Mathematics. — *A Representation of a quadrifold set of Twisted Cubics on the Points of a Linear Four-dimensional Space.* By J. W. A. VAN KOL. (Communicated by Prof. HENDRIK DE VRIES).

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§ 1. The twisted cubics k^3 that pass through two given points H_1 and H_2 and cut two given lines a_1 and a_2 twice, may be represented on the points of a linear four-dimensional space R_4 in the following way. In R_4 we choose two quadratic spaces $\Omega^2{}_1$ and $\Omega^2{}_2$ that have a double line l_1 resp. l_2. We suppose a projective correspondence to be established between the points of a_1 and the planes in $\Omega^2{}_1$ and another one between the points of a_2 and the planes in $\Omega^2{}_2$. Let a curve k^3 cut a_1 in A_1 and A'_1 and a_2 in A_2 and A'_2 and let R_1, R'_1, R_2 and R'_2 be the spaces that touch $\Omega^2{}_1$ resp. $\Omega^2{}_2$ along the planes associated to the said points. To k^3 we shall associate as image point the point where the plane of intersection of R_1 and R'_1 and that of R_2 and R'_2 cut each other. Inversely an arbitrary point in R_4 is the image of one curve k^3.

§ 2. Through an arbitrary point of l_1 resp. l_2 there pass two tangent spaces of $\Omega^2{}_1$ resp. $\Omega^2{}_2$. In this way in $\Omega^2{}_1$ and $\Omega^2{}_2$ there are defined quadratic involutions of planes to which quadratic involutions of points I_1 and I_2 on a_1 resp. a_2, are associated. Each of the ∞^3 curves k^3 that cut a_1 resp. a_2 in a pair of points of I_1 resp. I_2, has its image point on l_2 resp. l_1.

l_1 and l_2 are cardinal lines; an arbitrary point P of l_1 e.g. is the image of each of the ∞^2 curves k^3 that pass through the points of a_2 which are associated to the planes where $\Omega^2{}_2$ is touched by its spaces of contact through P.

The transversal t_1 resp. t_2 of a_1 and a_2 through H_1 resp. H_2 is completed by the conics through H_2 resp. H_1 that cut a_1, a_2 and t_1 resp. t_2, to ∞^3 curves k^3 that are represented in the points of the plane of intersection σ_1 resp. σ_2 of the spaces which touch $\Omega^2{}_1$ and $\Omega^2{}_2$ in the planes associated to the points of intersection of a_1 and a_2 with t_1 resp. t_2.

There are two singular planes σ_1 and σ_2 both of which cut l_1 and l_2; an arbitrary point P of σ_1 is the image of the ∞^1 curves k^3 formed by t_1 and the conics that pass through H_2, cut t_1 and cut a_1 and a_2 in the points corresponding to the planes where $\Omega^2{}_1$ an $\Omega^2{}_2$ are touched by its spaces of contact through P which are different from the spaces of contact $l_1\sigma_1$ and $l_2\sigma_1$.

$\sigma_1\sigma_2$ is a cardinal point that represents the ∞^2 curves k^3 formed by t_1, t_2 and the transversals of a_1 and a_2.

§ 3. Our set contains ∞^2 curves k^3 that are singular for the representation, viz. the curves k^3 that cut a_1 in a pair of points of l_1 and a_2 in a pair of points of l_2. Each of these curves k^3 has ∞^1 image points, viz. all the points of a transversal of l_1 and l_2.

§ 4. $\Omega^2{}_1$ and $\Omega^2{}_2$ are the loci of the image points of the curves k^3 that touch a_1 resp. a_2.

The surface of intersection O^4 of $\Omega^2{}_1$ and $\Omega^2{}_2$ is the locus of the image points of the curves k^3 that touch a_1 as well as a_2.

§ 5. Let us investigate the representation of the system Σ_1 of the curves k^3 that have a given chord b. The curves of Σ_1 cut a_1 as well as a_2 in pairs of points of a quadratic involution. To these quadratic point involutions on a_1 and a_2 there correspond quadratic plane involutions in $\Omega^2{}_1$ resp. $\Omega^2{}_2$. These involutions have the property that two spaces which touch $\Omega^2{}_1$ resp. $\Omega^2{}_2$ in planes that correspond to each other through this involution, have a plane of intersection lying in a fixed space through l_1 resp. l_2.

The plane of intersection α_b of these spaces is apparently the image plane of Σ_1.

Two planes α_{b_1} and α_{b_2} cut each other in one point. Hence:

There is one twisted cubic that passes through two given points and has four given chords.

O^4 and α_b cut each other in four points.

There are four twisted cubics that pass through two given points, have a given chord and touch two given lines.

