Mathematics. — *The Representations of a Linear Ray Complex on Point-Space that Associate Quadratic Surfaces to the Bilinear Congruences of the Complex.* By G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES.)

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§ 1. Through the well known representation of the lines l of space on the points P of a quadratic variety V_4 in R_5, to a linear complex C a quadratic variety V_3 lying in a four-dimensional space R_4 is associated. A representation of the lines l of C on the points L of space associates a point L of R_3 to any point P of V_3 and inversely.

Through the one-one correspondence (P, L) the spacial sections Φ of V_3, on which the bilinear congruences of C are represented, are transformed into surfaces Ω of point-space. These ∞^4 surfaces Ω form a linear system Σ. Indeed, four points of V_3 define a spacial section Φ so that four points L of space define one surface Ω.

To the surfaces Φ of V_3 lying in the spaces of a pencil with base-plane β, there correspond the surfaces Ω of a pencil in Σ. The base curve of this pencil consists of the curve associated to the intersection βV_3, i.e. of the image of a quadratic scroll of C, and of one or more possible base curves of Σ.

If to any space R_3 of R_4 we associate the surface Ω of Σ that corresponds to the intersection Φ of R_3 and V_3, we get a collinear correspondence between the spaces R_3 of R_4 and the surfaces Ω of Σ.

Consequently to the spaces R_3 through a line r of R_4 there correspond the surfaces of a net of Σ. Besides the fixed base curves and base points of Σ the surfaces of this net have two variable points in common, the images of the points of intersection of r and V_3.

To the spaces R_3 through a point P of V_3 there are associated the surfaces Ω of a linear complex in Σ. Besides the base curves and base points of Σ these surfaces have the point L in common that corresponds to P.

In order to find representations of a linear complex C on three-dimensional point-space, we must, therefore, in the first place determine linear systems of ∞^4 surfaces with such base curves and base points that three arbitrary surfaces of the system have two more points in common.

We shall now start from such a system Σ and we suppose a collinear correspondence to exist between the surfaces Ω of Σ and the spaces R_3 of R_4. A point L of space carries ∞^3 surfaces Ω, which form a linear complex. To these surfaces Ω there correspond the spaces R_3 of R_4 through a point P of R_4. If we associate this point P to L, the points P form a three dimensional quadratic variety V_3 in R_4. For the image points L of the points P of the intersection Φ of a space R_3 and V_3 form the surface Ω associated to R_3. Further the surfaces Ω corresponding to the spaces R_3 through a line r, have two points outside the base curves and base points of Σ in common, so that the surfaces Φ in the said spaces R_3 have two points in common. Accordingly a line r of R_4 cuts V_3 in two points.

Now if the lines l of a linear complex C can be represented on the points P of V_3, by representing each line l on the point L that is associated to the point P corresponding to l, we find a representation of the lines l of C on the points L of space.

We can find systems Σ by the aid of the remark that the surfaces of such a system that contain a point of Σ outside the base curves and base points, form a homaloid complex. Inversely from a homaloid complex which contains one or more isolated base points, we can deduce a system Σ by omitting one of these base points.

§ 2. We find three systems Σ of quadratic surfaces, which, perhaps, can lead to a representation of a linear complex on point-space. They consist of:

1. the quadratic surfaces through a conic k^2;
2. the quadratic surfaces through a line q and two points O_1 and O_2;
3. the quadratic surfaces through three given points O, O_1 and O_2 that touch a plane ω at O [1]).

§ 3. We shall first choose for Σ the system of the ∞^4 quadratic surfaces that contain a given conic k^2, and we shall suppose a collinear correspondence to exist between the surfaces Ω of Σ and the spaces R_3 of R_4. In this way we get a one-one correspondence between the points L of space and the points P of a quadratic variety V_3 in R_4.

The plane α of k^2 forms a surface Ω with any plane of space. The ∞^3 surfaces Ω that consist of α and an arbitrary plane of space, form a linear complex belonging to Σ, the surfaces of which have all points of α in common. To these surfaces correspond the spaces R_3 through a point A of V_3. This point is a cardinal point for our representation; the associated points L form the plane α.

To the spaces R_3 through a line a containing A there correspond the surfaces Ω consisting of α and the planes of a sheaf. The vertex of this sheaf is the point L corresponding to the point of intersection of a and V_3 different from A. Hence A is a single point of V_3.

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[1]) Cf. STURM, Geometrische Verwandtschaften, IV, § 134.

Any point L of k^2 is singular for our representation as any spacial section Φ must contain a point associated to L. Any space R_3 contains one point of the locus of the points that are associated to L; hence the points corresponding to L form a line k. This line passes through A, because L belongs to α.

