int \cos a \frac{\Delta_0}{r^2} df \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

We may express this formula in the following way : If we apply to every surface element df of the geoid a mass equal to $\frac{\varrho''}{2\pi\gamma} \Delta_0 df$, the reflection of the plumb-line is the resultant of the attractions, exerted by these masses in A if these attractions act along the radii towards df and are equal to the mass divided by the square of the distance.

The decrease of the influence of a certain anomaly with the distance is therefore greater than for the formula of STOKES : This formula reduces for the neighbouring region to :

$$N = \frac{1}{2\pi\gamma} \int \frac{\Delta_0}{r} df \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

so that the influence on N decreases only in inverse ratio to the first power of r.

Formula (6) is in harmony with formula (5) as this last formula indeed represents the gradient of the former.

The property that the influence of a certain anomaly decreases more quickly with the distance than for the formula of STOKES, gives the advantage that the formula for the deflection of the plumb-line is more independent of the anomalies in other parts of the earth's surface, so that the fact, that as yet the gravity is only very imperfectly known over the globe, need not necessarily prevent its application.

The formula for the deflection of the plumb-line may be useful for connecting the results of both types of geodetic operations : the astronomic observations and the determinations of the gravity. Besides it may be of use, as well as the formula of STOKES itself, to study the local deviations of the geoid for the central area of a region where the gravity has been determined. The anomalies in the other parts of the globe will generally have an equable effect for the whole central area : for the deflection of the plumb-line for instance a small and nearly constant amount. So by neglecting the influence of the anomalies in those distant region we will get a result for the shape of the geoid, which will perhaps be slightly wrong in position, but which will not be much deformed. We may even put the question, if the determination of the geoid along this line will not better answer the purpose, than that, which is founded on the direct observations of the plumb-line deflection. The disadvantage of the uncertainty of the gravity in other parts of the globe, may perhaps be more than compensated by the uncertainty, resulting from the fact that in using this last method, N is determined by integrating a quantity, which is only known in isolated spots.

A provisional computation for the Netherlands has shown, that the differences of the plumb-line deflections for the three stations in the central part : Urk, Wolberg, and Utrecht, are in good harmony with the values which have been deduced from the gravity anomalies : the deviations do not exceed $0''.5$. For this computation only the gravity-anomalies in the Netherlands have been used.

It appears doubtful if the accuracy of the result can ever attain such a perfection, that the formula could be used to controll the results of the triangulations by giving for each astronomical station an equation for the latitude and longitude components of the plumb-line deflection. Still it may give a useful control for traverse surveys, of which the errors are so much greater, as is well known : We may considerably improve in this way the control of these surveys by astronomical observations. For this purpose it is of course necessary to have a sufficient number of gravity stations in an area of somewhat greater extension than the area of the survey.

Besides the formulae for the deflection of the plumb-line, it is a simple matter to deduce other formulae from the formula of STOKES by further differentiation. In this way we may derive formulae, expressing the second differential coefficients of N in the gravity anomalies or in the gradient of these anomalies. These second differential coefficients may be considered as the difference of the reciprocal values of the radii of curvature of the geoid and the spheroid, so that the curvature of the geoid may be determined in this way. We will not take up this question in this paper and we will only remark that in this way a formula may be found, connecting the two series of quantities, given by the torsion balance : the horizontal gradients of the gravity and the curvature data. These data are therefore not independent, although the curvature cannot be brought in connection with a single value of the gradient but with the gradient field over the whole earth.

The next paragraphs will treat of the neglected terms of the formula of STOKES and will prove the validity of this formula and of the formulae found by differentiating it, as well for the determination of the general shape of the geoid as for that of the local irregularities. Besides they will treat of some formulae about the gravity field outside the geoid, which we will need for our purpose.

§ 2. *Gravity anomaly in an arbitrary point outside a sphere, which encloses all the masses and on which the anomaly is known.*

Before looking into the question of the validity of the formula of STOKES, we will derive the solution of the above problem, which will be wanted for that research. Practically this will also give the formula for expressing the gravity anomaly outside the geoid in the anomalies on the geoid : the error committed by replacing the geoid by the sphere with a radius equal to the mean earth-radius is of the same order as the error in the STOKES formula i.e. the percentual error will not exceed the order of the flattening.