§ 6. Let us call the image surface of the system Σ_2 of the curves k^3 that pass through a given point P, O_P. We determine the degree of O_P by examining the intersection of it and a plane α that touches $\Omega^2{}_1$ as well as $\Omega^2{}_2$. As there is one curve k^3 of Σ_2 that passes through a given point of a_1 as well as through a given point of a_2, α cuts O_P besides in the points $\alpha\, l_1$ and $\alpha\, l_2$ in one more point. l_1 and l_2 are single lines of O_P as through two given points of l_1 and l_2 there passes one curve k^3 of Σ_2. As, accordingly, α cuts O_P in all in three points, O_P is a cubic surface. We can show that O_P has *one conic that passes through the points $\sigma_1\, l_1$, $\sigma_1\, l_2$* and *$\sigma_1\, \sigma_2$ in common with σ_1 and one conic that passes through $\sigma_2\, l_1$, $\sigma_2\, l_2$ and $\sigma_1\, \sigma_2$ with σ_2.*

O_P and α_b have one point in common besides the points $\alpha_b\, l_1$ and $\alpha_b\, l_2$. Hence:

There is one twisted cubic that passes through three given points and has three given chords.

By applying the method indicated in § 8 we find that O_P and O_Q cut each other outside l_1 and l_2 in singular points only, whence:

There is no twisted cubic that passes through four given points and has two given chords.

The intersection of O^4 and O_P gives:

There are four twisted cubics that pass through three given points and touch two given lines.

§ 7. Let Ω_l be the image space of the system Σ_3 of the curves k^3 that cut a given line l. We determine the degree of Ω_l by means of the intersection with a line p that touches $\Omega_1{}^2$ as well as $\Omega_2{}^2$. p is the locus of the image points of the curves k^3 that pass through a definite point A_1 of a_1, through a definite point A_2 of a_2, and cut a_1 and a_2 outside A_1 resp. A_2 in points that correspond to each other through a certain projective correspondence between the points of a_1 and those of a_2. The number of points of intersection of p and Ω_l is, therefore, equal to twice the number of curves of Σ_3 that pass through two given points of a_1 as well as through a given point of a_2. This number is equal to two as the twisted cubics that pass through five given points and cut a given line, form a surface of the fifth degree that has triple points in the given points. Ω_l is, accordingly, of the fourth degree. We can show that l_1 and l_2 are double lines and that σ_1 and σ_2 are single planes of Ω_l.

§ 8. The intersection of Ω_l and Ω_m consists of σ_1, σ_2 and a surface O_{lm} of the degree 14, which is evidently the image surface of the system Σ_4 of the curves k^3 that cut two given lines l and m. l_1 and l_2 are quadruple lines of O_{lm} and σ_i has a curve of the sixth order that has triple points in the points $\sigma_i l_1$ and $\sigma_i l_2$ and a double point in the point $\sigma_1 \sigma_2$ in common with O_{lm}.

The intersection of O_{lm} successively with a_b and O^4 gives:

There are six twisted cubics that pass through two given points, have three given chords and cut two given lines.

There are 24 twisted cubics that pass through two given points, touch two given lines and cut two other given lines.

According to a theorem of PIERI[1]) the number of points of intersection of O_{lm} and O_P outside l_1 and l_2 is found by subtracting from the product of the degrees of O_{lm} and O_P the product of the multiplicities of l_1 on O_{lm} and O_P, the product of the multiplicities of l_2 on O_{lm} and O_P and the classes of the envelopes of the spaces through l_1 or l_2 that touch O_{lm} and O_P at the same point of one of these lines. The class of the envelope of the spaces through l_1 that touch O_{lm} and O_P at the same point of l_1, is equal to the number of spaces that pass

[1]) Rend. del Circolo Mat. di Palermo, t. V, 1891.

through an arbitrary point S and through l_1 and touch O_{lm} and O_P at the same point of l_1. It is easily proved that an arbitrary space through l_1 cuts O_P along l_1 and a conic that cuts l_1 once; accordingly this space touches O_P once, viz. in the point of intersection of l_1 and the said conic. An arbitrary space through l_1 cuts O_{lm} along the line l_1, which must be counted four times, and a curve of the tenth order that cuts l_1 in six points; consequently this space touches O_{lm} six times, viz. in the points of intersection of l_1 and the said curve of the tenth order. To an arbitrary point L_1 of l_1 we shall now associate the six points L'_1 of l_1 where O_{lm} is touched by the space that is defined by S and the plane touching O_P at L_1. Inversely through this correspondence there are associated to an arbitrary point L'_1 the four points L_1 where O_P is touched by the four spaces that are defined by S and the four planes touching O_{lm} at L'_1. The (4, 6)-correspondence between the points L_1 and L'_1 arising in this way, has 10 coincidences, hence the class in question is ten. Consequently the number of points where O_P and O_{lm} cut each other outside l_1 and l_2, is equal to $3 \times 14 - 2.1.4 - 2.10 = 14$. This number contains 4 points where the intersections of O_P and O_{lm} with σ_1 cut each other outside the points $l_1\sigma_1$, $l_2\sigma_1$ and $\sigma_1\sigma_2$, 4 points where the intersections of O_P and O_{lm} cut each other outside the points $l_1\sigma_2$, $l_2\sigma_2$ and $\sigma_1\sigma_2$ and the point $\sigma_1\sigma_2$ itself, which must be counted twice. There remain, accordingly, 4 points that are neither singular nor cardinal points. Thus we have found the following number, which, however, may be derived more simply in a direct way:

