Mathematics. — *On* HILBERT's *Function, Series of Composition of Ideals and a generalisation of the Theorem of* BEZOUT. By BARTEL L. VAN DER WAERDEN. (Communicated by Prof. R. WEITZENBÖCK.)

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By the aid of his "chains of syzygies" HILBERT [1]) has shown that the number of linear equations which the coefficients of a polynomial of the degree ϱ must satisfy in order to belong to a given *H*-ideal ("Formenmodul") may be represented by a whole rational function of ϱ, the "characteristic function", if ϱ has a sufficiently high value. LASKER [2]) has proved this result again in a simpler way, and has shown that the degree of the characteristic function is connected with the dimension of the ideal. If we write the function as a sum of binomial coefficients $\binom{\varrho}{k}$ multiplied by constants c_k, these c are integer numbers that are characteristic for the ideal.

Another system of such numbers, among which the *exponent* and the *length* are the most important ones, can be defined not only for the polynomial domains but also for arbitrary primary ideals in arbitrary „ring" domains, where HILBERT's basis theorem holds good. These numbers are exclusively dependent on the domain of restclasses, not on the initial domain. In what follows we shall only investigate the "length", i.e. the length of a series of composition of primary ideals which terminates in a given primary ideal.

This length is closely connected with the characteristic function. In the simplest case: when the *H*-ideal is primary with only one zero in projective space, the characteristic function is a constant and equal to the length. The coefficient of highest index of the characteristic function, for wich we use the name "degree" of the ideal, is always equal to the sum of the degrees of the primary components of the largest dimension and the degree of a primary ideal is equal to the length multiplied by the degree of the corresponding prime ideal, which latter is again equal to the degree of the manifold of zeros of this prime ideal in projective space.

The latter fact gives importance to the theory of the characteristic function for geometry: the theory makes it possible to establish a relation

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[1]) D. HILBERT, Ueber die Theorie der algebraïschen Formen, Math. Ann. **36**, (1890) p. 473.

[2]) E. LASKER, Zur Theorie der Moduln und Ideale, Math. Ann. **60**, (1905) p. 20; Cf. also A. OSTROWSKI, Abh. Math. Sem. Hamburg, **1**, (1922) p. 281 and F. S. MACAULAY, Proc. London Mathem. Soc. **26**, (1926) p. 531.

between the degree of an algebraic manifold and the degree of the intersection of this manifold with a spread $f=0$. This relation is a generalisation of the theorem of BEZOUT. [1])

All these things are indicated partly in the paper of LASKER, partly in a report of an address in Göttingen by E. NOETHER [2]); however, they have not yet been investigated with the necessary precision. It seemed, therefore, desirable to treat the whole connected complex again with the most modern and most simple methods (especially without chains of syzygies and without the theory of elimination).

In what follows we shall at once treat the more general case in which the functions under consideration are homogeneous in more than one system of variables. This gives nothing essentially new, nor is it more difficult, and the results have importance for certain geometrical applications, which LASKER has already pointed out. In order not to be drowned in indices I shall introduce 2 systems of variables only; it will at once be clear how the results read for more than 2 systems of variables.

I shall only suppose the principal notions exposed in my "Nullstellentheorie der Polynomideale" [3]) to be known.

After an introductory part I, which contains only formal trivialities, part II gives a theory of the characteristic function entirely based on LASKER's method of argumentation. III deals in a general way with series of composition for primary ideals, independently of the preceding. For this part a manuscript of E. NOETHER has been gratefully made use of. IV connects the different ways of investigation and V gives the geometrical application.

I. *Homogeneous ideals and multifold-projective spaces.*

§ 1. Let Γ be a field and $x_0, \ldots, x_n, y_0, \ldots, y_m$ variables. By *forms* we shall understand such polynomials in $R=\Gamma[x_0, \ldots, y_m]$ as are homogeneous in $x_0, \ldots, x_n$ as well as in $y_0, \ldots, y_m$. A *homogeneous ideal* or *H-ideal* in R is an ideal that together with any polynomial f always contains all homogeneous additive component parts of f. Evidently a homogeneous ideal has a homogeneous basis, for the polynomials of an arbitrary basis can be decomposed into homogeneous component

[1]) This generalisation differs from the one which I published in Math. Ann. **99**, p. 497: there we have to do with the far more difficult case that an M_r and an M_{n-r} in projective space P_n intersect, whereas in what follows here the intersection of an M_r and an M_{n-1} is investigated. This separation is due to methodical reasons: the simplest case of the intersection of a curve M_1 with a spread M_{n-1} may be treated with the characteristic function only; by the aid of the notion length the theory can easily be extended to M_r en M_{n-1}; but for the other generalisation M_r and M_{n-r} these two notions are certainly insufficient; in this case we need besides the characteristic function also the general notion of multiplicity as has been indicated in the paper in question in the Math. Ann.

[2]) Jahresberichte Deutsche Mathematiker-Vereinigung **34**, (1925) p. 101.

[3]) Math. Ann. **96**, (1926) p. 183, §§ 1—5.

parts. Conversely any ideal for which a basis consisting of forms can be found, is homogeneous.

§ 2. By a *point of Cartesian space* C_{n+m} we understand a system of elements $\{\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m\}$ of Γ or of a field containing Γ. By a point $x = \{\xi_0, \ldots, \xi_n; \eta_0, \ldots, \eta_m\}$ of the *twofold-projective space* $P_{n,m}$ we understand a class of systems of elements $\{\lambda\xi_0, \ldots, \lambda\xi_n; \mu\xi_0, \ldots, \mu\xi_n\}$; where $\xi_0, \ldots, \xi_n, \eta_0, \ldots, \eta_m$ are fixed elements of Γ or of a field containing Γ, so that not all $\xi_i = 0$ and not all $\eta_i = 0$, and where λ and μ describe all elements of Γ or of the larger field in question. If it is necessary to indicate more closely which larger field Ω we have in view, we write $C_{n+m}(\Omega)$ resp. $P_{n,m}(\Omega)$. The $(n+1)+(m+1)$ elements ξ_i, η_k are called *coordinates* of the point. The elements $\lambda\xi_i, \mu\eta_k$ ($\lambda \neq 0$, $\mu \neq 0$) may also be used as coordinates of the same point. The points $\{\xi, \eta\}$ of $P_{n,m}$ are evidently in a one-one correspondence with the pairs of points $\{\xi\}, \{\eta\}$ of two single projective spaces P_n, P_m.

