Mathematics. — *On the Motion of a Plane Fixed System with Two Degrees of Freedom.* (Second Communication [1])). By Prof. W. VAN DER WOUDE.

(Communicated at the meeting of March 31, 1928).

§ 1. By the motion of a fixed system we always understand, at least in purely kinematical considerations, the motion of two fixed systems relative to each other; already in the usual indication of the problem we notice the peculiar lack of symmetry that strikes us in the further treatment [2]). In the simplest case, where this motion depends on one parameter, this lack is very little troublesome; it seems to me that already in the next case — the motion depending on two parameters — it is certainly worth while to pass on to a more symmetrical representation.

As in this case we choose a system of axes that is not fixed to either of the systems, the formulas are in the beginning slightly more complicated than in the usual method; this disadvantage disappears, however, as soon as we give this system of axes the movement that is prescribed by the problem so that the symmetry remains intact.

The followed method is briefly this: With the exception of a few special cases (§ 3) there always exists a definite line d, that is not fixed to either of the systems and that is the locus of the possible poles of rotation; this line may be chosen as the X-axis of a system of axes. It is then obvious that a definite point must be chosen as origin. In this way the formulas for the motion have become so simple that the known conclusions may be read at once.

For the sake of a more outward consequence the usual expressions "fixed" and "movable system" have been replaced by "the systems Σ_1 and Σ_2".

§ 2. Let OXY be a rectangular system of axes; the coordinates (x, y) of any point always relate to *this* system; whenever there is question of the components of a vector we always mean the projections of this

[1]) The earlier communication (these Proceedings, Vol. **29**, p. 652) gives a list of literature. I received for perusal an article on this subject of Dr. H. J. E. BETH, which will appear in the next number of the Nieuw Archief voor Wiskunde (2e reeks, deel XV, vierde stuk). The method of Dr. BETH, however, is entirely different from mine. Dr. BETH has always treated also the non holonomous cases; I restrict myself in this paper to the holonomous cases, although an extension would not be difficult.

[2]) Except in a few chapters of:

R. BRICARD. Leçons de Cinématique. Tome I. Paris, Gauthier—Villars; 1926.

vector on *this* system. Let further Σ_1 and Σ_2 be two plane fixed systems the motion of which relative to each other depends on two parameters u and v; we shall assume that the motion of both relative to OXY also depends on u and v, that, however, it is possible that the motion of one, e.g. of Σ_1, depends on one parameter, e.g. of u, but that in this case the motion of Σ_2 relative to OXY depends either on both u and v or on v only.

For convenience' sake there follows here first a summary of the formulas that express the displacements occurring in this motion. The elementary displacement relative to OXY of a point (x, y) that is fixed to Σ_i $(i = 1, 2)$ is defined by

$$\left.\begin{aligned} \delta x &= (\xi_1^{(i)} - \omega_1^{(i)}\, y)\, du + (\xi_2^{(i)} - \omega_2^{(i)}\, y)\, dv \\ \delta y &= (\eta_1^{(i)} + \omega_1^{(i)}\, x)\, du + (\eta_2^{(i)} + \omega_2^{(i)}\, x)\, dv \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here

$$\xi_1^{(i)},\ \xi_2^{(i)},\ \eta_1^{(i)},\ \eta_2^{(i)},\ \omega_1^{(i)},\ \omega_2^{(i)}$$

have the well known signification; between these quantities exist the relations [1])

$$\frac{\partial \omega_1^{(i)}}{\partial v} = \frac{\partial \omega_2^{(i)}}{\partial u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2^{\alpha})$$

$$\left.\begin{aligned} \frac{\partial \xi_1^{(i)}}{\partial v} - \frac{\partial \xi_2^{(i)}}{\partial u} &= \eta_2^{(i)}\, \omega_1^{(i)} - \eta_1^{(i)}\, \omega_2^{(i)} \\ \frac{\partial \eta_1^{(i)}}{\partial v} - \frac{\partial \eta_2^{(i)}}{\partial u} &= -\, \xi_2^{(i)}\, \omega_1^{(i)} + \xi_1^{(i)} \omega_2^{(i)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2^{\beta})$$

