Mathematics. — *Linear Congruences of Twisted Cubics that Cut at least one Fixed Line Twice.* By Prof. JAN DE VRIES.

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§ 1. If the curves ϱ^3 of a congruence Γ cut a fixed line b twice, and through an arbitrary point there passes only one ϱ^3, Γ may be represented on the points of a plane β through b. As the image of ϱ^3 we consider the point R outside b that it has in common with β.

If the curves of Γ pass through *four* fixed points A_k [1]) the curves that cut b in B, lie on the quadratic cone β^2 that has the lines b and BA_k as generatrices (any ϱ^3 has seven points in common with β^2). This system is represented on a line r of β.

Let k be a chord of a ϱ^3. The hyperboloid H through three of the points A that contains b and k, passes through ϱ^3. It contains ∞^1 curves ϱ^3; for if we project H out of one of its points O on a plane, this contains a pencil of curves c^3 all of which have a double point in one of the two cardinal points of the representation and pass through the second cardinal point and the images of the points A. These c^3 are the images of curves ϱ^3 through A that have the lines of the scroll to which b belongs, as bisecants. This *pencil* of ϱ^3 is represented on the line r that H has still in common with β.

§ 2. To Γ belong four systems Σ_k of figures each of which consists of a line and a conic ϱ^2. The system Σ_1 is formed by the conics in the plane α_{234} that have the points A_2, A_3, A_4 and the transit of b as basis and each of which is completed by a ray t_1 of the plane pencil about A_1 in the plane A_1b. It is represented on a line c_1.

The four lines c_k form a *quadrilateral*; each of the six angular points is the image of a figure that belongs to two of the systems Σ and, accordingly, consists of *three* lines. Hence Γ contains *six* figures each of which consists of two crossing lines and one line that cuts them.

The system Λ of the ϱ^3 that cut a line l, is represented on a *conic* λ^2. For the image line of a conic β^2 contains the images R of the two ϱ^3 belonging to β^2 that rest on l. On two lines l_1 and l_2 there rest, therefore, four ϱ^3 and the system Γ lies on a surface Λ^4. On this b is a *double line*. A ϱ^3 outside Λ^4 can only cut Λ^4 on b and in A_k; hence Λ^4 has *double points* in A_k.

[1]) Another representation of the congruence ($4A$, b) has been treated by Dr. G. SCHAAKE in a paper in these Proceedings, Vol. **28**, p. 776.

An arbitrary line l cannot be a chord of a ϱ^3 belonging to Γ; for the scrolls on the hyperboloids H together form a complex. The complex-cone projects a ϱ^3, is therefore quadratic, and the A_k are *cardinal points; the complex* is accordingly *tetrahedral.*

§ 3. The ϱ^3 of an H define an I^3 on the conic that H has in common with a plane φ; *four* of these curves touch φ. Accordingly on the image line r there lie four points R originating from curves ϱ^3 that belong to the system Φ which is formed by the ϱ^3 that touch a plane φ. Consequently the *image curve* of Φ is a φ^4.

As φ^4 has eight points R in common with a λ^2, the system Φ lies on a surface Φ^8 on which b is *quadruple* and the A_k are *quadruple* points.

The points of contact of the ϱ^3 lie on a curve φ^3 that has a double point on b. For a plane φ through l has besides a λ^3 in common with Λ^4 and any point of intersection of λ^3 and l is a point of contact of a ϱ^3 of Φ. Between the curves φ^3 and φ^4 there exists a (1,1)-correspondence; hence φ^4 has three double points. Consequently φ is a *plane of osculation* for three curves of Γ.

§ 4. Let us now consider the congruence Γ of which the curves ϱ^3 pass through the cardinal points A' and A'' and have the lines b_1, b_2, b_3 as cardinal chords (congruence of STUYVAERT).

It contains *six* systems Σ of composite figures. The line a''_{23} through A'' that cuts b_2 and b_3, forms figures ϱ^3 with any conic ϱ^2 in the plane $A_1 b_1$ which passes through A_1 and rests on b_2, b_3, a''_{23}. If the image plane β passes through b_1, the point $S' \equiv a''_{23}\beta$ is the image of all ϱ^3 of this system Σ'_1; this point is, therefore, *singular* for the representation.

Analogously the *singular* point $S'' \equiv a'_{23}\beta$ is the image of the system Σ''_1 formed by a'_{23} and the conics in the plane $A'' b_1$.