Let T be the difference of the potentials caused in the same point by the actual earth and by the theoretical earth ; by the last we mean one of the infinite number of possible mass-distributions inside the geoid with the same total mass as the earth and with an outside potential surface coinciding with the spheroid. Only values of T in points outside both mass-distributions will be considered, that is to say in points outside the geoid or on the geoid.

Let further :

$a =$ aequatorial radius of the spheroid,
$f =$ flattening of the spheroid,
$g =$ gravity in an arbitrary point, caused by the actual earth,
$\gamma =$ gravity in an arbitrary point, caused by the theoretical earth,
$\varrho =$ radius of an arbitrary point towards the centre of gravity of geoid and spheroid,
$g_0, \gamma_0, \varrho_0 =$ values of g, γ and ϱ on the geoid,
$g_s, \gamma_s, \varrho_s =$ values of g, γ and ϱ on the spheroid,
$N_0 =$ distance between geoid and spheroid,
$R =$ radius of the sphere, which is considered,
$\Delta_0 = g_0 - \gamma_s =$ gravity anomaly on the geoid.

As we suppose N_0 to be of the order of (af^2), Δ_0 of the order of (gf^2) and the angle between ϱ and g or γ of the order of (f), we have:

$$N_0 = \frac{T_0}{\gamma_0} + (af^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^A)$$

$$g_0 - \gamma_0 = -\frac{\partial T_0}{\partial \varrho} + (gf^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^B)$$

$$\gamma_0 - \gamma_s = N_0 \frac{\partial \gamma_0}{\partial \varrho} + (gf^4) = \frac{\partial \gamma_0}{\partial \varrho}\frac{T_0}{\gamma_0} + (gf^4) \quad . \quad . \quad . \quad . \quad (7^C)$$

Therefore:

$$\Delta_0 = -\frac{\partial T_0}{\partial \varrho} + \frac{\partial \gamma_0}{\partial \varrho}\frac{T_0}{\gamma_0} + (gf^4)\,^{1)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^D)$$

In the same way as the anomaly Δ_0 has been defined for the geoid, we may define the anomaly Δ for a point outside the geoid as the difference of the actual gravity in this point and the gravity of the theoretical earth in a point in the same vertical, where the potential is equal to the potential of the actual earth in the first point. By rough approximation we may say that the second point is at the same distance outside the spheroid as the first point is outside the geoid. We have again:

$$\Delta = -\frac{\partial T}{\partial \varrho} + \frac{\partial \gamma}{\partial \varrho}\frac{T}{\gamma} + (gf^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^E)$$

If the spheroid were a sphere, the differential-quotient $\frac{\partial \gamma}{\partial \varrho}$ would be:

$$\frac{\partial \gamma}{\partial \varrho} = -2\frac{\gamma}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For a spheroid it has to be multiplied by a factor $1 + (f)$ (see e.g. HELMERT, Theor. d. h. Geod. II, page 94) and so we get:

$$\Delta = -\frac{\partial T}{\partial \varrho} - 2\frac{T}{\varrho} + (gf^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

with a corresponding equation for Δ_0. As Δ is of the order of (gf^2) the last term in (9) is small with regard to the others. It will in fact not surpass 0.0001 or 0.0002 cm in Δ. We will neglect it as well as all the last terms of the formulae (7).

The problem which we want to solve is to express Δ in Δ_R, Δ_R representing the value of Δ on the sphere. It is well known that for a point outside the attracting masses, T can be written in the shape:

$$T = \frac{k^2}{\varrho}\left[\frac{K_2}{\varrho^2} + \ldots + \frac{K_n}{\varrho^n} + \ldots\right]. \quad . \quad . \quad . \quad . \quad . \quad (10^A)$$

1) In the original paper (Kon. Akad. v. Wet. Amsterdam, Dl. **37** N°. 1) the second term of the right member of (7^D) has been neglected. The reasoning remains the same when taking it into account.

in which k^2 is the gravitational constant and $K_2 \dots K_n \dots$ are spherical harmonics of the $2^{nd} \dots n^{th}$-degree. There are no spherical harmonics of zero and first degree, because the total mass of the actual and of the theoretical earths are the same, so that the differential masses, which cause the potential T, have a total mass zero, and because the centres of gravity of the masses of the actual and the theoretical earths coincide with the origin of the radius ϱ.