In the same way the elementary displacement relative to Σ_1 of a point (x, y) that is fixed to Σ_2 is given by

$$\left.\begin{aligned} \delta x &= (\xi_1 - \omega_1\, y)\, du + (\xi_2 - \omega_2\, y)\, dv \\ \delta y &= (\eta_1 + \omega_1\, x)\, du + (\eta_2 + \omega_2\, x)\, dv \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad (3)$$

ω_1 and ω_2 are the rotations of Σ_2 in its motion relative to Σ_1; ξ_1 is the projection on OX of the vector that expresses the velocity relative to Σ_1 of the point (o, o) fixed to Σ_2. If we consider (x, y) as a point that is fixed to Σ_1 and if in (3) we replace ξ_1 by $-\xi_1$ etc. the displacement relative to Σ_2 of a point fixed to Σ_1 is expressed by these equations.

In this case it follows from (1) and (3) that

$$\left.\begin{aligned} \xi_1 = \xi_1^{(2)} - \xi_1^{(1)};\ \xi_2 = \xi_2^{(2)} - \xi_2^{(1)};\ \eta_1 = \eta_1^{(2)} - \eta_1^{1};\ \eta_2 = \eta_2^{2} - \eta_2^{(1)} \\ \omega_1 = \omega_1^{(2)} - \omega_1^{(1)};\ \omega_2 = \omega_2^{(2)} - \omega_2^{(1)} \end{aligned}\right\} \quad . \quad (4)$$

[1]) Cf. e.g. G. DARBOUX: Théorie des Surfaces I, p. 67, 71 (Gauthier—Villars, Paris), or L. P. EISENHART: A Treatise on Differential Geometry, p. 168, 170 (Ginn and Co., Boston, New York, London). Our formulas (2) however are only identical with the cited ones when we replace $\xi_1^{(i)}$ by $-\xi_1^{(i)}$ etc. as l.c. always the inverse motion, the motion of OXY relative to Σ_1, is considered.

The equations (2^{β}) do not hold good for $\xi_1 \ldots . \omega_2$. (2^{α}) remains valid; we can see this by filling in $i=1$ and $i=2$ in (2^{α}) and by subtracting the two equations from each other. It appears that

$$\frac{\partial \omega_1}{\partial v} = \frac{\partial \omega_2}{\partial u} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

§ 3. We start from (3) and we consider, therefore, the motion of Σ_2 relative to Σ_1. We shall call the possible movements depending on two parameters the system $[\mathfrak{M}^2]$.

The locus of the poles is found from:

$$\delta x = \delta y = 0$$

and has, therefore, as equation

$$\left\| \begin{matrix} \xi_1 - \omega_1 y & \xi_2 - \omega_2 y \\ \eta_1 + \omega_1 x & \eta_2 + \omega_2 x \end{matrix} \right\| = 0$$

or

$$(\xi_1 \omega_2 - \xi_2 \omega_1) x + (\eta_1 \omega_2 - \eta_2 \omega_1) y + \xi_1 \eta_2 - \xi_2 \eta_1 = 0$$

If for the moment we exclude the cases where

$$\omega_1 = \omega_2 = 0$$

or

$$\left\| \begin{matrix} \xi_1 \, \eta_1 \, \omega_1 \\ \xi_2 \, \eta_2 \, \omega_2 \end{matrix} \right\| = 0,$$

(6) represents a straight line d.

It is evident that

$$\omega_1 = \omega_2 = 0$$

means that $[\mathfrak{M}^2]$ contains only translations and that

$$\left\| \begin{matrix} \xi_1 \, \eta_1 \, \omega_1 \\ \xi_2 \, \eta_2 \, \omega_2 \end{matrix} \right\| = 0$$

indicates that the system $[\mathfrak{M}^2]$ depends on one parameter only.

In the future we shall always exclude these cases.

§ 4. We shall now make the condition that d coincide with OX; for this it is necessary and sufficient that

$$\xi_1 = \xi_2 = 0$$

For the sake of a further simplification we first remark that owing to (5) we can introduce a new variable θ through

$$2\, d\theta = \omega_1 \, du + \omega_2 \, dv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^a)$$

We denote an integrating factor of $\eta_1 \, du + \eta_2 \, dv$ by $\frac{1}{2H(u, v)}$ so that we can put

$$2H \, d\tau = \eta_1 \, du + \eta_2 \, dv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^b)$$

We shall further introduce in H the variables θ and τ defined through (7^a) and (7^b) but for a constant, in stead of u and v.