The systems $\Sigma'_2, \Sigma''_2, \Sigma'_3, \Sigma''_3$ are represented on lines c'_2, c''_2, c'_3, c''_3.

The ϱ^3 through the point of intersection S_2 of b_2 and β lie on the hyperboloid H_2 that contains b_1, b_3, S_2, A' and A''; they have the *singular point* S_2 as image.

Analogously $S_3 \equiv b_3\beta$ is the image of the ϱ^3 on the hyperboloid H_3 that contains b_1, b_2, S_3, A' and A''.

The system Σ'_2 consists of the line a''_{13} and the conics in the plane $A' b_2$ through A' that rest on b_1, b_3 and a''_{13}. To (ϱ^2) belongs the pair of lines of which a'_{23} is one of the lines, which, accordingly, with a'_{13} forms a figure that belongs at the same time to Σ''_1. The *image line* c'_2 contains, therefore, the point S''; it contains at the same time the point $S_2 \equiv b_2\beta$.

Analogously $c''_2 \equiv S' S_2$, $c'_3 \equiv S'' S_3$, $c''_3 \equiv S' S_3$.

§ 5. The conics through A' and A'' that rest on b_1, b_2, b_3, lie on a *dimonoid* Δ^4 with double torsal line $A'A''$. They are completed to figures of Γ by the transversals t of b_1, b_2, b_3. On $\triangle^4$ lie six pairs of lines each of which has one of the lines a'_{kl}, a''_{kl} as component part. The *image curve* δ^3 of the system contains, therefore, the points S' and S'', and evidently also the points S_2 and S_3.

The double point of δ^3 lies on $A'A''$ and is the image of the ϱ^3 that consists of $A'A''$ and the two lines t that cut $A'A''$.

Each of the points S', S'' is the image of three figures, each of which consists of three lines, each of the points S_2 and S_3 is the image of two such figures; finally also the points $c'_3\,c''_2$ and $c'_2\,c''_3$ are the images of figures consisting of *three* lines. Hence Γ contains *thirteen* ϱ^3 formed by three lines.

§ 6. A line l cuts two ϱ^3 of H_2 and two of H_3, hence S_2 and S_3 are double points of the image curve λ of the system Λ. This evidently contains the points S' and S'', for l cuts one ϱ^2 of each of the systems Σ. Accordingly the curve λ has the double point S_2, the point S'' and a point R in common with the image line c'_2; it is, therefore, a $\lambda^4\,(S_3{}^2\,S_2{}^2\,S'\,S'')$. Being a rational curve it has a *third* double point; this is the image of a ϱ^3 that cuts l twice. Any line is, therefore, a *bisecant* of one curve ϱ^3.

Two curves λ^4 have six points R in common; accordingly on two lines there rest six ϱ^3 and the curves ϱ^3 that are cut by l, form a surface Λ^6. On this b_2 and b_3, but then also b_1, are *double lines* and A', A'' are *triple* points.

Two surfaces Λ^6 have the double lines b, the six lines a'_{kl}, a''_{kl} and six curves ϱ^3 in common. The curves $\lambda^4(S_2{}^2\,S_3{}^2\,S'\,S'')$ and $\delta^3\,(S_2\,S_3\,S'\,S'')$ have six points R in common; these are the images of six composite ϱ^3; of four of these ϱ^3 the conic rests on l, of the other two the line t.

§ 7. A hyperboloid H contains again four ϱ^3 that touch a plane φ, a system Σ contains two of them. The *image curve* of the system Φ is, therefore, a $\varphi^8\,(S_2{}^4\,S_3{}^4\,S'^2\,S''^2)$. It has 12 points R in common with a λ^4; accordingly the system lies on a surface Φ^{12} on which the lines b are *quadruple*, the lines a'_{kl}, a''_{kl} *double*.

A plane φ through l has still a curve λ^5 in common with Λ^6; this cuts l in the two points of support of the ϱ^3 for which l is a chord and in the points of contact of three curves ϱ^3 with φ. Hence the locus of the points of contact is a curve φ^3 of the genus 1. Accordingly also φ^8 has the genus 1; it has, therefore, *six* double points besides the point S. Consequently a plane φ is *osculated* by *six* curves ϱ^3.

§ 8. Let us now consider the Γ of which the ϱ^3 pass through the point A and have four fixed bisecants b_k (congruence of GODEAUX).