The intersection of two surfaces Ω to which two spacial sections Φ are associated, i.e. a conic ω^2 that cuts k^2 twice, is the representation of a plane section φ^2 of V_3. As a conic ω^2 has two points in common with k^2, a plane of R_4 cuts the locus of the points P that correspond to the points L of k^2, twice. Accordingly the locus of the lines k of V_3 through A is a quadratic surface; it is the quadratic cone $\varkappa$ along which the space of contact to V_3 at A cuts this variety.

Let us consider a line r through a point P of $\varkappa$. The intersections of V_3 and the planes through P are represented on conics ω^2 through the point of k^2 that corresponds to P. All the surfaces Ω corresponding to the spaces R_3 through P touch, therefore, a fixed tangent plane of $\varkappa$ at L and form a complex. A point outside the tangent plane defines a net of the complex of which the surfaces are associated to the spaces R_3 through a line r that contains P. As the net has one isolated base point, the line r has one point in common with V_3 outside P. Any point P of $\varkappa$ is, accordingly, a single point of V_3.

A point P of V_3 outside $\varkappa$ is always a single point, as the point L corresponding to P together with a point L' for which LL'' does not cut k^2, defines a net of surfaces Ω with two isolated base points so that through P we can always draw a line that cuts V_3 outside P.

Accordingly we have here a general quadratic variety V_3, which may always be considered as the image of a general linear complex C.

We have already seen that a quadratic surface Φ and a conic φ^2 of V_3 are resp. represented on a quadratic surface Ω through k^2 and a conic ω^2 that cuts k^2 twice.

A plane γ of space which forms a surface Ω with α, is the image of a quadratic surface of V_3 containing A. Inversely from a surface Ω associated to such a surface the plane α splits off so that there remains a plane γ as image plane.

A line c of space being the intersection of two planes γ is the image of a conic φ_2 of V_3 trough A. Inversely such a conic being the intersection of two surfaces Φ through A is represented on a line c.

A line f of V_3 that has one point in common with a surface Φ containing A, is represented on a line of space. As f cuts one of the generatrices of $\varkappa$, the image line has one point in common with k^2. Inversely a line cutting k^2 cuts a surface Φ in one point that is not singular for the representation so that this line is the image of a line of V_3.

For the representation of the lines l of C on the points L of space we now find the following properties. C contains one cardinal line a, to which all points of the plane a are associated. There is a conic k^2 of singular image points. To a point of k^2 there correspond the lines l of a plane pencil of C containing a. The lines l corresponding to the points of k^2, form the special bilinear congruence of the lines of C that cut a.

A plane pencil of C is represented on a line that cuts k^2, a scroll of C on a conic that cuts k^2 twice. To a bilinear congruence of C there corresponds a quadratic surface Ω through k^2, which becomes a cone when the congruence is special.

A line of point-space is the image of a scroll of C containing a, a plane of the point-space the image of a bilinear congruence of C containing a.

In this way we have arrived at the representation of NÖTHER-KLEIN of the rays of a linear complex on the points of space.

§ 4. Let us now choose for Σ the system of the quadratic surfaces that contain a line q and two points O_1 and O_2. We get again a one-one correspondence between the points L of space and the points P of a quadratic variety V_3 in R_4 by the aid of a collinear correspondence between the surfaces Ω of Σ and the spaces R_3 of R_4. The surfaces of Σ that contain the line $O_1 O_2$, form a linear complex. The point T of R_4 associated to this complex is a singular point of the said correspondence on V_3 to which the whole line $O_1 O_2$ is associated. To the spaces R_3 through a line r containing T there correspond the surfaces Ω of a net of Σ. All these surfaces contain the lines q and $O_1 O_2$ and accordingly besides the base elements of Σ they have only points of $O_1 O_2$ in common, so that the line r in consideration cuts the variety V_3 in T only. The point T is, therefore, a double point of V_3.

A line o_1 through O_1 which cuts q and, therefore, lies in the plane $O_1 q \equiv \omega_1$, has two fixed points in common with all quadratic surfaces. Ω. Hence there are ∞^3 surfaces Ω that contain a line o_1. They form a linear complex belonging to Σ. The corresponding P of R_4 is a singular point on V_3 of the correspondence to which the line o_1 is associated. The points P corresponding to the lines o_1 form a line s_1 of V_3, as any surface Ω contains one line o_1 and consequently any spacial section of V_3 one of the said points P.

In the same way it appears that there is a second singular line s_2 on V_3. To a point of s_2 there corresponds a line o_2 in the plane $O_2 q \equiv \omega_2$, passing through O_2.