If $\xi_0 \neq 0$, $\eta_0 \neq 0$, we can associate a point

$$\overline{X} = \left\{ \frac{\xi_1}{\xi_0}, \ldots, \frac{\xi_n}{\xi_0}; \frac{\eta_1}{\eta_0}, \ldots, \frac{\eta_m}{\eta_0} \right\}$$

of C_{n+m} to the point $X = \{\xi, \eta\}$ of $P_{n,m}$. In this way the points X of $P_{n,m}$ for which $\xi_0 \neq 0$, $\eta_0 \neq 0$, are in a one-one correspondence with the points $\overline{X}$ of C_{n+m}. The elements $\frac{\xi_i}{\xi_0}, \frac{\eta_i}{\eta_0}$ are called *non-homogeneous coordinates* of the point X. If a finite number of points is given, we can always attain by a linear transformation of coordinates that we have for all these points at the same time $\xi_0\eta_0 \neq 0$. This is at once possible if the field Γ has an infinite number of elements; in the other case the field must first be extended, e.g. by adjunction of variables.

§ 3. By *zeros* of an ideal M in R we understand such systems of values $\{\xi_0, \ldots, \xi_n, \eta_0, \ldots, \eta_m\}$ for which all polynomial, of the ideal become $= 0$. The elements ξ_i, η_i may be taken from an arbitrary domain containing Γ. All zeros of an ideal constitute together the *manifold* of the ideal in C_{n+m+2}.

Let M be an H-ideal in R. We suppose that M has zeros where not all $\xi_i = 0$ and not all $\eta_i = 0$. If $\{\xi, \eta\}$ is such a zero, all systems of elements $\{\lambda\xi, \mu\eta\}$ are zeros and these may be united to a "*class of zeros*", which is a point of the projective space $P_{n,m}$. The manifold of M may, therefore, also be considered as an algebraic manifold in $P_{n,m}$.

We shall now show that between the H-ideals in $R = \Gamma[x_0, \ldots, x_n, y_0, \ldots, y_m]$ and the ideals in $\overline{R} = \Gamma[x_1, \ldots, x_n, y_1, \ldots, y_m]$ there exists a correspondence analogous to that between the points X of $P_{n,m}$ and the points $\overline{X}$ of C_{n+m} (§ 2).

§ 4. If in a polynomial $f(x, y)$ of R we put $x_0 = 1, y_0 = 1$, we get a polynomial $\bar{f}$ of $\bar{R}$. In this way we get any polynomial $\bar{f}$ of $\bar{R}$ at least once. Sums $f+g$ and products $f.g$ pass into sums $\bar{f}+\bar{g}$ and products $\bar{f}\cdot\bar{g}$. (Homomorphism or meroedric isomorphism). Hence an ideal M in R transforms into an ideal $\bar{M}$ in $\bar{R}$.

If as a special case for M we choose an H-ideal, we need not take all polynomials f of M for the determination of the polynomials $\bar{f}$ of $\bar{M}$, but we can restrict ourselves to the forms of f. For we can split up any polynomial f of M into homogeneous component parts which belong likewise to M, and we can multiply these component parts by such factors $x_0^{\varrho} y_0^{\tau}$ that they get the same degree. In this case the sum is a form of M, and for $x_0 = 1$ it gives the same as the original polynomial.

§ 5. If conversely an ideal $\bar{M}$ of $\bar{R}$ is given, all forms of R that through the substitution $x_0 = 1$ are transformed into polynomials of $\bar{M}$, generate an H-ideal M_0, which, apparently, through the substitution $x_0 = 1$, $y_0 = 1$ again transforms into the ideal $\bar{M}$. The H-ideal M_0 constructed in this way, is called *the H-ideal equivalent to* $\bar{M}$.

§ 6. The following formal relations are easily verified:

1. If $M = f_1, \ldots, f_r)$, then $\bar{M} = (\bar{f}_1, \ldots, \bar{f}_r)$.
2. $\overline{(M, N)} = \bar{M}, \bar{N})$ (Consequence of 1).
3. $\overline{M.N} = \bar{M}.\bar{N}$ (Consequence of 1).
4. If M is primary and P the corresponding prime ideal, then $\bar{M}$ is primary, and $\bar{P}$ is the corresponding prime ideal. (If M is prime, $\bar{M}$ is also prime).
5. If $\bar{M}$ is primary and $\bar{P}$ the corresponding prime ideal, M_0 is also primary and P_0 the corresponding prime ideal. (If $\bar{M}$ is prime, M_0 is also prime).

§ 7. Different H-ideals M can give the same ideal $\bar{M}$; among these H-ideals the H-ideal M_0 that is equivalent to $\bar{M}$ plays a special part. M_0 can be defined directly by means of M:

THEOREM 1. *M_0 is the aggregate of all the polynomials f that satisfy a congruence of the form*

$$x_0^{\varrho}\, y_0^{\sigma}\, f \equiv 0\ (M)$$

PROOF. As M_0 is an H-ideal, it is sufficient to prove the theorem for the forms in M_0. The forms in M_0 are those forms f_0 which by the substitution $x_0 = 1$ pass into polynomials $\bar{f}$ of $\bar{M}$, and these polynomials arise in their turn by the substitution $x_0 = 1$ from forms

f of M. Now if two forms f_0 and f by the substitution $x_0 = 1$, $y_0 = 1$ lead to the same polynomial $\bar{f}$, they only differ from each other by factors x_0 and y_0 and conversely. From this follows what was to be proved.

As a consequence of this theorem if in all zeros $\{\xi, \eta\}$ of the ideal M we always have either $\xi_0 = 0$ or $\eta_0 = 0$, then M_0 is the unity-ideal. For if the polynomial $x_0 y_0$ becomes equal to zero in all zeros of M, we have, according to HILBERT's theorem of zeros,

$$x_0^\lambda \, y_0^\lambda \, . \, 1 \equiv 0 \, (M)$$

and consequently 1 belongs to M_0.