Let the image plane β again pass through b_1. Γ contains the system Σ_1 formed by conics ϱ^2 in the plane Ab_1 and transversals t_1 of b_2, b_3, b_4; it is represented on the *conic* $\gamma_1{}^2$ in which the hyperboloid H_1 (through b_2, b_3, b_4) cuts β.

The analogous system Σ_2 is represented on a line c_2, the systems Σ_3 and Σ_4 on lines c_3 and c_4.

The curves through A and $S_2 \equiv b_2 \beta$ that have b_1, b_3, b_4 as chords and rest on l, form (§ 6) a surface O^6 on which S_2 is a *triple* point. It contains therefore three ϱ^3 that cut b_2 once more and accordingly belong to Γ. Consequently the ϱ^3 of Γ through S_2 form a *cubic surface* $O_2{}^3$. Analogously the *singular* points $S_3 \equiv b_3 \beta$ and $S_4 \equiv b_4 \beta$ are images of systems that lie on surfaces $O_3{}^3$ and $O_4{}^3$.

The image lines c_2, c_3, c_4 evidently contain resp. the *singular* points S_2, S_3, S_4.

Three of the ϱ^3 that cut l belong to Σ_2 for l rests on one ϱ^2 and on two lines t_2. The image curve λ has three points R and the triple point S_2 in common with c_2; it is, therefore, a $\lambda^6 (S_2{}^3 S_3{}^3 S_4{}^3)$. It contains also a *double point*, image of a ϱ^3 that cuts l twice.

On two lines l there rest nine ϱ^3 and the system Λ lies on a Λ^9 with *triple* lines b_1, b_2, b_3, b_4 and *triple* point A.

§ 9. Let $\triangle_1$ be the surface formed by the ϱ^2 of Γ that rest on the four lines b and on their transversal t'; analogously $\triangle_2$ the surface corresponding to the second transversal t''. Evidently the systems $\triangle_1$ and $\triangle_2$ have only one ϱ^3 in common, the figure consisting of t', t'' and their transversal through A. Accordingly their image curves have one point R in common; as they contain the points S_2, S_3, S_4 they are *conics* $\delta_1{}^2$ and $\delta_2{}^2$.

$\triangle_1$ contains a ϱ^2 that passes through the point $b_2 t'$; the ϱ^3 which it forms with t', belongs at the same time to Σ_2; its image point R lies, therefore, also on c_2. In fact this has only one point R in common with $\delta_1{}^2$ outside S_2.

The curves $\delta_1{}^2 (3\,S)$ and $\lambda^6 (3\,S^3)$ have three points R in common; hence $\triangle_2$ is a *cubic scroll* with directrices t', t'' ($\triangle_1{}^3$ has one more line in common with the plane of a ϱ^2).

There are *seven* ϱ^3 composed of three lines; one of them consists of t', t'' and a; the other six are represented in the angular points of the quadrilateral formed by the lines c.

§ 10. The system Σ_2 contains two ϱ^2 that touch a given plane φ and two figures ϱ^3 of which the point $\varrho^2 t_2$ lies in φ; each of these must be counted twice.

The surface $O_2{}^3$ has a curve φ^3 of genus one in common with φ; the I^3 which the curves ϱ^3 define on this, has six coincidences; S_2 is, therefore, sextuple on the image curve of the system Φ. Of this curve c_2 contains the 6-fold point S_2 and 6 points R; accordingly it is a φ^{12} $(3\,S^6)$. It has 18 points R in common with a λ^6 $(3\,S^3)$; the system Φ lies, therefore, on a Φ^{18} with *sextuple* lines b and *sextuple* point A.

§ 11. In the representation of the congruence Γ that has *five* cardinal chords b, the points S_2, S_3, S_4 and S_5 in which b_2, b_3, b_4, b_5 cut β, are *singular* [1]).

The ϱ^3 through S_2 that have b_1, b_3, b_4 and b_5 as chords and rest on l, form (§ 8) an O^9 on which S_2 is a *triple* point. Accordingly there are *six* ϱ^3 of this system that cut b_2 once more. Hence the ϱ^3 of Γ that are represented in S_2, form a *surface* $\beta_2{}^6$.

The hyperboloid H_{234} with directrices b_2, b_3, b_4 contains ∞^1 curves of Γ; they all pass through the four points in which H_{234} is cut by b_1 and b_5. The image curve of this system is a *conic* $\gamma^2{}_{234}$ through the singular points S_2, S_3 and S_4.