Consequently the double point T of V_3 and the points of two crossing lines s_1 and s_2 are singular for our representation. To a point of s_1 or of s_2 there correspond resp. a line of (O_1, ω_1) or of (O_2, ω_2) and inversely.

A plane section φ^2 of V_3 is represented on the cubic section ω^3 different from q of two surfaces Ω. The ∞^6 curves ω^3 are the twisted cubics through O_1 and O_2 that cut the line q twice.

The point O_1 is a cardinal point of our correspondence; the points P corresponding to this point form a plane σ_1 as any plane section φ that is represented on a curve ω^3 through O_1, contains one point P corresponding to O_1. This plane σ_1 passes through T because O_1 belongs to $O_1 O_2$ and through s_1 as the images of the lines o trough O_1 must belong to the plane corresponding to O_1.

In the same way it appears that the point O_2 is a cardinal point for our correspondence, to which the plane $Ts_2 \equiv \sigma_2$ is associated.

Also the points of the lines q are singular for the correspondence (L, P). As any surface Ω passes through the line q, any spacial section Φ of V_3 contains one of the points P that are associated to a definite point L of q, so that to such a point L a line t is associated. The lines t associated to the points of q, form a quadratic scroll τ^2. For a curve ω^3 cuts the line q twice, so that a plane section φ^2 of V_3 cuts two lines t.

Any line t has one point in common with the line s_1, viz. the image of the line o_1 that passes through the point L corresponding to t. We see, therefore, that the line s_1 and also the line s_2 belong to the scroll σ^2 connected with τ^2. The quadratic surface Φ_1 containing τ^2 is the intersection of V_3 and the space (s_1, s_2).

The surfaces Ω corresponding to the spaces R_3 through a point P of Φ_1 form the complex of the surfaces of Σ that touch a plane λ through q at the point L that corresponds to the line t through P. If for λ we choose a definite plane through q and if we make L describe the line q, P describes a line of the scroll σ^2 connected with τ^2. For any surface Ω touches the plane λ at one point L of q and accordingly any spacial section Φ cuts the locus of the points P once.

In point-space we have two cardinal points O_1 *and* O_2 *to which the planes* $Ts_1 \equiv \sigma_1$ *and* $Ts_2 \equiv \sigma_2$ *are associated. The points of the line* q *are singular. To a point* L *of* q *there corresponds a line* t*. The lines* t *form the intersection* Φ_1 *of* V_3 *and the space* $(s_1\, s_2)$.

Among the surfaces Ω there are ∞^2 surfaces that are degenerate in a plane λ through q and a plane μ through O_1O_2. To the surfaces Ω that contain a fixed plane μ there correspond the spaces R_3 of a pencil of which the axial plane τ is the locus of the points of V_3 that are associated to the points of μ so that τ belongs entirely to V_3. The plane τ passes through T and the line t of τ^2 that is associated to the point of intersection of μ and q.

In the same way it appears that to a plane λ there corresponds a plane σ of V_3. This plane σ also passes through T because λ generally cuts the line O_1O_2 in a point that is not singular for the representation, and passes through the line s that corresponds to the plane λ in the way indicated above. Among the planes σ are the planes σ_1 and σ_2 that correspond resp. to the planes q_1O_1 and q_2O_2.

The variety V_3 is formed by the lines p that project the image surface

Φ_1 out of T. Such a line p that is the intersection of a plane τ and a plane σ, is represented on the line of intersection b of a plane μ and a plane λ, i.e. a generatrix of the bilinear congruence B of the lines that cut q and O_1O_2. Inversely a line of B as the intersection of a plane λ and a plane μ is the image of the line of intersection of a plane σ and a plane τ, hence of a line p of V_3 through T. The points of intersection of b with O_1O_2 and q are resp. associated to the point T on p and the point of intersection of p and Φ_1.

On the hypercone V_3 a special linear complex C with axis a can always be represented. For the representation (l, L) of the lines l of C on the points L of space that arise through combination of the correspondence (l, P) between C and V_3 and the correspondence (P, L), we find the following properties if we suppose the sheaves of rays and the fields of rays of C to be represented resp. on the planes σ and on the planes τ of V_3.

In C among the lines that are singular for the representation we find in the first place the axis a that is represented on the points of O_1O_2. Further there are two plane pencils (F_1, φ_1) and (F_2, φ_2) of singular lines l of C that together with a define sheaves of rays of C and of which the vertices F_1 and F_2 are, accordingly, points of a; φ_1 and φ_2 are arbitrary planes passing resp. through F_1 and F_2.

A line l of (F_1, φ_1) or (F_2, φ_2) is represented resp. on a generatrix of the plane pencil (O_1, ω_1) or (O_2, ω_2). To the rays l of (F_1, φ_1) and (F_2, φ_2) there correspond resp. the points L of the planes ω_1 and ω_2.