§ 8. THEOREM 2. *If M is given as an intersection* (*L. C. M.*) *of primary ideals:*

$$M = [Q_1, \ldots, Q_r]$$

and if among the ideals $Q_1, \ldots, Q_r$ only $Q_1, \ldots, Q_s$ have the property that in all their zeros $\xi_0 \eta_0 = 0$, we have

$$M_0 = [Q_{s+1}, \ldots, Q_r] \qquad (\textit{resp. } M_0 = (1) \textit{ for } s = r).$$

PROOF. A polynomial f belongs only then to M_0 if (for a sufficiently large ϱ and σ) $x_0^\varrho y_0^\sigma f$ belongs to M, i.e. to all Q_i $(i = 1, \ldots, r)$. For $i = 1, \ldots, s$ the condition $x_0^\varrho y_0^\sigma f \equiv 0 \, (Q_i)$ for sufficiently large ϱ, σ is no condition at all for f, for a power $x_0^\lambda y_0^\lambda$ lies ipso facto in Q_i (see final remark in § 7). For $i = s+1, \ldots, r$ $x_0^\varrho y_0^\sigma f \equiv 0 \, (Q_i)$ is equivalent to $f \equiv 0 \, (Q_i)$, as $x_0^\varrho y_0^\sigma$ does not contain the manifold of Q. Hence $x_0^\varrho y_0^\sigma f \equiv 0 \, (M)$ is equivalent to $f \equiv 0 \, (Q_i)$ $(i = s+1, \ldots, r)$, q. e. d.

§ 9. We now choose among the ideals $Q_1, \ldots, Q_r$ those that have only zeros $\{0, \ldots, 0;\ \eta_0, \ldots, \eta_m\}$ or $\{\xi_0, \ldots, \xi_m;\ 0, \ldots, 0\}$, in other words that have no zeros at all in projective space $P_{n,m}$. We shall call such ideals *projectively irrelevant*. For each of the other Q_i we can find a zero where at least one $\xi_i \neq 0$ and at least one $\eta_k \neq 0$. By means of a linear transformation (if necessary after extension of the field Γ) we can ensure that in this finite number of points at the same time $\xi_0 \neq 0$ and $\eta_0 \neq 0$ hold. In what follows we shall always suppose this transformation to have been applied beforehand. By so doing we ensure that the projectively irrelevant components of M are the only ones whose zeros are contained in $\xi_0 \eta_0 = 0$. We can now formulate theorem 2 in the following way:

If M is given as an intersection of primary ideals, we get M_0 by omitting the projectively irrelevant components.

§ 10. If $X = \{\lambda\,\xi, \mu\,\eta\}$ is a class of zeros of M in $P_{n,m}$, and if $\xi_0 =|= 0$, $\eta_0 =|= 0$, we can suppose $\xi_0 = 1$, $\eta_0 = 1$. In this case $\overline{X} = \{\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m\}$ is a zero of $\overline{M}$ in C_{n+m}.

If, conversely, $\overline{X}$ is a zero of $\overline{M}$, K is a class of zeros of M. We get, therefore, the manifold of $\overline{M}$ in C_{n+m} from that of M in $P_{n,m}$, by finding the points $\overline{X}$ in C_{n+m} corresponding to all points X of the latter manifold where $\xi_0 \neq 0$ and $\eta_0 \neq 0$.

§ 11. If, especially, $M = P$ is a prime ideal that is not projectively irrelevant, we have, according to the theorem of § 9, $P_0 = P$. Further we have:

THEOREM 3. *If P is a prime ideal that has not exclusively zeros of the form $\{0, \dots, 0; \eta_0, \dots, \eta_m\}$ or $\{\xi_1, \dots, \xi_n; 0, \dots, 0\}$, $\overline{P}$ is again prime, and each general zero $\{\xi_1, \dots, \xi_n : \eta_1, \dots, \eta_m\}$ of P yields a general zero $\{\lambda, \lambda\xi_1, \dots, \lambda\,\xi_n; \mu, \mu\eta_1, \dots, \mu\eta_m\}$ of P where λ and μ are variables* [1]).

PROOF. We have already seen (§ 6) that P is prime; in the same way that $\{\lambda, \lambda\,\xi_1, \dots, \lambda\,\xi_n; \mu, \mu\,\eta_1, \dots, \mu\,\eta_n\}$ is a zero of P (§ 10). If f is a polynomial in R and if $f(\lambda, \lambda\,\xi_1, \dots, \lambda\,\xi_n; \mu, \mu\,\eta_1, \dots, \mu\,\eta_m) = 0$, we can in the first place decompose f into homogeneous component parts:

$$f = \Sigma\, f_i\,.$$

If γ_i, δ_i are the degrees of f_i, we have

$$\Sigma\, f_i\,(\lambda, \lambda\xi_1, \dots, \lambda\xi_n; \mu, \mu\eta_1, \dots, \mu\eta_m) = 0$$
$$\Sigma\, \lambda^{\gamma_i}\,\mu^{\delta_i} f_i\,(1, \xi_1, \dots, \xi_n; 1, \eta_1, \dots, \eta_m) = 0$$

identically in λ, μ, hence

$$f_i\,(1, \xi_1, \dots, \xi_n; 1, \eta_1, \dots, \eta_m) = 0$$
$$\overline{f_i} = f_i\,(1, x_1, \dots, x_n; 1, y_1, \dots, y_m) \equiv 0\,(\overline{P})$$
$$f_i \equiv 0\,(P_0) \equiv 0\,(P)$$
$$f = \Sigma\, f_i \equiv 0\,(P).$$

Consequently $\{\lambda_1\,\lambda\,\xi_1, \dots, \lambda\,\xi_n; \mu, \mu\,\eta_1, \dots, \mu\,\eta_m\}$ is a general zero of P.

§ 12. If r is the dimension of $\overline{P}$, $r + 2$ is the dimension of P because by the adjunction of the variables λ, μ the degree of transcendence of the general zero is augmented by 2. For the determination of the number of dimensions of the manifold of P in projective space, the parameters λ, μ are usually considered as not essential and by the number of dimensions not $r + 2$, is understood, but r. Accordingly the dimension of the manifold in the twofold-projective space differs 2 unities from the dimension of the corresponding H-ideal [2]).