The analogous hyperboloid H_{145} is represented on a *line* c_{145}. The hyperboloids H_{145} and H_{234} have a ϱ^3 besides b_4 in common that has the second point of intersection of c_{145} and $\gamma^2{}_{234}$ as image. The curves $\gamma^2{}_{234}$ and $\gamma^2{}_{235}$ have two points R in common; these are the images of two figures that consist of a ϱ^2 and one of the transversals t', t'' of b_2, b_3, b_4, b_5 and are, accordingly, *singular points* for the representation; we shall indicate them by S' and S''. That they are *singular* appears thus: any ϱ^2 that has b_1 as chord and rests on b_2, b_3, b_4, b_5 and t', forms with t' a ϱ^3 belonging to Γ; all these figures are represented in the point of intersection S' of t' and β. They form a surface $O'_1{}^4$ with double line b_1. [2])

The analogous system $O'_2{}^4$ with *double line* b_2 is represented on the curve $\omega_2{}^3\,(S_2{}^2)$ which β has in common with this surface. $O'_2{}^4$ contains the lines t' and t''; for the plane $b_2\,t'$ contains one line that cuts b_1 and the transversal t'_{1345} and with this and with t' forms a ϱ^3.

Accordingly the image curve of this system is an $\omega_2{}^3\,(S^2{}_2 S_3 S_4 S_5 S' S'')$.

§ 12. The system Λ has an image curve of which S' and S'' are quadruple points, the points S_2, S_3, S_4, S_5 are sextuple points. Besides S', S'', S_2, S_3, S_4 $\gamma^2{}_{234}$ contains the images R of the two ϱ^3 of H_{234} that rest on l; accordingly it has 28 points in common with the *image curve* λ and consequently it is a $\lambda^{14}\,(S_2{}^6\,S_3{}^6\,S_4{}^6\,S_5{}^6\,S'^4\,S''^4)$.

The line $S_4 S_5$, image of H_{145}, contains two points R; also this shows that the order of λ is 14.

Two curves λ have 20 points R in common; accordingly on two lines l there rest 20 curves ϱ^3.

As λ^{14} is rational it has six double points outside the points S; there are, therefore, *six* curves ϱ^3 that have *six* given lines as chords.

The system Λ lies on a surface Λ^{20} on which the five lines b are *sextuple* and the ten transversals t are *quadruple*.

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[1]) I have treated another representation of this congruence in my paper: "The Congruence of the twisted Cubics that cut five given lines twice". (These Proceedings, Vol. **31**, p. 454).

[2]) L.c. p. 454.

§ 13. Let us also consider the congruence Γ with cardinal chord b and the cardinal points A_1, A_2, A_3 of which the curves ϱ^3 rest on the line c_1 that passes through A_1 and the line c_2 that passes trough A_2 [1]).

The curves through the point $S_1(c_2\beta)$ and the points A that cut b twice and rest on l, form an O^4 with double point A_2 (§ 2); it contains *two* ϱ^3 that cut c_2 once more. Accordingly the curves of Γ that pass through S_1, form a *quadratic surface* $O_1{}^2$. Analogously there is an $O_2{}^2$ of which the curves ϱ^3 are represented in the *singular* point $S_2(c_2\beta)$.

In the plane A_3b there lies one (ϱ^2) of which any individual is completed to a ϱ^3 by the line $a'_{12}(A_1 A_2)$. All these ϱ^3 have the *singular* point $S(a_{12}\beta)$ as image.

To this system Σ_3 there belong three figures ϱ^3 that consist of three lines. In the first place the system of a_{12}, the transversal through A_3 of a_{12} and b and a line in A_3b that cuts c_1 and c_2. This figure may also be considered as the system of a pair of lines in the plane $a_{123}(A_1A_2A_3)$ and a transversal of b, c_1, c_2. It belongs, therefore, at the same time to the system Σ_{123} of the ϱ^3 that consist of a ϱ^2 in α_{123} and a transversal of b, c_1, c_2. Accordingly the image of Σ_{123} is a *line* d_{123} through S.

Σ_3 contains also the ϱ^3 that consists of a_{12}, A_3C_1 and $A_{12}C_2(C_1, C_2$ and A_{12} are base points of the pencil (ϱ^2)). As a_{12} forms a pair of lines in the plane A_1c_2 with $A_{12}C_2$, this ϱ^3 belongs at the same time to the system Σ_2 of which the ϱ^3 consist of a ϱ^2 in A_1c_2 and the line a'_3 through A_3 cutting c_1 and b.