In the space of the points L there lie two cardinal points, viz. O_1 and O_2. The lines l of C that are associated to O_1 and O_2 resp. form the sheaves of rays F_1 and F_2, i.e. the sheaves that are defined by a and the plane pencil (F_1, φ_1) and by a and the plane pencil (F_2, φ_2).

Also the points of $q \equiv \omega_1\omega_2$ are singular to which belong the rays of the bilinear congruence that has a and $\varphi_1\varphi_2 \equiv f$ as directrices. To a point of q are associated the rays of a plane pencil of this congruence that defines a field of rays together with a, i.e. the rays of a plane pencil with vertex on f in a plane through a.

The quadratic scrolls of C, that are those which have a as directrix, to which the conics of V_3 are associated, are represented on the twisted cubics ω^3 through O_1 and O_2 that cut q twice.

Among the ∞^6 curves ω^3 there are ∞^5 which are degenerate in a conic ω^2 through O_1 and O_2 and a line ω that cuts ω^2 and q in different points. These composite curves ω^3 correspond to the intersections of V_3 and the tangent planes of this variety so that to ω^2 as well as to ω a plane pencil in C is associated. As ω^2 lies in a plane through $O_1 O_2$, to which there corresponds a plane τ of V_3, this conic is the image of a pencil w, in a plane through a and the line ω that lies in a plane through q, represents a plane pencil w' with vertex on a. A conic ω^2 and a line ω that cut each other, are the images of two plane pencils

w and w' that have a line in common so that the plane of w passes through the vertex of w'.

A plane pencil of C in a plane through a is represented on a conic ω^2 through O_1 and O_2 cutting q, a plane pencil of C with vertex on a on a line ω cutting q.

To a plane pencil containing a there corresponds a line b of the bilinear congruence B that has q and $O_1 O_2$ as directrices. Such a line forms a conic ω^2 together with $O_1 O_2$.

If the vertex of w lies in φ_1 a generatrix of (O_1, ω_1) splits off from ω^2 and, accordingly, there remains a line through O_2. In the same way it appears that a plane pencil ω with vertex in φ_2 is represented on a line through O_1.

A sheaf of C with vertex on a is represented on a plane λ through q, a field of rays of C, of which, accordingly, the plane passes through a, on a plane μ through $O_1 O_2$. The image points L of the rays l of a bilinear congruence of C, of which, accordingly, one of the directrices coincides with a, form a quadratic surface Ω through O_1, O_2 and q.

If the directrix different from a lies in φ_1, the plane ω_1 splits off from Ω and, accordingly, there remains a plane through O_2. In the same way it appears that a bilinear congruence of C with directrix in φ_2 is represented on a plane through O_1.

A line g of points L is the image of a quadratic scroll γ of C. For g has two points in common with a surface Ω and, accordingly, γ contains two generatrices that cut an arbitrary straight line. As g cuts one generatrix of each of the plane pencils (O_1, ω_1) and (O_2, ω_2), γ has a generatrix in common with each of the plane pencils (F_1, φ_1) and (F_2, φ_2).

When g cuts the line $O_1 O_2$ g lies in a plane μ. In this case g is the image of the system of the tangents to a conic that touches a and the planes φ_1 and φ_2.

A plane α of point-space is the image of a congruence A (2,1) of C, because a conic ω^2 has two points in common with α and a line ω one point. The plane α cuts all generatrices of the plane pencils (O_1, ω_1) and (O_2, ω_2) and also the line $O_1 O_2$. Consequently $A(2,1)$ contains the plane pencils (F_1, φ_1) and (F_2, φ_2) and also the line a. The congruence consists of the lines that cut a and touch a cone that has a as tangent and the planes φ_1 and φ_2 as tangent planes so that the vertex of the cone lies on the line $\varphi_1\varphi_2$.

Let us now in point-space choose a curve k of the order n that has an o_1-fold point in O_1, an o_2-fold point in O_2 and that cuts the line q k times. Such a curve cuts a surface Ω, the planes ω_1 and ω_2 and a plane λ resp. in $2n—o_1—o_2—k$, $n—o_1—k$, $n—o_2—k$ and $n—k$ points that are not singular for the representation. Hence this curve is the image of a scroll of the degree $2n—o_1—o_2—k$ that contains resp. $n—o_1—k$ and $n—o_2—k$ generatrices of (F_1, φ_1) and (F_2, φ_2) and has a as $(n—k)$-fold directrix. If the curve has p more points in common with O_1O_2, the line a is besides a p-fold torsal line of the corresponding scroll.