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[1]) For the notion of a "general zero" of a prime ideal see "Nullstellentheorie der Polynomideale" § 3, Math. Ann. **96** (1926) p. 183.

[2]) In the ordinary (single-) projective space this difference is 1.

The number r, which for H-ideals generally plays a larger part than the real dimension $r+2$, may be called the *reduced dimension* of P.

Projectively irrelevant ideals P have no reduced dimension, because, in this case $\overline{P}=(1)$, hence $\overline{P}$ has no dimension. For complete inductions it is sometimes desirable to assign the reduced dimension -1 to the projectively irrelevant ideals (just as we can assign the dimension -1 to the unit ideal and to the empty point class).

The reduced dimension of a primary ideal is defined as that of the corresponding prime ideal, and that of an arbitrary ideal as the largest of the reduced dimensions of the primary components.

II. *The characteristic function of* HILBERT.

§ 13. HILBERT and LASKER have investigated as a function of ϱ the number of independent linear equations which a form of degree ϱ in $x_1, \ldots, x_n$ must satisfy in order to belong to a given H-ideal. The principal result is that for sufficiently large values of ϱ this number can be represented by a whole rational function, the "characteristic function". LASKER gives a few indications how the theorems proved by him can be extended to multiple homogeneous ideals. As these indications are not quite clear in every respect, I shall treat the extension in question again. The methods for the proofs are LASKER's. In order to understand this §, however, it is not necessary to be acquainted with LASKER's investigations.

§ 14. *Definitions.* Let Γ be again a field and $R=\Gamma[x_0, \ldots, x_n; y_0, \ldots, y_m]$. By $\varphi(\varrho, \sigma)$ we indicate the number of power products of degree ϱ in the x, σ in the y, by $\varphi(\varrho, \sigma;\ A)$ the number of linearly independent forms of the degrees ϱ, σ in a given H-ideal A, and finally by $\chi(\varrho, \varrho, A)$ the number of mod. A linearly independent forms of the degrees ϱ, σ or, what amounts to the same, the number of independent linear equations which forms of the degrees ϱ, σ must satisfy in order to belong to the ideal A.

§ 15. Apparently

$$\varphi(\varrho, \sigma) = \varphi(\varrho, \sigma; (1)) \quad \ldots \ldots \ldots \quad (1)$$

$$\chi(\varrho, \sigma; A) = \varphi(\varrho, \sigma) - \varphi(\varrho, \sigma; A) \quad \ldots \ldots \quad (2)$$

§ 16. THEOREM 4. *If A, B are two H-ideals, (A, B) their sum or G. C. M., $[A, B]$ their intersection, we have*

$$\varphi(\varrho, \sigma; (A, B)) = \varphi(\varrho, \sigma: A) + \varphi(\varrho, \sigma; B) - \varphi(\varrho, \sigma; [A, B]) \quad . \quad (3)$$

$$\chi(\varrho, \sigma; (A, B)) = \chi(\varrho, \sigma; A) + \chi(\varrho, \sigma; B) - \chi(\varrho, \sigma: [A, B]) \quad . \quad (4)$$

(3) is at once evident, for the forms of the degrees ϱ, σ in the ideal (A, B) form a linear set which is additively composed of the linear sets

of the forms of the same degrees in A and B; these sets have all forms of the same degrees in $[A, B]$ in common.

(4) follows from (2) and (3).

§ 17. THEOREM 5. *If f is a form of the degrees γ, δ and if f is relatively prime with respect to A (i.e. that $fg \equiv O(A)$ implies $g \equiv O(A)$), we have*

$$\chi(\varrho, \sigma; (A, f)) = \chi(\varrho, \sigma; A) - \chi(\varrho - \gamma, \sigma - \delta; A) \quad . \quad . \quad . \quad (5)$$

PROOF. The form f generates an ideal F. We shall first determine the functions $\varphi(\varrho, \sigma; F)$ and $\varphi(\varrho, \sigma; [A, F])$.

All forms of the degree ϱ, σ in F have the form $f \cdot g$, where g is a form of the degrees $\varrho - \gamma, \sigma - \delta$. Hence

$$\varphi(\varrho, \sigma; F) = \varphi(\varrho - \gamma, \sigma - \delta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The forms of the degree ϱ, σ in $[A, F]$ must lie in F (and consequently have the form $f \cdot g$) as well as in A; this gives

$$fg \equiv 0\,(A)$$

This congruence implies $g \equiv 0\,(A)$ and inversely. Hence the number of linearly independent forms $f \cdot g$ of the degrees ϱ, σ in $[A, F]$ is equal to the number of linearly independent forms g of the degrees $\varrho - \gamma, \ \sigma - \delta$ in A. Hence:

$$\varphi(\varrho, \sigma; [A, F]) = \varphi(\varrho - \gamma, \sigma - \delta; A) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

According to (4) we have:

$$\chi(\varrho, \sigma; (A, F)) = \chi(\varrho, \sigma; A) + \chi(\varrho, \sigma; F) - \chi(\varrho, \sigma; [A, F])$$

according to (2):

$$= \chi(\varrho, \sigma; A) - \varphi(\varrho, \sigma: F) + \varphi(\varrho, \sigma; [A, F])$$

according to (6), (7):

$$= \chi(\varrho, \sigma; A) - \varphi(\varrho - \gamma, \sigma - \delta) + \varphi(\varrho - \gamma, \sigma - \delta; A)$$

according to (2):

$$= \chi(\varrho, \sigma: A) - \chi(\varrho - \gamma, \sigma - \delta; A).$$

§ 18. THEOREM 6. *For projectively irrelevent ideals (i.e. for those ideals A that have no other zeros than $\{0, \ldots, 0 : \eta_0, \ldots, \eta_n\}$ and $\{\xi_1, \ldots, \xi_n ; 0, \ldots, 0\}$ and for sufficiently large ϱ, σ we have:*

$$\chi(\varrho, \sigma; A) = 0.$$

PROOF. All polynomials of the ideal

$$(x_0, \ldots, x_n) \cdot (y_0, \ldots, y_n)$$

become zero in the zeros of A; hence according to HILBERT's theorem of zeros we have for a certain λ

$$(x_0, \ldots, x_n)^\lambda \cdot (y_0, \ldots, y_n)^\lambda \equiv 0\,(A)$$

i.e. all power products whose degrees are $\geqq \lambda$, lie in A. Consequently the same holds good for arbitrary forms whose degrees are $\geqq \lambda$.