The system Σ_2 is represented on the *line* d_2 that joins the points S and $S_2(c_2\beta)$.

Analogously the *line* $d_1(SS_1)$ is the image of the system Σ_1. The conics ϱ^2 in A_3c_1 through A_1 and A_3 that cut b and c_2 and are associated to a line t_2 through A_2 that rests on b, form a system Σ' that has a *line* d' through S_1 as image.

Analogously there is a *line* d'' through S_2 that represents the system Σ''.

The lines b, c_2, a'_2 (line through A_2 cutting b and c_1), A_1 and A_3 define a hyperboloid H_2; it contains the system Σ_{13} of the figures that consist of a'_2 and a conic through A_1 and A_3. This system is represented on a *line* d_{13} through the point $S_2(c_2\beta)$.

Analogously there is a system Σ_{23} of figures that consist of the line a''_1 (through A_1 cutting b and c_2) and a conic through A_2 an A_3, and that lie on the hyperboloid H_1 that is defined by b, c_1, a''_1, A_2 and A_3. The *image line* d_{23} contains the point $S_1(c_1\beta)$.

§ 14. The image curve λ of the system Λ of the ϱ^3 that rest on l, passes through S and has double points in S_1 and S_2. The system Σ_1

[1]) That this congruence is *linear* appears thus. Let P be an arbitrary point; the lines A_1A_2, A_1A_3, A_1P and c_1 form the basis of a pencil of quadratic cones. Analogously A_2A_1, A_2A_3, A_2P and c_2 define a similar pencil. The two pencils define an I^2 on b each; there is, therefore, one ϱ^3 of Γ that has b as chord.

contains one ϱ^3 that cuts l; hence the image line d_1 has a point R, the point S and twice the point S_1 in common with λ, which is, accordingly, a $\lambda^4(S S_1^2 S_2^2)$. Being a rational curve it has still a *double point*, the image of the ϱ^3 that has l as chord.

Two curves λ^4 have *seven* points R in common; accordingly the system Λ lies on a surface Λ^7.

The intersection of Λ^7 with α_{123} consists of three ϱ^2 and the line a_{12}; for l rests on three figures of Σ_{123} and on one ϱ^2 of Σ_3. Hence A_3 is a *triple* point and A_1 and A_2 are *quadruple* points on Λ^7, c_1 and c_2 are *double lines* and b is a *triple line* (points of intersection of Λ^7 and an arbitrary ϱ^3). Further a'_2 and a''_1 are *double lines* and Λ^7 contains the lines a_{12}, a'_3 and a''_3 and 14 conics, 3 rays t_{12}, 2 rays t_1 and 2 rays t_2.

The image curve of the system Φ has S_1 as quadruple point (system O_1^2), S as double point (system Σ_1); the image line $d_1(S_1 S)$ of Σ_1 contains two more points R; hence the image curve of Φ is a $\varphi^8(S_1^4 S_2^4 S^2)$. In connection with $\lambda^4(S_1^2 S_2^2 S)$ we find that the curves of Φ lie on a surface Φ^{14}. [1])

§ 15. Let us also consider the congruence of the ϱ^3 that has b_1 and b_2 as cardinal chords, A_1 and A_2 as cardinal points, and where each of the ϱ^3 cuts the line c_1 through A_1 and the line c_2 through A_2 once more.

The image plane β is again passed through b_1. The point $S(b_2 \beta)$ is *singular*. The ϱ^3 through A_1, A_2, S that cut c_1, c_2 once more, have b_1 as chord and rest on l, form a surface Λ^7 with triple point S (§ 14). Accordingly this contains *four* ϱ^3 that cut b_2 once more, and the curves represented in S form a surface O^4.

Also $S_1(c_1 \beta)$ is *singular* and is the image of the ϱ^3 that lie on a hyperboloid H_1 which is defined by A_1, A_2, S_1, b_1 and b_2.

Analogously $S_2(c_2 \beta)$ is *singular* and is the image of a system ϱ^3 on the hyperboloid H_2 through A_1, A_2, S_2, b_1, b_2.

The plane $A_1 b_1$ contains a (ϱ^2) of which the ϱ^2 pass through A_1 and rest on b_2, c_2 and on the transversal a'_{22} through A_2 of b_2 and c_1. Each of them forms with a'_{22} a ϱ^3 of Γ (system Σ_{22}) and is represented in the *singular* point $S_{22}(a'_{22} \beta)$.