We find the image curve of a scroll of C of the degree ν that has resp. ω_1 and ω_2 generatrices in common with (F_1, φ_1) and (F_2, φ_2) and of which a is a $\varkappa$-fold directrix by solving the quantities n, o_1, o_2 and k out of the equations

$$\begin{aligned} 2n - o_1 - o_2 - k &= \nu \\ n - o_1 - k &= \omega_1 \\ n - o_2 - k &= \omega_2 \\ n - k &= \varkappa . \end{aligned}$$

We find that the image curve is of the order $\nu + \varkappa - \omega_1 - \omega_2$, has resp. a $(\varkappa - \omega_1)$- and a $(\varkappa - \omega_2)$-fold point in O_1 and O_2 and cuts q $\nu - \omega_1 - \omega_2$ times.

For an arbitrary cone of C of the degree n we have $\nu = n$ and $\varkappa = \omega_1 = \omega_2 = 0$. Hence such a cone is represented on a curve of the order n that cuts q n times. It lies in a plane through q, as all generatrices of the cone pass through the same point of a.

To a curve of the n^{th}-class lying in a plane through a there corresponds a curve of the order $2n$ in a plane through $O_1 O_2$ with n-fold points in O_1, O_2 and on q.

We shall further consider a surface of the degree m that has a v_1-fold point in O_1 and a v_2-fold point in O_2 and of which q is an r-fold line. Such a surface has resp. $2m - v_1 - v_2 - r$, $m - r$, $m - v_1 - r$ and $m - v_2 - r$ points that are not singular for the representation in common with a conic ω^2, a line ω, a generatrix of (O_1, ω_1) and a generatrix of (O_2, ω_2). Hence the said surface is the image of a congruence $(2m - v_1 - v_2 - r, m - r)$ of which the rays of (F_1, φ_1) and (F_2, φ_2) are resp. $(m - v_1 - r)$- and $(m - v_2 - r)$-fold lines.

As the surface cuts $O_1 O_2$ in $m - v_1 - v_2$ points outside O_1 and O_2, a is an $(m - v_1 - v_2)$-fold line of the congruence.

The image surface of a congruence (μ, ϱ) that has the rays of (F_1, φ_1) and (F_2, φ_2) resp. as φ_1- and φ_2-fold lines, is of the degree $\mu + \varrho - \varphi_1 - \varphi_2$; it has a $(\varrho - \varphi_1)$-fold point in O_1, a $(\varrho - \varphi_2)$-fold point in O_2, and q is a $(\mu - \varphi_1 - \varphi_2)$-fold line of the image surface.

We get a representation of the kind that has been investigated, in the following way. We choose two points O_1 and O_2 on the axis a of C. Further we consider two planes φ_1 and φ_2 that are cut by $O_1 O_2$ resp. in the points F_1 and F_2.

A line l of C that cuts φ_1 and φ_2 resp. in S_1 and S_2, is represented in the point of intersection L of the lines $O_1 S_2$ and $O_2 S_1$.

This representation has been treated by me in the "Nieuw Archief voor Wiskunde", 2de reeks, deel XIV, p. 330.

If we suppose that to the sheaves and fields of rays of C there correspond resp. the planes τ and the planes σ, we get a representation that arises from the one we have investigated if in stead of C we choose the reciprocal figure.

§ 5. Finally we choose for Σ the ∞^4-system of the quadratic surfaces Ω that pass through three given points O, O_1 and O_2 and touch a given plane ω at O. We shall first investigate the one-one correspondence between the points P of space and the points L of a quadratic variety V_3 in R_4 which we get by the aid of a collinear correspondence between the surfaces Ω of Σ and the spaces R_3 of R_4.

The plane $OO_1\,O_2$ contains a pencil of conics σ^2 that pass through O_1 and O_2 and touch the plane ω at O. Each of these conics has four fixed points in common with the surfaces Ω so that the surfaces Ω containing a conic σ^2 form a linear complex. To this a point S_2 of V_3 is associated, which is, accordingly, singular for our correspondence. The locus of the points S_2 is a line s_2 of V_3, as any surface Ω contains one conic σ^2 and, therefore, in any spacial section Φ of V_3 there lies one point S_2.

To a plane α of V_3 there corresponds a biquadratic surface ψ^4 of V_3. For to a plane section φ^2 of V_3 an intersection k^4 of two surfaces Ω is associated, which cuts α in four points so that a conic φ^2 has four points in common with the surface corresponding to α.

Because any conic σ^2 cuts a plane α twice, all surfaces ψ^4 have the line s_2 as double line.