We have now:

$$g - \gamma = -\frac{\partial T}{\partial \varrho} = \frac{k^2}{\varrho}\left[3\frac{K_2}{\varrho^2} + \dots + (n+1)\frac{K_n}{\varrho^n} + \dots\right] \quad . \quad (10^B)$$

$$\Delta = -\frac{\partial T}{\partial \varrho} - 2\frac{T}{\varrho} = \frac{k^2}{\varrho^2}\left[\frac{K_2}{\varrho^2} + \dots + (n-1)\frac{K_n}{\varrho^n} + \dots\right] \quad . \quad (10^C)$$

and for $\varrho = R$:

$$\Delta_R = \frac{k^2}{R^2}\left[\frac{K_2}{R^2} + \dots + (n-1)\frac{K_n}{R^n} + \dots\right] \quad . \quad . \quad . \quad (10^D)$$

$$T_R = \frac{k^2}{R}\left[\frac{K_2}{R^2} + \dots \qquad + \frac{K_n}{R^n} + \dots\right] \quad . \quad . \quad . \quad (10^E)$$

When Δ_R is known we can develop Δ_R into a series of spherical harmonics, which gives us $K_2 \dots K_n \dots$ and by substituting these values in the formula for Δ, the problem is solved because formula (10^C) is convergent when $\varrho \geqq R$. We can however deduce an equation which allows a simpler computation, by expressing Δ directly in Δ_R. This gives besides a more useful formula for the case Δ_R is not completely known over the whole globe, which is of course the actual state of things. The deduction is quite analogous to the method of STOKES, who solved the problem of computing T_R (and thereby $N_R = T_R/\gamma$) from Δ_R, by expressing T_R directly in Δ_R, so that the elaborate way of first computing all $K_2 \dots K_n \dots$ can be avoided.

The spherical harmonic of the n^{th} degree of Δ_R is:

$$\frac{2n+1}{4\pi}\int_0^\sigma P_n \, \Delta_R \, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in which: $d\sigma$ = surface-element corresponding to Δ_R of a sphere with radius 1.

P_n = LEGENDRE's spherical harmonic of the angle ψ between the radius ϱ of the point A where we want Δ and the radius R of the point P where we have Δ_R.

This gives:

$$\Delta = \sum_2^\infty \left(\frac{R}{\varrho}\right)^{n+2} \times \frac{(2n+1)}{4\pi}\int_0^\sigma P_n \, \Delta_R \, d\sigma$$

which is convergent because $\varrho \geqq R$. Therefore:

$$\triangle = \frac{1}{4\pi} \int_0^\sigma \left[\sum_2^\infty (2\,n+1)\, P_n \left(\frac{R}{\varrho}\right)^{n+2} \right] \triangle_R \, d\sigma$$

or

$$\triangle = \frac{1}{4\pi} \int_0^\sigma S_\triangle \triangle_R \, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12^A)$$

with

$$S_\triangle = \sum_2^\infty (2\,n+1\, P_n \left(\frac{R}{\varrho}\right)^{n+2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12^B)$$

Now $S_\triangle$ can be expressed in R, ϱ and ψ by making use of the well-known formula:

$$\sum_2^\infty P_n \left(\frac{R}{\varrho}\right)^n = \frac{\varrho}{r} - 1 - \frac{R}{\varrho} \cos \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in which r is the distance AP, given by:

$$r = (\varrho^2 - 2\,\varrho\, R \cos \psi + R^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Multiplying (13) by $\left(\frac{R}{\varrho}\right)^{\frac{1}{2}}$, differentiating according to $\frac{R}{\varrho}$ and multiplying by $2 \left(\frac{R}{\varrho}\right)^{2\frac{1}{2}}$ makes the first member equal to $S_\triangle$, and we find:

$$S_\triangle = \frac{R^2 (\varrho^2 - R^2)}{\varrho r^3} - \frac{R^2}{\varrho^2} - 3\,\frac{R^3}{\varrho^3} \cos \psi \quad . \quad . \quad . \quad . \quad . \quad (15)$$

(12^A) combined with (15) gives $\triangle$ expressed in $\triangle_R$.