§ 19. THEOREM 7. *If A is an H-ideal and d its reduced dimension, $\chi(\varrho, \sigma; A)$ is represented for high values of ϱ and σ by an expression of the form:*

$$\chi(\varrho, \sigma; A) = \sum_{i+j \leqq d} a_{ij} \binom{\varrho}{i} \binom{\sigma}{i} \quad (\varrho \geqq \varrho_0, \sigma \geqq \sigma_0) \quad . \quad . \quad . \quad (8)$$

with whole rational coefficients a_{ij}.

This expression is called the *characteristic function* of the ideal A. If the binomial coefficients are written in full, it may also be written as a polynomial in ϱ and σ.

PROOF. If $d = -1$, i. e. if A is projectively irrelevant, then according to theorem 6 we have $\chi(\varrho, \sigma; A) = 0$ for high values of ϱ and σ. If, therefore, we consider 0 as a polynomial of the degree -1 in ϱ and σ, the theorem holds good for $d = -1$. In the proof which follows, it will become clear that in fact we can begin the complete induction at $d = -1$.

We shall now assume that $d \geqq 0$ and that the theorem has been proved for all reduced dimensions $< d$. The given ideal A is the intersection of primary ideals

$$A = [Q_1, \ldots, Q_r]$$

and all Q_i have the reduced dimension d at most. If we suppose for a moment that the theorem has already been proved for primary ideals of the same dimension and (if $r > 1$) for ideals with fewer than r primary components, there follows from (4):

$$\chi(\varrho, \sigma; A) = \chi(\varrho, \sigma; [[Q_1, \ldots, Q_{r-1}], Q_r]])$$
$$= \chi(\sigma, \sigma; [Q_1, \ldots, Q_{r-1}]) + \chi(\varrho, \sigma; Q_r) - \chi(\varrho, \sigma; ([Q_1, \ldots, Q_{r-1}], Q_r))$$

The former two terms on the right-hand side refer to ideals with fewer primary components than A, and the latter term to an ideal of fewer dimensions, hence all functions, on the right-hand side have the form (8), so that the same is valid for the left-hand side. It is, therefore, only necessary to prove the theorem for primary ideals. Let A be primary and let P be the corresponding prime ideal. We choose two linear forms $l_1 = \sum_0^n u_i x_i$, $l_2 = \sum_0^m v_i y_i$ that do not belong to P. The ideals (A, l_1) and (A, l_2) have the reduced dimensions $d-1$ at most, and the forms l_1, l_2 are relatively prime with respect to A. Hence (5) holds good:

$$\left.\begin{aligned} \chi(\varrho, \sigma; (A, l_1)) &= \chi(\varrho, \sigma; A) - \chi(\varrho - 1, \sigma; A) \\ \chi(\varrho, \sigma; (A, l_2)) &= \chi(\varrho, \sigma; A) - \chi(\varrho, \sigma - 1; A) \end{aligned}\right\} \quad . \quad . \quad . \quad (9)$$

According to the supposition we have for $\varrho \geqq \varrho_0, \sigma \geqq \sigma_0$:

$$\chi(\varrho, \sigma : (A, l_1)) = \sum_{i+j \leqq d-1} b_{ij} \binom{\varrho}{i} \binom{\sigma}{j}$$

$$\chi(\varrho, \sigma; (A, l_2)) = \sum_{i+j \leqq d-1} c_{ij} \binom{\varrho}{i} \binom{\sigma}{j}$$

On summation we get from (9_1):

$$\chi(\varrho, \sigma : A) - \chi(\varrho_0, \sigma ; A) = \sum_{\lambda=\varrho_0+1}^{\varrho} \chi(\lambda, \sigma ; (A, l_1))$$
$$= \sum_{i+j \leqq d-1} \sum_{\lambda=\varrho_0+1}^{\varrho} b_{ij} \binom{\lambda}{i} \binom{\sigma}{j}$$
$$= \sum_{i+j \leqq d-1} b_{ij} \left\{ \binom{\varrho+1}{i+1} - \binom{\varrho_0+1}{i+1} \right\} \binom{\sigma}{j}$$

and in the same way from (9_2):

$$\chi(\varrho_0, \sigma ; A) - \chi(\varrho_0, \sigma_0 ; A) = \sum_{i+j \leqq d-1} c_{ij} \binom{\varrho_0}{i} \left\{ \binom{\sigma+1}{j+1} - \binom{\sigma_0+1}{j+1} \right\}$$

If, finally, we add these two equations we find:

$$\left.\begin{aligned} \chi(\varrho, \sigma ; A) = \chi(\varrho_0, \sigma_0 ; A) + \sum_{i+j \leqq d-1} b_{ij} \binom{\varrho+1}{i+1} \binom{\sigma}{j} - \\ - \sum_{i+j \leqq d-1} b_{ij} \binom{\varrho_0+1}{i+1} \binom{\sigma}{j} + \sum_{i+j \leqq d-1} c_{ij} \binom{\varrho_0}{i} \binom{\sigma+1}{j+1} - \sum_{i+j \leqq d-1} c_{ij} \binom{\varrho_0}{i} \binom{\sigma_0+1}{j+1} \end{aligned}\right\} \quad (10)$$

The right hand side has in fact the form (8), if we take into consideration that $\binom{\varrho+1}{i+1} = \binom{\varrho}{i+1} + \binom{\varrho}{i}$, etc.

§ 20. Among the numbers a_{ij} we shall pay special attention to those for which $i + j = d$, i.e. to the coefficients of the terms of highest degree in the characteristic function. These a_{ij} are called the *degrees* of the ideal A; their geometrical signification will appear later. For the degrees we introduce the symbol $a_{ij}(A)$.