The displacement relative to Σ_1 of a point (x, y) of Σ_2 is now expressed by

$$\left.\begin{aligned} \delta x &= \qquad\quad - 2\, y\, d\theta \\ \delta y &= 2\, H\, d\tau + 2\, x\, d\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

It is evident that $d\theta = 0$ means a translation and $d\tau = 0$ a rotation round the origin; also that θ is twice the angle between two lines one of which is fixed to Σ_1 and the other to Σ_2 and that τ may be replaced by any function of τ without the form of (I) changing.

We shall now represent the displacement relative to OXY of a point $(x\, y)$ that is fixed to Σ_1 by

$$\left.\begin{aligned} \delta x &= (U_1 - \Omega_1\, y)\, d\tau + [U_2 - (\Omega_2 - 1)\, y]\, d\theta \\ \delta y &= (V_1 - H + \Omega_1 x)\, d\tau + [V_2 + (\Omega_2 - 1)\, x]\, d\theta \end{aligned}\right\}; \quad . \quad . \quad . \quad (8)$$

accordingly for the displacement relative to OXY of a point of Σ_2 we have

$$\left.\begin{aligned} \delta x &= (U_1 - \Omega_1\, y)\, d\tau + [U_2 - (\Omega_2 + 1)\, y]\, d\theta \\ \delta y &= (V_1 + H + \Omega_1 x)\, d\tau + [V_2 + [\Omega_2 + 1)\, x]\, d\theta \end{aligned}\right\}. \quad . \quad . \quad . \quad (9)$$

From (8) as well as from (9) there follow relations (see 2^α and 2^β) between U_1, U_2, V_1, V_2, Ω_1, Ω_2 and H; through addition and subtraction these are simplified to

$$\left.\begin{aligned} &\frac{\partial \Omega_1}{\partial \theta} = \frac{\partial \Omega_2}{\partial \tau} && \frac{\partial U_1}{\partial \theta} - \frac{\partial U_2}{\partial \tau} = V_2\, \Omega_1 - V_1\, \Omega_2 - H \\ &V_1 + H\Omega_2 = 0 && \\ &\frac{\partial H}{\partial \theta} = U_1 && \frac{\partial V_1}{\partial \theta} - \frac{\partial V_2}{\partial \tau} = U_1\, \Omega_2 \;\; - U_2\, \Omega_1 \end{aligned}\right\} \quad . \quad . \quad (10)$$

§ 5. Let a definite displacement out of $[\mathfrak{M}^2]$ be defined by

$$\frac{d\theta}{d\tau} = \lambda\,;$$

if $(x, 0)$ is the pole of rotation for the motion of Σ_1 and Σ_2 relative to each other, we have

$$H + \lambda\, x = 0$$

Now in the system Σ_1 d turns about the point $(x', 0)$ for which in (9)

$$\delta\, y = 0$$

hence about the point that is defined by

$$V_1 - H + \Omega_1\, x' + \lambda\, [V_2 + (\Omega_2 - 1)\, x'] = 0\,;$$

in the system Σ_2 d turns about the point $(x'', 0)$ that is defined by

$$V_1 + H + \Omega_1\, x'' + \lambda\, [V_2 + (\Omega_2 + 1)\, x''] = 0.$$

The former two points coincide when

$$\Omega_1 x^2 + (V_1 - \Omega_2 H) x - V_2 H = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

if this is the case $(x, 0)$ and $(x'', 0)$ also coincide as might be expected. Through (11) two points are defined — at least if $\Omega_1 \pm 0$ —; we can call them the *stationary poles of rotation* (for the given position). For the moment we shall further put

$$\Omega_1 \neq 0.$$

In the future we shall always choose the middle between these stationary poles of rotation as origin of our system of coordinates of which so far we had defined the X-axis, not the origin. Now we have always

$$V_1 - H \Omega_2 = 0.$$

In connection with one of the formulas (10) it follows from this that

$$V_1 = \Omega_2 = 0.$$

It is impossible that H is identically equal to zero as in this case the motion of Σ_1 and Σ_2 relative to each other would only have one degree of freedom.