There are four twisted cubics that pass through three given points, have two given chords and cut two given lines.

If we apply the method indicated above to two surfaces O_{lm} and O_{no}, we find:

There are 36 twisted cubics that pass through two given points, have two given chords and cut four given lines.

§ 9. The intersection of O_{lm} and Ω_n consists of the lines l_1 and l_2, which must be counted eight times, two curves of the sixth order lying resp. in σ_1 and σ_2 and a curve k_{lmn} of the order 28 that is the image of the system Σ_5 of the curves k^3 that cut three given lines l, m and n. k_{lmn} cuts l_1 and l_2 in 14 points, as the number of points of intersection of k_{lmn} and l_1 as well as the number of points of intersection outside l_1 of k_{lmn} and a tangent space of $\Omega^2{}_1$ is equal to the number of curves of Σ_5 that pass through a given point of a_1. The number of points of intersection of k_{lmn} and σ_1 is equal to the number of conics that pass through H_2 and cut the six lines a_1, a_2, t_1, l, m and n (in different points). The conics that pass through H_2 and cut a_1, a_2, l, m and n form a surface of the degree 18 [1]) that is cut by t_1 outside the points of inter-

[1]) Cf. SCHUBERT, Kalkül der abzählenden Geometrie, p. 96, where the numbers of conics $P\nu^6 = 18$ and $P^2\nu^4 = 4$ are derived.

section of t_1 with a_1 and a_2, which are quadruple lines of the surface, in ten points. Accordingly σ_1 and σ_2 are cut by k_{lmn} in ten points.

The intersection of $\Omega^2{}_1$ and k_{lmn} gives:

There are 28 twisted cubics that pass through two given points, have a given chord, cut three given lines and touch another given line.

§ 10. We can further investigate the representations of several other systems, as the systems of the curves k^3 that touch one, two or three given planes, that touch a given plane and at the same time cut one or two given lines, that touch a given plane and at the same time pass through a given point and others.

The numbers that may be deduced in this way and those already found above are the following ones:

$P^4B^2 = 0$	$P^3B^2\nu^2 = 4$	$P^2B^3\nu^2 = 6$	$P^2B^2\nu^4 = 36$
$P^3B^3 = 1$	$P^3B^2\nu\varrho = 8$	$P^2B^3\nu\varrho = 12$	$P^2B^2\nu^3\varrho = 72$
$P^2B^4 = 1$	$P^3B^2\varrho^2 = 16$	$P^2B^3\varrho^2 = 24$	$P^2B^2\nu^2\varrho^2 = 144$
$P^3T^2 = 4$			$P^2B^2\nu\varrho^3 = 288$
$P^2BT^2 = 4$			$P^2B^2\varrho^4 = 576$
$P^3BT\nu = 4$	$P^2B^2T\nu = 4$	$P^2T^2\nu^2 = 24$	$P^2BT\nu^3 = 28$
$P^3BT = 8$	$P^2B^2T\varrho = 8$	$P^2T^2\nu\varrho = 48$	$P^2BT\nu^2\varrho = 56$
		$P^2T^2\varrho^2 = 96$	$P^2BT\nu\varrho^2 = 112$
			$P^2BT\varrho^3 = 224$

Here P indicates the condition that a twisted cubic pass through a given point, B that it have a given chord, ν that it cut a given line, T that it touch a given line and ϱ that it touch a given plane.

§ 11. From the above we can derive properties of different surfaces formed by systems of ∞^1 curves k^3 [1]).

The curves k^3 that touch a_1 and a_2 and cut a given line l, form a surface of the degree 24 that has 12-fold points in H_1 and H_2; a_1 and a_2 are eightfold lines and l is a quadruple line of this surface.

The curves k^3 that touch a_1 and cut two given lines l and m, form a surface of the degree 28 that has 14-fold points in H_1 and H_2; a_1 is an eightfold line, a_2 is a twelvefold line and l and m are quadruple lines of this surface. Etc.

1) Cf. also these Proceedings **30**, p. 1016 (1927).