From the proof given above the following theorems follow directly:

THEOREM 8. *Any degree a_{ij} of an H-ideal is the sum of the corresponding degrees of the primary components with the same reduced dimension.*

THEOREM 9. *If Q primary, P the corresponding prime ideal, l_1 a linear form in x_i, and $l_1 \not\equiv 0\ (P)$, we have*

$$a_{ij}(Q) = a_{i-1,j}(Q, l_1). \quad \ldots \ldots \quad (11)$$

In the same way if l_2 is a linear form in the y_i and $l_2 \not\equiv 0\ (P)$:

$$a_{ij}(Q) = a_{i,j-1}(Q, l_2). \quad \ldots \ldots \quad (12)$$

A generalisation of theorem 9 is:

THEOREM 10. *If Q is a primary ideal, P the corresponding prime ideal, f a form of the degrees γ, δ, and $f \not\equiv 0\ (P)$, we have:*

$$a_{ij}(Q, f) = \gamma \cdot a_{i+1,j}(Q) + \delta \cdot a_{i,j+1}(Q)$$

PROOF. The suppositions of theorem 5 are fulfilled, accordingly (5) is valid for $A = Q$. If we expand the two members of (5) according to theorem 7 with respect to binomial coefficients, and if we compare the coefficients of $\varrho^i \sigma^j$ on the two sides, we find the result in question directly.

§ 21. THEOREM 11. *The degrees of an H-ideal are* $\geqq 0$.

PROOF. For $d = 0$ the characteristic function becomes a constant a_{00}. In this case the ideal has a finite number of zeros in the projective space (with coordinates from Γ or a field containing Γ). There are polynomials of an arbitrarily high degree which are in at least one of these zeros $\neq 0$, and which, therefore, do not belong to the ideal. Hence the characteristic function a_{00} is in this case even > 0.

Now suppose the theorem to be valid for arbitrary ideals of less than d reduced dimensions. In this case for primary ideals of d reduced dimensions the theorem follows by formulas (11) and (12). For arbitrary ideals of d dimensions the same follows from theorem 8.

§ 22. THEOREM 12. *The characteristic function of an H-ideal A is the same as that of the ideal A_0 that is found by omitting the projectively-irrelevant primary components from A (cf. § 8).*

PROOF. Put

$$M = [Q_1, \ldots, Q_r],$$
$$Q_1, \ldots, Q_s \text{ projectively irrelevant},$$

hence

$$M_0 = [Q_{s+1}, \ldots, Q_r].$$

If we put $M_1 = [Q_1, \ldots, Q_s]$, we have $M = [M_0, M_1]$.

Now (4) implies:

$$\chi(M; \varrho, \sigma) = \chi(M_0; \varrho, \sigma) + \chi(M_1; \varrho, \sigma) - \chi((M_0; M_1); \varrho, \sigma)$$
$$= \chi(M_0; \varrho, \sigma) \qquad (\varrho \geqq \varrho_0, \sigma \geqq \sigma_0),$$

because the characteristic functions of the projectively-irrelevant ideals M_1 and (M_0, M_1) are zero (§ 18).

III. *Series of Composition of Primary Ideals.* [1]

§ 23. Let Q be a primary ideal in a domain (German: „Ring") R, let P be the corresponding prime ideal, and suppose $P \neq R$. By a *series of composition* of Q we understand a finite series of primary ideals

$$(Q_0 = R); \quad Q_1 = P; \quad Q_2; \ldots; \quad Q_l = Q,$$

(where the first term $Q_0 = R$ may be reckoned to belong to the series or not) with the following properties:

1) Q_i is primary and P is the corresponding prime ideal $(i = 1, \ldots, l)$
2) $Q_{i+1} \neq Q_i \qquad (i = 1, \ldots, l-1)$
3) $Q_{i+1} \equiv 0 \, (Q_i) \qquad (i = 1, \ldots, l-1)$.
4) It is impossible to insert a primary ideal between Q_i and Q_{i+1} so that the properties 1), 2), 3) remain valid.

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[1]) This § is a partial elaboration of ideas of E. NOETHER's; cf. her address in Göttingen on the numbers of HILBERT, Jahresbericht D.M.V. **34**, (1925), p. 101. Cf. also W. KRULL, verallgemeinerte Abelsche Gruppen, Sitzungsberichte Heidelberger Akad. 1926.

We shall now prove *the existence of a series of composition for any primary ideal* Q on the supposition that the corresponding prime ideal P has a *finite basis.*

§ 24. We shall first simplify the problem by passing from the ring R to the ring of restclasses $R' = R/Q$. Every ideal A, divisor of Q, gives rise to an ideal $A' = A/Q$ in R'. Especially P leads to a prime ideal P', and any primary ideal Q_i corresponding to P leads to a primary ideal Q'_i corresponding to P'. Q itself leads to $Q' = Q/Q$: the zero ideal in R'. Conversely A is uniquely defined by A': members of A are the members of all restclasses with respect to Q in A'. Accordingly it is sufficient to prove the existence of a series of composition

$$Q'_1 = P', \quad Q'_2, \ldots, Q'_l = (0).$$

The basis of P leads to a basis of P'.

We shall now again pass from the ring R' to a domain R^* consisting of all fractions $\frac{a}{b}$, $b \not\equiv 0(P')$. We put $\frac{a}{b} = \frac{a'}{b'}$, if $a\,b' = a'\,b$.

Addition and multiplication in this domain are as usually defined by the formulas

$$\frac{a}{b} + \frac{a'}{b'} = \frac{a\,b' + b\,a'}{b\,b'}$$

$$\frac{a}{b} \cdot \frac{a'}{b'} = \frac{a\,a'}{b\,b'} \text{ [1]}$$

The members $b \not\equiv 0\,(P)$ are no zero factors, for as the zero ideal Q' is primary and P the corresponding prime ideal, the relations $bc = 0$, $c \neq 0$ always implie $b \equiv 0\,(P)$.