Analogously the *singular* point S_{12} lying on the transversal a''_{12} through A_1 of b_2 and c_2 is the image of the system Σ_{12} of which the conics ϱ^2 lie in the plane $A_2 b_1$.

The system Σ'_{21} formed by the line a'_{21} (through A_2 and cutting b_1, c_1) with a (ϱ^2) in the plane $A_1 b_2$ is represented on the points of the *line* d'_{21} that $A_1 b_2$ has in common with β. This line contains the points $S(b_2 \beta)$ and $S_{12}(a''_{12} \beta)$; to Σ'_2 there also belongs a figure that contains a'_{21} and a''_{12}.

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[1]) The congruence can also be represented on the field of points of the plane $A_1 c_1$.

Analogously Σ''_{11} formed by a''_{11} (through A_1 cutting b_1, c_1) with a (ϱ^2) in the plane $A_2 b_2$, has an *image line* d''_{11} that passes through S_{22} and S.

§ 16. Any line t_1 that rests on b_1, b_2 and c_1, forms a ϱ^3 with a ϱ^2 in $A_1 c_2$ that passes through A_1, A_2 and cuts b_1, b_2, t_1 (system Σ_1). The *image line* d_1 passes through S_2 ($c_2\,\beta$).

Analogously any transversal t_2 of b_1, b_2, c_2 forms a ϱ^3 with a ϱ^2 in A_2, c_1 through A_1, A_2 that cuts b_1, b_2, t_2. This system Σ_2 has as image a *line* d_2 through S_1 (c_1, β).

Let t be one of the transversals of b_1, b_2, c_1, c_2. Any ϱ^2 through A_1, A_2 that cuts b_1, b_2 and t, forms a ϱ^2 of Γ with t. The ϱ^2 lie on the *hyperboloid* H that is defined by A_1, A_2, b_1, b_2 and a point of t. This system Σ is represented on the points of the *line* d which H has besides in common with β; d contains the point S.

Analogously the second transversal, t^*, of b_1, b_2, c_1, c_2 defines a system Σ^* with *image line* d^* that passes through S.

The line a_1 through A_1 cutting b_1, b_2 forms a ϱ^3 with any ϱ^2 that has c_2 as chord and rests on c_1, b_1, b_2, a_1. The conics through A_2 and a point of c_2 that cut b_1, b_2 and a_1, form a surface O^2; through the second point of intersection of O^2 and c_1 passes one of these ϱ^2. Hence c_2 is a single line on the locus of the ϱ^2 that are completed by a_1 to curves of Γ, and this is a cubic *monoid* $O_1{}^3$ with double point A^2. This system Σ', is represented on a *conic* $\delta_1{}^2$ that passes through the points S, S_1, S_2 and through S_{22} (a'_{22} lies on $O_1{}^3$).

Analogously the system Σ'', lying on the *monoid* $O_2{}^3$, is represented on a conic $\delta_2{}^2$ that contains the points S, S_1, S_2 and S_{12}.

§ 17. The system Σ_1 contains a figure consisting of a''_{12}, a line t_1 and the transversal through A_2 of b_1 and t_1. The transversal forms a pair of lines of the plane $A_2 b_1$ with t_1; accordingly the ϱ^3 belongs at the same time to the system Σ_{12}. Hence the *image line* d_1 of Σ_1 joins the points S_2 and S_{12}.

Σ_1 contains three ϱ^3 of the system Λ. As l cuts two ϱ^3 of the system H_2 and one ϱ^3 of Σ_{12}, the image curve of Λ is a λ^6. On l there rest four ϱ^3 of the system O^4; hence Λ has a λ^6 ($S^4 S_1{}^2 S_2{}^2 S_{12} S_{22}$) as *image*.

Two curves λ^6 have *ten* points R in common; consequently the curves that rest on l, form a Λ^{10}. On this b_1, b_2 are *quadruple*, c_1, c_2, t, t^* *double*, a_1, a_2 *triple*; besides Λ^{10} contains the lines $a'_{22}, a''_{12}, a'_{21}, a''_{11}$, three lines t_1, three lines t_2 and 20 conics. Finally by noticing the points of intersection with an arbitrary ϱ^3 of Γ, we find that A_1 and A_2 are *quintuple* points.

Two surfaces Λ^{10} have *ten* curves ϱ^3 in common besides the lines b, c, a, a', a'', t, t^*.
