Mathematics. — *Derivation of some Numbers relating to the System of Biquadratic Curves that pass through Five Points lying in an R_4.* By J. W. A. VAN KOL. (Communicated by Prof. HENDRIK DE VRIES.)

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§ 1. Let us consider the biquadratic involutory transformation in a linear four-dimensional space R_4 defined by the equations written in homogeneous coordinates:

$$x_i \, x_i' = \varrho \; (i = 1, \ldots, 5).$$

Let $A_1 \, A_2 \, A_3 \, A_4 \, A_5$ be the fundamental simplex.

Through this transformation to an angular point of the simplex there are associated ∞^3 points, viz. the points of the space defined by the other four angular points. To any point of a side of the simplex there correspond ∞^2 points, viz. the points of the "opposite" side plane. To any point of a side plane there correspond ∞^1 points, viz. the points of the "opposite" side.

An arbitrary linear space is transformed into a biquadratic space Ω^4 containing the side planes of the simplex; the sides of the simplex are double lines, and the angular points are triple points.

An arbitrary plane is transformed into a surface of the sixth degree F^6 containing the sides of the simplex and having triple points in the angular points.

An arbitrary line is transformed into a biquadratic (four-dimensional twisted) curve k^4 passing through $A_1, \ldots, A_5$. In this way we have defined a one-one representation of the curves k^4 passing through $A_1, \ldots, A_5$ on the lines of R_4. This representation shows at once that *through seven given points there passes one curve k^4.*

Through an arbitrary point there pass $4.6 - 5.3 = 9$ lines that cut a curve k^4 and a surface F^6 in different points. Hence: *There are nine curves k^4 that pass through six given points and cut a given plane and a given line.*

In the same way we can deduce: *There are three curves k^4 that pass through six given points, cut a given plane and cut a given line passing through one of the given points outside that point.*

And: *There are five curves k^4 that pass through six given points, cut a given line and cut a given plane passing through one of the given points outside that point.*

The curves k^4 that pass through six given points and cut a given plane, form a space of the ninth degree Ω^9 that has sextuple points in the given points; the given plane and the planes through three of the given points containing the curves k^4 that are degenerate in two conics which cut each other once, are single planes in this space; each of the spaces through four of the given points contains a surface of the fifth degree formed by twisted cubics that are parts of curves k^4 degenerate in a twisted cubic and a line which cut each other once.

The curves k^4 that pass through six given points and cut a given line, form a surface of the ninth degree [1]) which has quadruple points in the given points; this surface contains the joins of the given points, the given line, and the twisted cubics that pass through four of the given points and cut the join of the remaining two given points and the given line.

The lines through an arbitrary point P that touch a space Ω^4, form a "conical space" of the twelfth degree of which the lines PA_i ($i = 1, \ldots, 5$) are sextuple lines. This space is cut by a curve k^4 outside the points A_i in $4.12 - 5.6 = 18$ points. Hence:

There are eighteen curves k^4 that pass through six given points, cut a given line and touch a given linear space.

In the same way we can show: *There are six curves k^4 that pass through six given points, cut a given line through one of the given points outside that point, and touch a given linear space.*

The curves k^4 that pass through six given points $A_1, \ldots, A_6$ and touch a given linear space R_3, form a space of the eighteenth degree that has twelvefold points in $A_1, \ldots, A_6$; the planes through three of the given points are double planes in this space; each of the spaces through four of the given points contains a surface of the tenth degree formed by twisted cubics that are parts of curves k^4 which are degenerate in a twisted cubic and a line cutting each other once. The points of contact with R^3 lie on a quadratic surface [2]). For in the plane $\alpha \equiv A_1 A_2 A_3$ we can indicate two such points, viz. the points where the line αR_3 is touched by the conics passing through A_1, A_2, A_3 and the point of intersection of α and the plane $A_4 A_5 A_6$ and touching αR_3.

The transversals of two curves $k^4{}_1$ and $k^4{}_2$ that pass through $A_1, \ldots, A_5$, form a surface Ω^{11} of the degree eleven with sextuple points in $A_1, \ldots, A_5$. The number of points of intersection of Ω^{11} and an arbitrary line l is equal to the number of points where the axial projections of $k^4{}_1$ and $k^4{}_2$ out of l on an arbitrary plane cut each other outside the projections of $A_1, \ldots, A_5$, hence equal to $4^2 - 5 = 11$. The number of points of intersection outside A_1 of Ω^{11} and a line m through A_1 is equal to the number of points where the axial projections of $k^4{}_1$ and $k^4{}_2$ out of l on an arbitrary plane cut each other outside the axial projections of

[1]) Cf. Dr. G. SCHAAKE. A quadruple involution in space, these Proceedings **29**, p. 811 in which paper also several other of the results found above are derived in another way.