The prime ideal P' generates a prime ideal P^* in R^* consisting of all fractions $\frac{a}{b}$; $a \equiv 0\,(P')$. In the same way any primary ideal Q'_i in R' of which the corresponding prime ideal is P', generates a primary ideal Q^*_i in R^* of which the corresponding prime ideal is P^*. All this can be verified without difficulty. Conversely Q'_i is defined unambiguously by Q^*_i : Q'_i consists of all numerators of the fractions that appear in Q^*_i [2]. It is, therefore, sufficient to prove the existence of a series of composition

$$Q^*_1 = P^*; \quad Q^*_2; \ldots; \quad Q^*_l = Q^*$$

The ring R^* has this advantage over R that it has the following properties, which R did not necessarily possess:

I. There is a unity, viz. $\frac{b}{b}$ (b arbitrary, $\not\equiv 0\,(P')$).

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[1]) E. STEINITZ, Algebr. Theorie der Körper § 3, J. f. M. **137**, (1910). H. GRELL, Beziehungen zwischen den Idealen verschiedener Ringe § 1, (5), Math. Ann. **97**, (1927), p. 490.

[2]) H. GRELL, loc. cit., § 6.

II. The equation $a \, . \, x = 1$ is always solvable for $a \equiv\mid\equiv 0 \, (P^*)$.

III. Q^* is the zero ideal.

After multiplication by $\frac{b}{b}$ the basis elements of P' can be used as basis elements for P^*.

We shall henceforward omit the asterisks and suppose R to possess the properties I, II, III.

Now let ϱ be the exponent of the zero ideal Q, i.e. the smallest number with the property $P^\varrho = 0$.

If we prove that a series of composition from P^λ to $P^{\lambda+1}$ ($\lambda = 1, \ldots, \varrho - 1$) is possible, we need only place all these series one after another to obtain a series of composition from P to Q.

The powers of $P, P^2, \ldots . P^\varrho$ are all different, for $P^\lambda = P^{\lambda+1}$ would implie $P^{\varrho-1} = P^\varrho = 0$.

Further P^λ has a finite basis $(f_1, \ldots, f_r)$, so that we can write:

$$P^\lambda = (P^{\lambda+1}, f_1, \ldots, f_r) \quad \ldots \ldots \ldots \quad (1)$$

We shall suppose as many f_i's as possible to be omitted on the right-hand side, so that if we omitted one more of them (1) would become fals. Now I assert that the series of ideals

$$Q_0^{(\lambda)} = (P^{\lambda+1}, f_1, \ldots, f_r) = P^\lambda$$

$$Q_1^{(\lambda)} = (P^{\lambda+1}, f_1, \ldots, f_{r-1})$$

$$\ldots \ldots \ldots \ldots$$

$$Q_{r-1}^{(\lambda)} = (P^{\lambda+1}, f_1)$$

$$Q_r^{(\lambda)} = P^{\lambda+1}$$

is a series of composition from P^λ to $P^{\lambda+1}$. We must, therefore, prove the properties 1—4 (§ 23).

1. It appears thus that $Q_i^{(\lambda)}$ is primary and that P is the corresponding prime ideal:

a. From $ab \equiv 0 \, (Q_i^{(\lambda)})$ and $b = \mid = 0 \; (P)$ we find on multiplication by b^{-1}:

$$a \equiv 0 \, (Q_i^{(\lambda)})$$

b. $a \equiv 0 \, (P)$ implies $a^\varrho \equiv 0 \; (Q) \equiv 0 \; (Q_i^{(\lambda)})$.

c. $a^\sigma \equiv 0 \, (Q_i^{(\lambda)})$ implies $a^\sigma \equiv 0 \, (P)$, hence $a \equiv 0 \; (P)$.

2. $Q_i^{(\lambda)} = \mid = Q_{i+1}^{(\lambda)}$ is evident, for else

$$(Q^{(\lambda)}, f_1, \ldots, f_i) = (Q^{(\lambda)}, f_1, \ldots, f_{i-1})$$

and then f_i might be omitted from the equation (1).

3. $Q_{i+1}^{(\lambda)} \equiv 0 \; (Q_i^{(\lambda)})$ is evident.

4. If it were possible to insert an ideal Q' between $Q_i^{(\lambda)}$ and $Q_{i+1}^{(\lambda)}$:

$$Q_{i+1}^{(\lambda)} \equiv 0 \; (Q') \equiv 0 \; (Q_i^{(\lambda)})$$

$$Q_{i+1}^{(\lambda)} = \mid = Q' = \mid = Q_i^{(\lambda)}$$

Q' would contain a member f' which does not lie in $Q^{(\lambda)}_{i+1}$. This member lies in $Q^{(\lambda)}_i = (Q^{(\lambda)}_{i+1}, f_{r-i})$, hence

$$f' \equiv h . f_{r-i} \quad (Q^{(\lambda)}_{i+1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If h lies in P we have:

$$h . f_{r-i} \equiv 0 \, (P P^{\lambda}) \equiv 0 \, (P^{\lambda+1}) \equiv 0 \, (Q^{(\lambda)}_{i+1})$$

and, therefore, $f' \equiv 0 \, (Q^{(\lambda)}_{i+1})$, against the supposition.

If h does not lie in P we can multiply (2) by h^{-1}:

$$f_{r-i} \equiv h^{-1} f' \quad (Q^{(\lambda)}_{i+1})$$

$$f_{r-i} \equiv 0 \qquad (Q')$$

$$Q^{(\lambda)}_i = (Q^{(\lambda)}_{i+1}, f_{r-i}) \equiv 0 \, (Q')$$

$$Q^{(\lambda)}_i = Q',$$

against the supposition [1]).

§ 25. According to the proof the series of composition constructed by us has the following properties besides the postulated properties 1—4:

5. It is even impossible to insert an arbitrary ideal Q' with the properties $Q_{i+1} \equiv 0 \, (Q') \equiv 0 \, (Q_i)$, $Q_{i+1} =\!|\!= Q' =\!|\!= Q_i$ between Q_i and Q_{i+1}, whether Q' is primary or not.