The formulas (8), (9), and (10) are now greatly simplified. We have already found

$$V_1 = \Omega_2 = 0,$$

further in (10)

$$\frac{\partial \Omega_1}{\partial \theta} = 0$$

hence Ω_1 is a function of τ only. Accordingly we can again denote $\int \Omega_1 \, d\tau$ by a new variable; if this is again called τ we have — cf. (10) —

$$\frac{\partial H}{\partial \theta} = U_1 ; \quad \frac{\partial V_2}{\partial \tau} = U_2$$

$$\frac{\partial U_1}{\partial \theta} - \frac{\partial U_2}{\partial \tau} = V_2 - H.$$

SUMMARISING. The displacement relative to Σ_1 of a point of Σ_2, hence any displacement out of $[\mathfrak{M}^2]$, is expressed by

$$\left.\begin{aligned} \delta x &= \qquad\quad -2\, y d\theta \\ \delta y &= 2\, H d\tau + 2\, x d\theta \end{aligned}\right\} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (I)$$

the displacement relative to OXY of a point of Σ_1 by

$$\left.\begin{aligned} \delta x &= \left(\frac{\partial H}{\partial \theta} - y\right) d\tau + \left(\frac{\partial V_2}{\partial \tau} + y\right) d\theta \\ \delta y &= (-H + x)\, d\tau + (V_2 - x)\, d\theta \end{aligned}\right\} ; \quad . \quad . \quad . \quad . \qquad (II)$$

the displacement relative to OXY of a point of Σ_2 by

$$\left.\begin{aligned}\delta x &= \left(\frac{\partial H}{\partial \theta} - y\right) d\tau + \left(\frac{\partial V_2}{\partial \tau} - y\right) d\theta \\ \delta y &= (H + x)\, d\tau + (V_2 + x)\, d\theta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (III)$$

Between the functions $H(\tau, \theta)$ and $V_2(\tau, \theta)$ there only exists the relation

$$\frac{\partial^2 H}{\partial \theta^2} - \frac{\partial^2 V_2}{\partial \tau^2} + H - V_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IV)$$

§ 6. *Simple Results.* Let any displacement be given by

$$\frac{d\theta}{d\tau} = \lambda.$$

The pole of rotation for the displacement of Σ_1 and Σ_2 relative to each other is the point $P\left(-\frac{H}{\lambda}, 0\right)$; in the plane Σ_1 d turns round the point Q_1 $\left(\frac{H-\lambda V_2}{1-\lambda}, 0\right)$ for which

$$\delta y = 0;$$

in Σ_2 d turns round Q_2 $\left(\frac{-H-\lambda V_2}{1+\lambda}, 0\right)$.

In any position there exists a projective correspondence between P and Q_1 and also between P and Q_2 [1]).

Special cases:

1. The three points coincide in the stationary poles of rotation.
2. If O is the pole of rotation V_2 and $-V_2$ are the abscissae of Q_1 and Q_2.
3. If Σ_1 and Σ_2 have a translation relative to each other (P lies at infinity; $\lambda = 0$), d turns round Q_1 (H, o) in Σ_1, round $(-H, o)$ in Σ_2.
4. If a has a translation in Σ_1, $(-H, o)$ is the pole of rotation; if d has a translation in Σ_2, (H, o) is the pole of rotation.

O is always the middle between the found pairs of points.

The motion of d in Σ_1 depends on one parameter only in the case that

$$-H + V_2 = 0$$

for then in (II)

$$\delta y = 0$$

for any point ($x = 0$) if the displacement is defined by

$$d\tau - d\theta = 0;$$

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[1]) Cf. BETH l.c. who derives a complete classification of the movements with two parameters, including the non-holonomous ones, from the projective relations between P, Q_1 and Q_2.

in this case all points of d move on d and d is a fixed line in Σ_1. For any other displacement d turns in Σ_1 about the point (H, o).

But on the same condition the motion of d in Σ_2 depends on only one parameter; the displacement defined by

$$d\tau + d\theta = 0$$

leaves d at rest in Σ_2; for any other displacement it turns about the point $(-H, o)$. This gives the

THEOREM OF KOENIGS. *If the displacement of d in Σ_1 depends on only one parameter, this is also the case with the motion of d in Σ_2. The displacements that leave d at rest in Σ_1 and those that leave it at rest in Σ_2, are different.*

A second interesting special case is the following one. Suppose

$$V_2 = 0.$$

In this case the two stationary poles of rotation coincide in O; they correspond to the displacement for which

$$d\tau = 0.$$

In order to examine the displacement of O relative to Σ_1 and Σ_2 we have only to calculate δx and δy in (II) and (III) for $O\,(o, o)$ and to replace them by their opposites. Then evidently

$$\delta x = \delta y = 0.$$

The origin is accordingly at rest in Σ_1 and Σ_2, in other words:

If V_2 is identically equal to zero the system of movements $[\mathfrak{M}^2]$ of the systems Σ_1 and Σ_2 relative to each other contains a finite rotation about a point that is fixed to Σ_1 and to Σ_2.