Still we have to be careful if we want to use these formulae for determining the difference between $\triangle$ and $\triangle_R$: If $\triangle_R$ does not contain spherical harmonics of zero and first degree as has been assumed, the difference can indeed be found simply by subtracting $\triangle_R$ from formula (12^A), but if in the formula for $\triangle_R$ those spherical harmonics are not zero, we must first subtract those terms from $\triangle_R$ before taking the difference with $\triangle$. $\triangle$ is certainly free from those terms even if $\triangle_R$ is substituted in (12^A) without correction, because $S_\triangle$ does not contain spherical harmonics of zero and first degree. The terms of zero and first degree in $\triangle_R$ are easily found by applying formula (11) and in this way we find for the difference $\triangle - \triangle_R = \delta$:

$$\delta = \frac{1}{4\pi} \int_0^\sigma S_\delta \triangle_R \, d\sigma - \triangle_R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16^A)$$

with:

$$S_\delta = \frac{R^2 (\varrho^2 - R^2)}{\varrho r^3} + \frac{(\varrho^2 - R^2)}{\varrho^2} + 3\,\frac{(\varrho^3 - R^3)}{\varrho^3} \cos \psi \quad . \quad . \quad . \quad . \quad (16^B)$$

in which formulae $\triangle_R$ may be substituted as it is, without taking away its terms of zero and first degree. δ will not contain those terms.

Formula (16^B) converges towards zero when $\varrho - R$ is converging towards zero. If the difference $\varrho - R$ which we will represent by h, is small, we can write, neglecting second and higher powers of h in S_δ:

$$S_\delta = \frac{h}{R}\left[2\frac{R^3}{r^3} + 2 + 9\cos\psi\right] \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16^C)$$

Applying these formulae to the geoid in stead of to a sphere with radius R, we have to substitute for R the mean earth's radius and for h the elevation above the geoid.

For giving an idea of the magnitude of the variation of $\triangle$ outside the geoid, we will apply the formulae to a special case.

Supposing a circular patch of anomalies, defined by the formula:

$$\triangle_R = \triangle_C\left(1 - \frac{u^2}{l^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17^A)$$

in which $\triangle_C$ is the value in the centre, u the horizontal radius from $\triangle_R$ to this centre and l the outer limiting radius. Suppose l small with regard to the earth's radius. In this case we can neglect the second and third term of (16^C). We find for a point at an elevation h above the centre:

$$\delta_h = -2\frac{h}{l}\left[\sqrt{1 + \frac{h^2}{l^2}} - \frac{h}{l}\right]\triangle_C \quad . \quad . \quad . \quad . \quad . \qquad (17^B)$$

For $h = 0.36\ l$ we get $\delta_h = 0.5\ \triangle_C$, that is to say that the anomaly has diminished to half its value for an elevation about one sixth of the diameter of the patch.

This allows the conclusion, that, generally speaking, the effect of the diminution of $\triangle$ with the elevation is too small to warrant a corresponding reduction in bringing back the result of a gravity determination to sea level: it will seldomly exceed 0.001 or 0.002 cm sec^{-2} per 1000 m elevation.

The formulae (12), (15), (16) and (17) may be applied to all other quantities which can be represented by a series like:

$$q = \frac{C}{\varrho^2}\left[a_2\frac{K_2}{\varrho^2} + \ldots . + a_n\frac{K_n}{\varrho^n} + \ldots .\right] \quad . \quad . \quad . \quad . \qquad (18)$$

in which c, $a_2 \ldots . a_n \ldots .$ are constants and therefore independent of the radius ϱ or the angle ψ. We have simply to substitute q and q_R for $\triangle$ and $\triangle_R$. In this way they may for instance be used to express the variation of $g - \gamma$ outside the geoid in $g_0 - \gamma_0$ on the geoid, $g - \gamma$ representing the difference of the actual and the theoretical gravity in the same point.

§ 3. *Potential outside the geoid.*

To express the variation of T outside the geoid in T_0 on the geoid, we have analogous formulae deduced in the same way:

$$T = \frac{1}{4\pi} \int_0^\tau S_T \, T_0 \, d\sigma \quad . \; . \; . \; . \; . \; . \; . \; . \quad (19^A)$$

$$S_T = \frac{R(\varrho^2 - R^2)}{r^3} - \frac{R}{\varrho} - 3\frac{R^2}{\varrho^2} \cos\psi . \quad . \; . \; . \; . \; . \quad (19^B)$$

For the difference $t = T - T_0$ we find:

$$t = \frac{1}{4\pi} \int_0^\sigma S_t \, T_0 \, d\sigma - T_0 \quad . \; . \; . \; . \; . \; . \; . \quad (20^A)$$

$$S_t = \frac{R(\varrho^2 - R^2)}{r_3} + \frac{(\varrho - R)}{\varrho} + 3\frac{(\varrho^2 - R^2)}{\varrho^2} \cos\psi \quad . \; . \; . \quad (20^B)$$

and for a small elevation h:

$$S_t = \frac{h}{R}\left[2\left(\frac{R}{r}\right)^3 + 1 + 6\cos\psi\right] . \quad . \; . \; . \; . \; . \quad (20^C)$$

For expressing the distance N in some outside point between corresponding potential surfaces of the actual and theoretical earths, we have to substitute for T: γN and for T_0: $\gamma_0 N_0$.