[2]) This surface is identical with the quadratic surface ω^2 mentioned in the paper of Dr. SCHAAKE cited above, p. 808.

$A_2, \dots, A_5$, hence equal to $3^2 - 4 = 5$. Ω^{11} is cut by a third curve $k^4{}_3$, which passes through $A_1, \dots, A_5$ outside these points in $4 . 11 - 5 . 6 = 14$ points. Consequently:

There are fourteen curves k^4 that pass through five given points and cut three given lines.

In a similar way we can show: *There are five curves k^4 that pass through five given points, cut two given lines, and cut one given line through one of the given points outside that point. There are ten curves k^4 that pass through five given points and cut three given lines two of which intersect each other, in different points.*

The curves k^4 that pass through five given points and cut two given lines, form a space of the fourteenth degree with ninefold points in the given points; the given lines are quadruple lines and the planes through three of the given points are single planes in this space; each of the spaces through four of the given points contains two surfaces of the fifth degree formed by twisted cubics that are parts of degenerate curves k^4.

§ 2. The congruence of the curves k^4 that pass through six given points $A_1, \dots, A_6$ and cut a given plane α, may be represented on α by associating to each of the curves k^4 of the congruence as image its point of intersection with α. This representation has twenty singular points, viz. the points S_{ikl} where α is cut by the planes $A_i\, A_k\, A_l$ ($i, k, l = 1, \dots 6$; i, k, l unequal). S_{ikl} is the image of the ∞^1 curves k^4 of the congruence that are degenerate in two conics cutting each other of which one individual passes through S_{ikl}.

From a number deduced in § 1 it follows that the system of the curves k^4 that cut besides a second given plane φ, is represented on a curve k_φ of the ninth order. k_φ passes through the points S_{ikl}. Two curves k_φ and k_ψ cut each other outside the singular points in 61 points.

Hence: *There are 61 curves k^4 that pass through six given points and cut three given planes.*

In the same way we can deduce:

The system of the curves k^4 that also touch a given linear space, is represented on a curve of the eighteenth order with double points in the points S_{ikl}.

There are 122 curves k^4 that pass through six given points, cut two given planes and touch a given linear space.

There are 244 curves k^4 that pass through six points, cut a given plane and touch two given linear spaces.

Let k be the image curve of the system of the curves k^4 that pass through $A_1, \dots, A_6$, cut α twice and, accordingly, have two image points each. The order of k is evidently equal to the number of curves k^4 that pass through $A_1, \dots, A_6$, cut a given line l and cut a given plane α through l outside l; hence it is also equal to the number of the points where the surface F^9 formed by the curves k^4 that pass through $A_1, \ldots, A_6$ and cut l, is cut by a outside l. According to a theorem of PIERI [1]) the latter number is equal to the product of the degrees of F^9 and a, diminished by the product of the multiplicities of l in F^9 and a, and by the class of the envelope of the linear spaces that touch F^9 and a in the same point of l and, accordingly, contain a.

As, according to § 1, the points of contact of the curves k^4 that pass through $A_1, \ldots, A_6$ and touch a given linear space R_3 through a, lie on a quadratic surface that is cut by l in two points, this class is equal to two. Consequently k is of the order $9 - 1 - 2 = 6$. k has single points in the points S_{ikl}. The intersection of k and the image curves found above gives:

There are seventeen curves k^4 that pass through six given points, cut one given plane twice and cut another given plane once.

There are 34 curves k^4 that pass through six given points, cut a given plane twice and touch a given linear space.

From the above we can also deduce properties of surfaces formed by systems of ∞^1 curves k^4, such as:

The curves k^4 that pass through six given points and cut a given plane twice, form a surface of the degree 17 that has sevenfold points in the given points and cuts the given plane along a curve of the sixth order; the joins of the given points are single lines of the surface; each plane through three of the given points is cut by the surface in three of these lines and a conic; each space through four of the given points is cut by the surface in six lines, four conics and a twisted cubic; the surface itself is a double surface of Ω^9 (§ 1).

The curves k^4 that pass through six given points and cut two given planes, form a surface of the degree 61 that has 26-fold points in the given points and cuts the given planes in curves of the ninth order; the joins of the given points are quintuple lines of the surface; each plane through three of the given points is cut by the surface in three of these quintuple lines and two conics; each space through four of the given points is cut by the surface in six quintuple lines, eight conics and five twisted cubics.