This leads to the problem: when does the system $[\mathfrak{M}^2]$ contain finite rotations about a point that is fixed to Σ_1 and to Σ_2?

The pole of rotation must be fixed in Σ_1 and Σ_2: d always passes through the pole, hence d turns about the same point in Σ_1 and Σ_2, which point is accordingly one of the stationary poles of rotation. The problem is therefore: is it possible that for the *finite* movement defined by

$$\sqrt{H}\, d\tau + \sqrt{V_2}\, d\theta = 0$$

the pole of rotation $(\pm \sqrt{H V_2}, o)$ is fixed to Σ_1. The vector of the displacement of a point (x, y) relative to Σ_1 is given by the components $dx - \delta x$, $dy - \delta y$ when δx and δy are taken from (II) and if for $\dfrac{d\theta}{d\tau}$ $(\pm \sqrt{H V_2}, o)$ is substituted.

It is therefore necessary that for one of the points $(\pm \sqrt{H V_2}, o)$

$$dx - \delta x = 0$$

or

$$3\left(H\sqrt{V_2}\,\frac{\partial V_2}{\partial \tau} + V_2\sqrt{H}\,\frac{\partial H}{\partial \theta}\right) + V_2\sqrt{\theta_2}\,\frac{\partial H}{\partial \tau} + H\sqrt{H}\,\frac{\partial V_2}{\partial \theta} = 0.$$

If H and V_2 satisfy this equation besides (IV), $[\mathfrak{M}^2]$ contains a finite rotation about a fixed point.

As might be expected this equation is always satisfied if $V_2 = 0$.

Does the system $[\mathfrak{M}^2]$ contain *finite* rectilinear translations? The translation is always parallel to d (cf. (I)); however we have seen that d turns about a point at finite distance in Σ_1 as well as in Σ_2 and, accordingly, has neither a fixed direction in Σ_1 nor in Σ_2. A finite rectilinear translation can, therefore, only be expected in the case that we have excluded until now; in that case it is in fact contained in $[\mathfrak{M}^2]$ (cf. §§ 5, 7; $\Omega_1 = 0$).

§ 7. We shall now briefly discuss the case that has so far been excluded where (cf. § 5)

$$\Omega_1 = 0$$

If we suppose in the first place

$$V_1 - \Omega_2 H \neq 0 \quad \ldots\ldots\ldots \quad (a)$$

it appears from (II) § 5 that:

there is only one stationary pole of rotation (or the other one lies at infinity).

If we choose this stationary pole of rotation (not at infinity) as origin we have:

$$V_2 = 0.$$

It appears further from (10) that Ω_2 is a function of θ only and $\Omega_2 H^2$ of τ only, hence

$$H = \frac{\varphi(\tau)}{\sqrt{\Omega_2}}$$

We shall introduce $\int \varphi(\tau)\, d\tau$ as new variable; if we call this variable again τ the form of (I) does not change and H is a function of θ only. Further

$$\Omega_2 = \frac{1}{H^2}, \quad V_1 = -\frac{1}{H}, \quad U_1 = \frac{\partial H}{\partial \theta}.$$

Any displacement relative to Σ_1 of a point of Σ_2 is still expressed by

$$\left.\begin{aligned} \delta x &= \qquad\qquad - 2\, y\, d\theta \\ \delta y &= 2\, H(\theta)\, d\tau + 2\, x\, d\theta \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\mathrm{I})$$

the displacement relative to OXY of a point of Σ_1 by

$$\left.\begin{aligned} \delta x &= \frac{\partial H}{\partial \theta} d\tau + \left[U_2 - \left(\frac{1}{H^2} - 1\right) y \right] d\theta \\ \delta y &= -\left(\frac{1}{H} + H\right) d\tau + \left(\frac{1}{H^2} - 1\right) x\, d\theta \end{aligned}\right\}; \quad \ldots \quad (\mathrm{II})$$

the displacement relative to OXY of a point of Σ_2 by

$$\left.\begin{aligned}\delta x &= \frac{\partial H}{\partial \theta}\,d\tau + \left[U_2 - \left(\frac{1}{H^2}+1\right)y\right]d\theta \\ \delta y &= -\left(\frac{1}{H}-H\right)d\tau + \left(\frac{1}{H^2}+1\right)x\,d\theta\end{aligned}\right\} \quad . \; . \; . \quad \text{(III)}$$