A generatrix o of the plane pencil (O, ω) touches all surfaces Ω at O. The surfaces Ω through o form, therefore, a linear complex. Hence to a line o there corresponds a point S_1 of V_3 that is singular for the correspondence (l, P). As any surface Ω contains two lines o, any spacial section of V_3 has two points in common with the locus of the points S_1. Accordingly the points S_1 form a conic $s_1{}^2$. All surfaces ψ^4 pass through $s_1{}^2$.

The generatrix of (O, ω) in the plane $OO_1\,O_2$ forms a conic σ^2 together with O_1O_2. To the linear complex of the surfaces Ω that contain this conic σ^2, there corresponds a point P which lies on s_2 as well as on $s_1{}^2$. Hence we see that s_2 and $s_1{}^2$ have one point S_1 in common.

The singular points of the correspondence (P, L) *on* V_3 *form a line* s_2 *and a conic* $s_1{}^2$ *that have a point* S *in common. To a point of* s_2 *there corresponds a conic* σ^2 *in the plane* $OO_1\,O_2$, *to a point of* $s_1{}^2$ *there corresponds a generatrix of the plane pencil* (O, ω).

The linear complex of the surfaces Ω through the conic σ^2 consisting of the lines OO_1 and OO_2, is formed by the quadratic cones with vertex in O that have OO_1 and OO_2 as generatrices. To this complex a point T of s_2 is associated. The surfaces Ω corresponding to the spaces R_3 that contain a line through T, form a net of cones with vertices in O that have the generatrices OO_1 and OO_2 in common. As this net does not generally contain any base points outside OO_1 and OO_2, an arbitrary line through T has no point in common with V_3 besides T. Accordingly the variety V_3 is a hypercone with vertex T.

When the cones of the said net have a line different from OO_1 and OO_2 in common, the corresponding line through T lies entirely on V_3. Also the inverse holds good. Hence a line of V_3 through T is represented on a line through O and inversely.

The cones Ω are the images of the quadratic cones in which the three-dimensional spaces through T cut V_3. Among the ∞^3 cones Ω there are ∞^2 degenerations, each of which is formed by a plane through OO_1 and a plane through OO_2. They correspond to the intersections of V_3 and the tangent spaces of this hypercone. The planes through OO_1 are associated to the planes σ of one of the systems of planes, the planes through OO_2 correspond to the planes τ of the other system of V_3.

A line s of V_3 in a plane σ is the intersection of this plane and a spacial section Φ of V_3. Consequently to such a line s there corresponds the intersection of a plane through OO_1 and a surface Ω, i.e. a conic s^2 through O and O_1 that touches ω at O. In the same way it is evident that to a line t of V_3 in a plane τ a conic t^2 is associated that passes through O and O_2 and touches ω at O.

As any base curve k^4 of a pencil of the complex Σ passes through O_1, any plane section φ^2 of V_3 contains one point corresponding to O_1. Accordingly O_1 is a cardinal point for our representation and the points corresponding to O_1 form a plane of V_3. This plane passes through s_2 because all conics σ^2 contain the point O. It is the plane τ through s_2, for it has no point different from T in common with an arbitrary plane τ that is represented on a plane through OO_2. We shall call the plane of V_3 corresponding to O_1 τ_1.

In the same way it appears that O_2 is a cardinal point for the correspondence (P, L) and that to this point the plane σ_1, the plane σ through s_2, is associated.

Any curve k^4 has a double point in O, because the surfaces Ω touch each other at O. Hence two of the points corresponding to O lie in an arbitrary plane section φ_2 of V_3. The point O is, therefore, a cardinal point and the points corresponding to O form a quadratic surface. As the conics σ^2 and the lines o all pass through O, the quadratic surface associated to O must contain the conic ${s_1}^2$ and the line s_2. It is, therefore, the intersection of V_3 and the space through ${s_1}^2$ and s_2, that is a quadratic cone $\varkappa$ with vertex T.

Consequently the space of the points L contains three cardinal points, viz. O, O_1 and O_2. The points of V_3 corresponding to O, O_1 and O_2 form resp. the quadratic cone $\varkappa$ that projects ${s_1}^2$ out of T, the plane τ_1 passing through s_2 and the plane σ_1 containing s_2.

An arbitrary special linear complex C with axis a can always be represented on the hypercone V_3. We shall suppose that this representation associates the planes σ to the sheaves of C and the planes τ of V_3

to the fields of C. In this case we find the following properties of the representation (l, L) that arises through combination of the correspondences (l, P) and (P, L).