§ 3. The congruence of the curves k^4 that pass through five given points $A_1, \ldots, A_5$ and cut a given line a twice, may be represented on a plane α in the following way. We choose a conic k^2 in α and we suppose a projective correspondence to be established between the points of a and those of k^2. To a curve k^4 of the congruence that cuts a in A and A', we shall now associate the point of intersection of the lines that touch k^2 in the points that are associated to A and A'.

This representation does not contain any singular elements.

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[1]) Cf. Rend. del Circolo Mat. di Palermo, t. **5**, 1891.

It follows from numbers that have been deduced in § 1, that the system of the curves k^4 of the congruence that cut a given plane, is represented on a curve of the third order and that the system of the curves k^4 of the congruence that touch a given linear space, is represented on a curve of the sixth order. From this we can deduce the following numbers:

There are nine curves k^4 that pass through five given points, cut a given line twice and cut two given planes. There are eighteen curves k^4 that pass through five given points, cut a given line twice, cut a given plane and touch a given linear space. There are thirty six curves k^4 that pass through five given points, cut a given line twice and touch two given linear spaces.

The curves k^4 that pass through five given points, cut a given line twice and cut a given plane, form a surface of the ninth degree with triple points in the given points; the given line is a triple line of the surface; each space through four of the given points is cut by the surface in three twisted cubics; the given plane is cut by the surface in a conic. The latter property follows from the number:

There are two curves k^4 that pass through five given points, cut a given line once and cut another given line twice.

This number may be deduced in the way indicated in § 1. But it follows also from the theorem that the curves k^4 which pass through five given points and cut a given line twice, lie in the quadratic space that has the given line as double line and passes through the five given points.

§ 4. The congruence of the curves k^4 that pass through five given points $A_1, \dots, A_5$ and cut two given lines a_1 and a_2, may be represented on a plane α in the following way. We choose two points K and L in α. We suppose a projective correspondence to be established between the points of a_1 and the rays of the plane pencil (K, α) and another one between the points of a_2 and the rays of the plane pencil (L, α). To a curve k^4 of the congruence which cuts a_1 in P and a_2 in Q, we associate the point of intersection of the rays that are associated to P and Q.

Let us call the points of a_1 and a_2 corresponding to the line KL that belongs to both plane pencils resp. A'_1 and A'_2. The curve k^4 of the congruence that passes through A'_1 and A'_2, is a singular curve; it has any point of KL as image.

K and L are singular points; K and L are resp. the images of the curves k^4 of the congruence that pass through A'_2 resp. A'_1.

The twisted cubic which passes through $A_1, \dots, A_4$ and cuts a_1 and a_2, is a part of ∞^1 degenerate curves k^4 that are represented in the same point S_5. In this way we find 5 singular points S_i $(i = 1, \dots, 5)$.

The conic that passes through A_1 and A_2, cuts a_1, a_2 and cuts the plane $A_3 A_4 A_5$, is a part of ∞^1 degenerate curves k^4 that are represented in the same point T_{12}. Thus we find 10 more singular points $T_{ik} (i, k = 1, \ldots, 5;\ i \neq k)$.

It follows from numbers that have already been found:

The system of the curves k^4 of the congruence that cut a given plane, is represented on a curve of the order 18 with ninefold points in K and L, triple points in S_i and single points in T_{ik}.

The system of the curves k^4 of the congruence that touch a given linear space, is represented in a curve of the order 36 with 18-fold points in K and L, sextuple points in S_i and double points in T_{ik},

This gives again the numbers:

There are 107, 214, 428 curves k^4 that pass through five given points, cut two given lines and besides resp. cut two given planes, cut one given plane and touch a given linear space, touch two given linear spaces.

Finally we can again indicate properties of surfaces and spaces formed by systems of ∞^1 resp. ∞^2 curves k^4, as:

The curves k^4 that pass through five given points, cut two given lines and cut a given plane, form a surface of the degree 107 with 45-fold points in the given points. The given lines and also the joins of the given points are ninefold lines of the surface. The latter results from the property that the twisted cubics which pass through three given points and cut three given lines, form a surface of the ninth degree [1]). The surface has further a curve of the fourteenth order in common with the given plane, three ninefold lines and a conic with each of the planes through three of the given points, and six ninefold lines, four conics, one triple and twelve single twisted cubics with each of the spaces through four of the given points.

1) Cf. Wiskundige opgaven, deel **14**, opg. 146.