Between the functions U_2 and H there exists the relation

$$\frac{\partial^2 H}{\partial \theta^2} - \frac{\partial^2 U^2}{\partial \tau^2} + \frac{1}{H^3} + H = 0.$$

This proves: *if always (i. e. for any pair of values of u and v) $\Omega_1 = 0$, the system $[\mathfrak{M}^2]$ contains a rectilinear finite translation of Σ_1 and Σ_2 relative to each other, corresponding to $d\theta = 0$. If besides $H = \pm 1$ the motion of d in Σ_1 as well as in Σ_2 depends on one parameter only.*

There remain the possibilities

$$\Omega_1 = 0, \quad V_1 - \Omega_2 H = 0, \quad V_2 \neq 0 \quad . \; . \; . \; . \; . \quad (\beta)$$

d does not contain any stationary pole of rotation.

$$\Omega_1 = 0, \quad V_1 - \Omega_2 H = 0, \quad V_2 = 0. \quad . \; . \; . \; . \; . \quad (\gamma)$$

Any point of d is a stationary pole of rotation.

If, accordingly, Ω_1, $V_1 - \Omega_2 H$ and V_2 are always equal to zero, about any point of d a finite rotation is possible. I intend to come back to this remarkable case in a later short paper.

In the following discussion of the quantities of the second order this case $\Omega_1 = o$ is again excluded.

§ 8. *The quantities of the second order.*

If we start from a given original position, for a given $\frac{d\theta}{d\tau}$ the tangent to the path in Σ_1 of any point of Σ_2 is defined (and inversely); the quantities of the second order, e.g. the radius of curvature of any point in its path, are not defined before also $\frac{d^2\theta}{d\tau^2}$ is given.

We shall now put

$$\frac{d\theta}{d\tau} = \lambda, \quad \frac{d^2\theta}{d\tau^2} = \lambda'$$

and we shall only consider the system of infinitesimal displacements where λ is kept constant and λ' is variable. In other words, *we choose movements from $[\mathfrak{M}^2]$ with a fixed pole of rotation* where to any point a definite tangent to its path is already assigned.

For the present we assume that the movement of Σ_2 relative to Σ_1 is given by the functions ξ, η, ω that depend on one parameter t (the time) and that the movement of Σ_1 relative to OXY is given by $\xi^{(1)}$, $\eta^{(1)}$, $\omega^{(1)}$ that also depend on t only. For the components of the velocity- and acceleration relative to Σ_1 of a point (x, y) of Σ_2 we have

$$v_x = \xi - \omega y, \quad v_y = \eta + \omega x.$$

$$J_x = \frac{dv_x}{dt} + \omega^{(1)} v_y = \frac{d\xi}{dt} + \omega^{(1)} \eta - (\eta^{(1)} + \eta)\,\omega - \frac{d\omega}{dt} y - \omega^2 x,$$

$$J_y = \frac{dv_x}{dt} - \omega^{(1)} v_x = \frac{d\eta}{dt} - \omega^{(1)} \xi + (\xi^{(1)} + \xi)\,\omega + \frac{d\omega}{dt}\, x - \omega^2 y$$

The radius of curvature of a point (x, y) of Σ_2 in its path in Σ_1 is given by

$$\frac{1}{R^2} = \frac{(V_x J_y - J_x V_y)^2}{V^6}$$

PROOF. If we use fixed axes we have

$$\frac{1}{R^2} = \frac{(x' y'' - y' x'')^2}{(x'^2 + y'^2)^3}$$

i.e. $\dfrac{1}{R^2}$ is equal to the square of the vector product of the velocity and the acceleration divided by the sixth power of the velocity. That is exactly what the above mentioned formula expresses.