For local irregularities in T or N of an extension, which is small with regard to the earth's radius, we can neglect the second and third terms of (20^C) and we find the same formula for the diminution with h as for Δ. This is also true for the general case, if Δ_R and T_0 are free of spherical harmonics of zero and first degree, because the difference between S_δ and S_t contains only spherical harmonics of zero and first degree.

Lastly we will ask to express T outside the geoid in the anomaly Δ_0 on the geoid, or if we replace again the geoid by the sphere with radius R, to express T in Δ_R. This is an enlargement of the problem of STOKES; by making the radius ϱ of the point, where we want T, equal to R, it is brought back to the identical problem.

Following the same way of deduction as for the formula for Δ we get:

$$T = \frac{R}{4\pi} \int_0^\sigma S_{\Delta T} \, \Delta_R \, d\sigma \quad . \; . \; . \; . \; . \; . \; . \; . \quad (21^A)$$

with

$$S_{\Delta T} = \sum_2^\infty \frac{(2n+1)}{(n-1)} P_n \left(\frac{R}{\varrho}\right)^{n+1} \quad . \; . \; . \; . \; . \; . \; . \quad (21^B)$$

and we find $S_{\triangle T}$ by multiplying (13) by $\left(\frac{R}{\varrho}\right)^{\frac{1}{2}}$, differentiating according to $\frac{R}{\varrho}$, multiplying by $2\left(\frac{R}{\varrho}\right)^{-1\frac{1}{2}}$, integrating according to $\frac{R}{\varrho}$ and multiplying by $\left(\frac{R}{\varrho}\right)^2$:

$$S_{\triangle T} = 2\frac{R}{r} + \frac{R}{\varrho} - 5\frac{R^2}{\varrho^2}\cos\psi - 3\frac{Rr}{\varrho^2} - 3\frac{R^2}{\varrho^2}\cos\psi \log nat \frac{[\varrho - R\cos\psi + r]}{2\varrho} \quad (21^C)$$

By putting $\varrho = R$, which gives $r = 2R \sin \frac{1}{2}\psi$, we get back to the formula of STOKES. The distance N is of course found by dividing T by γ. In this way the formula gives the outside potential surfaces of the earth, when the gravity anomaly on the geoid is known and provided the theoretical outside potential surfaces have been computed. N may of course be differentiated in the same way as has been done in the first paragraph; we find then the deflection of the plumb-line in a point outside the geoid, expressed in the gravity anomalies on the geoid, or in their horizontal gradient. Executing the same thing with the formula for N deduced from formula (19), we can get this plumb-line deflection expressed in the value of N_0 on the geoid or in the plumb-line deflections on the geoid.

§ 4. *The validity of the formula of* STOKES.

To find the order of magnitude of the neglected terms of the formula of STOKES, we suppose a fictitious earth with a mass equal to the total mass of the real earth, and of which the outside potential surface is a sphere with a radius R. The difference of R and the earth's radius is of the order of the flattening. The gravity on this sphere is, according to a potential theorem, constant over the whole surface.

To this fictitious earth is added a mass-distribution of positive and negative masses with a zero total mass, in such a way, that the combination of these masses with the fictitious earth gives a geoid on which the gravity anomalies (i.e. the gravity minus the above mentioned constant value) are the same as the gravity anomalies $\triangle_0$ of the real earth in corresponding points of the real geoid; for corresponding points we may for instance take points with the same geographical coordinates. We will henceforth indicate this added mass-distribution with the letter M.

If $\triangle_0$ is supposed to be known, the formulae of §§ 2 and 3 allow the complete determination of the outside gravity-field of this mass-distribution M without any further neglections than those, given by the last terms of the formulae (7); in the same way the formula of STOKES may give the distance N_R between the sphere and the geoid of the above combination without neglecting more than these terms of (7).