The linear complex C contains a plane pencil (A, α) of singular lines l one of which is a. The image points of an arbitrary generatrix of (A, α) form a conic σ^2 in the plane $O O_1 O_2$ that passes through O, O_1 and O_2 and touches ω at O. Accordingly to the lines of (A, α) there correspond the conics of a pencil Σ. The image points of the lines of (A, α) form the plane $O O_1 O_2$. The conic σ^2 corresponding to a is formed by the lines $O O_1$ and OO_2. Let the conic σ^2 that is formed by the line $O_1 O_2$ and the generatrix in $O O_1 O_2$ of the plane pencil (O, ω), be associated to the line c.

Further C contains a scroll σ^2 of singular rays that have the line c in common with (A, α). The image points of a generatrix of ϱ^2 form a ray of (O, ω). To the lines of ϱ^2 the points of the plane ω are associated.

In the space of the image points L we have three cardinal points, viz. O, O_1 and O_2. The lines l associated to O form the special bilinear congruence K with directrix a consisting of the generatrices of the plane pencils each of which is defined by a and a generatrix of ϱ^2. To O_1 there correspond the rays of the plane α, to O_2 the rays of the sheaf A.

A plane pencil w_1 of C with vertex on a is represented on a conic s^2 through O and O_1 that touches ω at O, a plane pencil w_2 of C in a plane through a on a conic t^2 through O and O_2 that touches ω at O. To a plane pencil w containing a a line through O is associated. Such a line forms a conic s^2 with $O O_1$, a conic t^2 with $O O_2$.

If we have a plane pencil w_1 of C containing a line of ϱ^2 of which the plane is a plane of contact that does not pass through a of the quadratic surface defined by ϱ^2, a generatrix of (O, ω) splits off from the image conic s^2; such a plane pencil is, therefore, represented on a line through O_1. In the same way it appears that a plane pencil w_2 of C of which the vertex lies on the quadratic surface defined by ϱ^2 but not on a, is represented on a line through O_2.

To a sheaf of rays of C, the vertex of which, accordingly, lies on a, there corresponds a plane through $O O_1$, to a field of rays of C, the plane of which, accordingly, passes through a, there corresponds a plane through OO_2. A bilinear congruence of C is represented on a quadratic surface Ω through O, O_1 and O_2 that touches ω at O. In particular the ∞^3 special bilinear congruences with vertex a are represented on the quadratic cones that contain $O O_1$ and $O O_2$. If the rays of (A, α) belong to such a special linear congruence the plane $O O_1 O_2$ splits off from the image cone and there remains a plane through O.

To a bilinear congruence of C that contains the scroll ϱ^2 and of which, accordingly, the directrix different from a belongs to the scroll connected with ϱ^2, the same as a, there corresponds a plane through $O_1 O_2$, as in this case the plane ω has split off from the image surface Ω.

A line g of points L has two points in common with a surface Ω, and cuts one conic σ^2 and one generatrix of (O, ω). A line g is, therefore, the image of a quadratic scroll γ^2 of C that has one generatrix in common with the plane pencil (A, a) and also with ϱ^2.

To a line g which cuts O_1O_2 there corresponds a scroll γ^2 through c that has one more generatrix in common with ϱ^2. The scrolls γ^2 that touch ϱ^2 along c, are represented on the lines through the point of intersection of O_1O_2 and ω.

If we have a line g that cuts OO_1, this lies in a plane through OO_1, so that all the generatrices of γ^2 belong to a sheaf C. In this case the common line of γ^2 and (A, a) coincides with a. A line cutting OO_1 is, therefore, the image of a cone containing a that touches α and contains the generatrix of ϱ^2 passing through its vertex. In the same way we see that a line g cutting OO_2 is the image of the system of tangents to a conic touching a at A that touches the generatrix of ϱ^2 lying in its plane.

A plane α of the points of space cuts a conic t^2, a conic s^2 and a conic σ^2 twice and a line of (O, ω) once. Such a plane is, therefore, the image of a congruence $\Gamma(2,2)$ that has the lines of (A, a) as double lines and contains the generatrices of ϱ^2.

If the plane passes through O_1, the field of rays a splits off from Γ and there remains, therefore, a congruence (2,1) containing the generatrices of (A, a) and the lines of ϱ^2. This congruence consists of the lines that cut a which touch an enveloping cone with vertex in α of the quadratic surface defined by ϱ^2, with the exception of the lines of α. In the same way it appears that a plane through O_2 is the image of the congruence (1,2) of the lines that cut a and a conic through A of the quadratic surface defined by ϱ^2 in different points.

We shall now consider a curve k^n of the order n that has resp. an o-, o_1- and o_2-fold point in O, O_1 and O_2. Let us suppose that r of the o branches through O of k^n touch ω at this point. The chosen curve cuts a surface Ω in $2n - o - o_1 - o_2 - r$ and a plane through OO_1 in $n - o - o_1$ points that are not singular for the representation and it cuts $n - o - o_1 - o_2$ conics σ^2 and $n - o - r$ lines of (O, ω) outside the base points of Σ.