In the same way it appears that the center of curvature in the path of $M(x, y)$ is the point

$$\mu\left(x - \frac{V_y}{V_x J_y - V_y J_x} V^2, \quad y + \frac{V_x}{V_x J_y - V_y J_x} V^2\right)$$

We now pass on to the case in question by the following substitutions (cf. the formulas (I) and (II), § 5)

$$\xi = 0, \quad \eta = 2H, \quad \omega = 2\lambda$$

$$\xi^{(1)} = \frac{\partial H}{\partial \theta} + \lambda \frac{\partial V_2}{\partial \tau}, \quad \eta^{(1)} = -H + \lambda V_2, \quad \omega^{(1)} = 1 - \lambda,$$

$$\frac{d}{dt} = \frac{\partial}{\partial \tau} + \lambda \frac{\partial}{\partial \theta}.$$

These entirely determine the elements of the second order.

§ 9. As a first example we shall determine the system of inflexional circles for these displacements.

First we find

$$V_x = -2\lambda y; \quad V_y = 2H + 2\lambda x$$

$$J_x = 2\dot{H} - 2\lambda^2 V_2 - 4\lambda^2 - 2\lambda' y$$

$$J_y = 2\frac{\partial H}{\partial \tau} + 4\lambda \frac{\partial H}{\partial \theta} + 2\lambda^2 \frac{\partial V_2}{\partial \tau} - 4\lambda^2 y + 2\lambda' x.$$

The equation of the inflexional circle, i.e. the locus of the points where the curve has an infinite radius of curvature, runs:

$$V_x J_y - V_y J_x = 0$$

or

$$2\lambda^3(x^2+y^2) - (\lambda H + 2\lambda^2 H - \lambda^3 V_2)\,x - \\ -\left(\lambda\frac{\partial H}{\partial \tau} + 2\lambda^2\frac{\partial H}{\partial \theta} + \lambda^3\frac{\partial V_2}{\partial \tau} - \lambda' H\right) y - H(H - \lambda^2 V_2) = 0.$$

As λ is a constant, λ' a variable parameter, this gives:

For all displacements about the same pole of rotation the inflexional circles form a pencil; the base points lie on d; one of them is the pole of rotation, the other a point $H\left(\frac{H-\lambda^2 V_2}{\lambda^2}, 0\right)$. *In all these displacements H describes a point of inflexion.*

We found above: if $\mu(\xi, \eta)$ is the center of curvature of $M(x, y)$ we have

$$\xi = x - \frac{V_y}{V_x J_y - J_x V_y} V^2, \quad \eta = y + \frac{V_x}{V_x J_y - V_y J_x} V^2.$$

We substitute the values indicated for V_x, V_y, J_x and J_y but at the same time, in order to simplify the formulas, we choose the pole $\left(-\frac{H}{\lambda}, 0\right)$ of the movement as new origin, i.e. we put

$$\bar{x} = x + \frac{H}{\lambda}, \quad y = \bar{y}$$

$$\bar{\xi} = \xi - \frac{H}{\lambda} = \bar{x} - \frac{V_y}{V_x J_y - V_y J_x} V^2, \quad \bar{\eta} = \eta.$$

Thus we find

$$\bar{\xi} = \frac{(2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y})\,\bar{x}}{8\lambda^3(\bar{x}^2+\bar{y}^2) + 2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y}},$$

$$\bar{\eta} = \frac{(2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y})\,\bar{y}}{8\lambda^3(\bar{x}^2+\bar{y}^2) + 2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y}}$$

Here B and C are functions of λ, τ and θ, i.e. in all the considered elementary displacements — where the initial position and the value $\lambda = \frac{\partial \theta}{\partial \tau}$ have been chosen — they have the same values; λ' is a parameter that assumes any value.

If for the moment we choose also λ' constant, any point M has a definite center of curvature μ; in this case the aforesaid formulas express a well known quadratic correspondence between M and μ; it is especially interesting that in the inverse movement M is the center of curvature of μ. Analytically this means that $\bar{x}$ and $\bar{y}$ are expressed in $\bar{\xi}$ and $\bar{\eta}$ through similar formulas.

The correspondence is quadratically involutory; to any line l described by M (or μ), there corresponds a conic as locus of μ (or M). The latter is sometimes called the conic of RIVALS, associated to l.

If now again we consider λ' as a parameter, it is at once evident [1]) that:

The conics of RIVALS *that in the different displacements about the same pole of rotation correspond to the same line, form a pencil. The points of d — and these only — always have the same center of curvature.*

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[1]) H. J. E. BETH, l.c.