We will now combine the mass-distribution M with the theoretical earth, which has been defined in the beginning of § 2, and of which the outside potential surface is the spheroid. This addition to the theoretical earth causes the potential surface to shift from the spheroid to a geoid, which will nearly coincide with the real geoid. If we suppose $\triangle_0$ to be known, the distances N'_0 between this geoid and the spheroid, and the anomaly $\triangle'_0$ on this geoid, may be computed with the formulae of §§ 2 and 3; we have to introduce in these formulae for h the distance between the speroid and the sphere.

We will now prove that the difference $\delta' = \triangle'_0 - \triangle_0$ is of the order of gf^3, i. e. of the order of the flattening with regard to $\triangle_0$ itself. If this is true we may neglect δ' as we have already neglected quantities of the same order in formula (9); these neglections are insignificant considering the accuracy of the determination of the gravity anomalies. And secondly; if we neglect the difference between the anomaly $\triangle'_0$ and the real anomaly $\triangle_0$ we may confound the geoid, which we have got by combining M with the theoretical earth, with the real geoid, so that the distance N'_0 can be considered to be also valid for this last geoid.

We need not doubt that δ' is of the order of gf^3 as far as δ' is given by the second and third terms of (16^C) in combination with (16^A), because $\frac{h}{R}$ is of the order of the flattening, while we assumed that $\triangle_0$ is of the order of gf^2. The only doubt, which might arise, concerns the effect of the first term of (16^C) for small r. We have seen in § 2, page 13, that because of this term, the difference δ, caused by local anomalies, may get a value of the order of $\frac{2h}{l} \triangle_0$, in which l is the horizontal extension of the anomaly. So we see that, locally, values of δ' may occur, exceeding the order of magnitude of gf^3. It is clear that these local values of δ' may be neglected for the determination of N'_0 in some point A of the spheroid which is far away; a more thorough investigation which we will not repeat here, confirms this opinion. The question is however, if they have no effect if they occur near to A.

In order to prove that this is not the case, we will suppose that our sphere with radius R is tangent to the spheroid in A, while the radius is supposed to coincide with the smallest radius of curvature of the spheroid in that point (see the supposition about this radius on page 1), so that the whole sphere is inside the spheroid. This is necessary if we want to apply the formulae of the previous paragraphs, because we assumed there: $\varrho \geqq R$. This supposition makes it of course impossible that the centre of the sphere should coincide with the centre of gravity of the earth, but it may easily be seen, that this only affects the deduction in this way, that ϱ in formula (8) for $\frac{\partial \gamma}{\partial \varrho}$ is not measured from the centre of gravity of the earth, but from the centre of the sphere at some distance of the order of fR. This means a deviation of the formula (9) of the order of gf^3, which does not exceed the term, which has already been neglected.

If we introduce in A an azimuthal coordinate α, representing the angle between the vertical plane through A in which the sphere and the spheroid osculate, and the vertical plane through A and through the point P, where we suppose that the value of δ' occurs, we find in P a distance h between the sphere and the spheroid, which may approximately be given by:

$$h = (f) \frac{r^2}{R} \sin^2 \alpha$$

in which (f) means a constant of the order of the flattening, while r represents again the distance AP.

We find thus that δ' is of the order of:

$$(f) \frac{r^2}{Rl} \times \triangle_P$$

in which $\triangle_P$ is the value of $\triangle_0$ in P. We see therefore that, even for small r, we need not fear that δ' would exceed the order of gf^3.

We may conclude that, if we neglect in $\triangle_0$ quantities of the order of gf^3, our problem is brought back to the determination in A of the distance N'_0, corresponding to the mass-distribution M, which has been defined in the beginning of this paragraph. As A is also a point of the sphere, N'_0 equals N_R, so that it can be computed by applying the formula of STOKES: we have to substitute in this formula the anomalies $\triangle_0$ and the radius R. We may notice however, that it will doubtless be somewhat better to substitute in this formula for R the mean earth's radius; we only chose a slightly different value for R in order to be able to apply the formulae of the previous paragraphs.

The conclusion at which we arrive, is that we are justified in using the formula of STOKES: the neglections in N will not exceed the order of Rf^3, i. e. one metre. We may apply the formula of STOKES as well for the determination of the general shape of the geoid as for the deduction of its local shape and obviously we may follow the same reasoning and arrive at the same conclusion for the formulae, derived from the formula of STOKES by differentiating it once or twice according to a direction tangent to the geoid.