Consequently the curve k^n is the image of a scroll λ belonging to C of the degree $2n - o - o_1 - o_2 - r$ that has a as $(n - o - o_1)$-fold directrix and has resp. $n - o - o_1 - o_2$ and $n - o - r$ generatrices in the plane pencil (A, a) and the scroll ϱ^2. As a plane through OO_2 cuts the curve in $n - o - o_2$ points that are not singular for the representation, the scroll λ has $n - o - o_2$ lines different from a in common with a field of rays containing a. Now such a field of rays contains in all $n - o_2 - r$ generatrices of λ. The line a is, therefore, an $(o - r)$-fold torsal generatrix of λ. The cuspidal points together with the planes of contact at the corresponding torsal lines define plane pencils that are represented on the tangents of k^n at O to the $o-r$ branches that do not touch ω. The r generatrices of (O, ω) that touch k^n, correspond to lines of ϱ^2, which, together with a, define plane pencils in each of which there lies a line of λ.

If k^n cuts the line OO_1 in one more point, a is a torsal generatrix of λ with α as corresponding torsal plane. For the cuspidal point corresponds to the plane through OO_1 that touches k^n at the point of intersection, and the torsal plane to the plane through OO_2 and the point of intersection, i.e. the plane OO_1O_2. If k^n has one more point in common with the line OO_2, we find that a is a torsal generatrix of λ with A as corresponding cuspidal point. If k^n touches the plane OO_1O_2 at O, a is a torsal generatrix of λ with A as corresponding cuspidal point and α as corresponding cuspidal plane.

Let us now investigate the image curve of a scroll of C of the degree ν that has a as α_1-fold directrix and as α_2-fold torsal generatrix, and that has resp. ω and ϱ generatrices in the plane pencil (A, α) and the scroll ϱ^2.

We find that such a scroll is represented on a curve of the order $2\nu-\varrho-2\omega-\alpha_2$, that has a $(\nu-\varrho-\omega)$-fold point in O, a $(\nu-\omega-\alpha_1-\alpha_2)$-fold point in O_1, an $(\alpha_1-\omega)$-fold point in O_2 and $\nu-\varrho-\omega-\alpha_2$ branches through O that touch ω at that point. This result only holds good when the α_2 cuspidal points and torsal planes corresponding to a are different resp. from A and α. If this is not the case the peculiarities of the image curve can be easily indicated by the aid of what has been found in the preceding paragraph.

If as a special case we choose a cone C of the n^{th}-degree, we have $\nu=n$ and $\alpha_1=\alpha_2=\omega=\varrho=0$. Such a cone is, accordingly, represented on a curve of the order $2n$ lying in a plane through OO_1 that has n-fold points in O and O_1. This curve touches itself at O as the n branches through this point all touch ω.

For a curve of the class n lying in a plane through a we have $\nu=\alpha_1=n$ and $\alpha_2=\varrho=\omega=0$. The system of tangents to such a curve is accordingly represented on a curve of the order $2n$ in a plane through OO_2 that has n-fold points in O and O_2 and of which the n branches through O_2 touch ω.

A surface of the degree m that has a p-fold point in O and q leaves touching ω at O, and for which O_1 and O_2 are resp. p_1-fold and p_2-fold points, has resp. $2m-p-p_2-q$, $2m-p-p_1-q$, $2m-p-p_1-p_2-q$, $m-p-q$ and $m-p$ points that are not singular for the representation, in common with a conic t^2, a conic s^2, a conic σ^2, a line of (O, ω) and a line passing through O. Such a surface is, therefore, the image of a congruence $(2m-p-p_2-q,\ 2m-p-p_1-q)$, for which the lines of (A, α) and of ϱ^2 are resp. $(2m-p-p_1-p_2-q)$- and $(m-p-q)$-fold lines and of which a plane pencil through a contains $m-p$ lines different from a.

If inversely we have a congruence (μ_1, μ_2) for which the lines of (A, a) are $\varkappa$-fold lines and the lines of ϱ^2 ϱ-fold lines, and that has π lines different from a in common with a plane pencil containing a, we find that this congruence is represented on a surface of the degree $\mu_1 + \mu_2 - \varkappa - \varrho$ that has a $(\mu_1 + \mu_2 - \varkappa - \varrho - \pi)$-fold point in O, $\pi - \varrho$ leaves touching ω at O and resp. a $(\mu_1 - \varkappa)$- and a $(\mu_2 - \varkappa)$-fold point in O_1 and O_2.