6. $$P Q_i \equiv 0 \, (Q_{i+1})$$

7. $$Q_i = (Q_{i+1}, f_i)$$

One might think that these properties are only consequences of the construction we have followed and that perhaps another series of composition does not possess these properties. We shall prove that, on the contrary, *these* 3 *properties are valid for any series of composition*, where we need not make the suppositions I, II, III for our ring R, but only the much less far reaching suppositions:

I*. *R contains a unity;*

II*. *The prime ideal P does not contain any true divisors besides R.*

PROOF of 5. Let Q' be an arbitrary ideal, $Q_{i+1} \equiv 0 \, (Q') \equiv 0 \, (Q_i)$. In order to reduce everything to 4. we shall prove that Q' most necessarilly be primary and that P is the corresponding prime ideal.

a. The relations $ab \equiv 0 \, (Q')$ and $b \equiv\!|\!\equiv 0 \, (P)$ imply in the first place $(b, P) =\!|\!= P$, hence $(b, P) = R$; accordingly any member of R has the form $h\,b + p \, (p \equiv 0 \, (P))$; especially:

$$1 = h\,b + p$$

$$a = h\,a\,b + a\,p \equiv a\,p \quad (Q')$$

$$a \equiv a\,p \equiv a\,p^2 \equiv \ldots \equiv a\,p^{\rho} \equiv 0 \, (Q').$$

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[1]) In the original Dutch paper the proofs of this § were not correct, at least not generally valid. A correction will be added. The English text given above is correct.

b. $a \equiv 0\,(P)$ implies $a^{\rho} \equiv 0\,(Q) \equiv 0\,(Q')$,

c. $a^{\sigma} \equiv 0\,(Q')$ implies $a^{\sigma} \equiv 0\,(P)$, hence $a \equiv 0\,(P)$.

PROOF of 6. We have

$$Q_{i+1} \equiv 0\,(PQ_i\,,\ Q_{i+1}) \equiv 0\,(Q_i)$$

hence, according to 5:

$$\left.\begin{array}{ll}\text{either} & (PQ_i\,,\ Q_{i+1}) = Q_i \\ \text{or} & (PQ_i\,,\ Q_{i+1}) = Q_{i+1}\end{array}\right\} \quad \ldots\ldots\ldots \quad (3)$$

In the first case

$$PQ_i \equiv Q_i\ (Q_{i+1})$$

$$Q_i \equiv PQ_i \equiv P^2Q_i \equiv \ldots \equiv P^{\rho}\, Q_i \equiv 0\,(Q_{i+1})$$

$$Q_i = Q_{i+1}$$

which is impossible. Hence only the second alternative of (3) remains; it implies

$$PQ_i \equiv 0\,(Q_{i+1}), \quad \text{q. e. d.}$$

PROOF of 7. Let f_i be a member of Q_i that does not belong to Q_{i+1}; then

$$(Q_{i+1}, f_i) \neq Q_{i+1}$$

and consequently, according to 5.:

$$(Q_{i+1}, f_i) = Q_i, \qquad \text{q.e.d.}$$

§ 26. According to the well known theorem of JORDAN—HÖLDER the existence of one single series of composition implies *that all series of composition with the properties* 2, 3, 5 *have the same lengths* l. In a domain with the properties I*, II* any series of composition for primary ideal has the properties 2, 3, 5 and all have, therefore, the same length; this result may be directly extended to arbitrary rings by means of the transformation $R \rightarrow R^*$.

The theorem of JORDAN—HÖLDER, however, is not strictly necessary for what follows; we avoid the use of this theorem by the following definition:

The *length* of a primary ideal Q is the length of the shortest series of composition for this ideal. In reality l is at the same time the length of any series of composition.

§ 27. *If a series of composition has the properties* 1—7, *the following modulus-isomorphism is valid*:

$$Q_i/Q_{i+1} \cong R/P \qquad (i = 0, \ldots, l-1) \quad \ldots\ldots \quad (4)$$

where both members must be considered as R-moduli [1] [2]. In terms of

[1]) An R-module is a set of members which together with a and b also contain $a+b$ and $a-b$ and together with a always $g\,.\,a$ where g is an arbitrary member of R. In our case those members are restclasses mod. Q_{i+1} resp. mod. P. A module-isomorphism is a one-one correspondence $a \rightarrow \bar{a}$ with the properties

$$a + b \rightarrow \bar{a} + \bar{b}, \quad g\,.\,a \rightarrow g\,.\,\bar{a}.$$

[2]) Cf. H. GRELL, Ordnungen in Zahl- und Funktionenkörpern, Math. Ann. **97**, (1927) p. 540.

the theory of groups we have to do with series of composition where all factor groups are isomorphic.

PROOF. As $Q_i = (Q_{i+1}, f_i)$ all members of Q_i are mod. Q_{i+1} multiples of f_i, i.e. any member of Q_i is $\equiv h \,.\, f_i\ (Q_{i+1})$. To any member h of R there corresponds a member $h \,.\, f_i$ of Q_i; if $h_1 - h_2 \equiv 0\ (P)$, we have $h_1 f_i - h_2 f_i \equiv 0\ (P\,Q_i) \equiv 0\ (Q_{i+1})$. If conversely $h_1 f_i - h_2 f_i \equiv 0\ (Q_{i+1})$ we have, according to $f_i \equiv\!\!|\!\!\equiv 0\ (Q_{i+1})$,

$$h_1 - h_2 \equiv 0\ (P).$$

If, therefore, for the members h of R we choose the congruence mod. P as definition of equality and for the members $h \,.\, f_i$ of Q_i the congruence mod. Q_{i+1}, the correspondence $h \rightarrow h f_i$ is a one-one correspondence. To $h_1 + h_2$ there corresponds $h_1 f_i + h_2 f_i$ and to $g \,.\, h$ there corresponds $g \,.\, h f_i$. In this way the isomorphism (4) for $i = 1, \ldots, l-1$ is proved. For $i = 0$ the members of (4) are identical.

IV. *The Degrees of an Ideal.*

§ 28. Let Q be a zero-dimensional primary ideal in $R = \Gamma[x_1, \ldots, x_n;\ y_1, \ldots, y_m]$. The H-ideal Q_0 equivalent to Q (§ 5) in $\Gamma\ [x_0, \ldots, x_n\ y_0, \ldots, y_m]$ has the reduced dimension zero (§ 13), and its characteristic function is, therefore, a constant $a_{00}\ (Q_0)$ that is at the same time the degree of Q_0. By the substitution $x_0 = 1$, $y_0 = 1$ the forms of the degrees ϱ, σ in Q_0 are transformed into all polynomials of $\overline{Q}$ of which the degrees are $\leqq \varrho, \leqq \sigma$; hence the number of the linearly independent polynomials of the degrees $\leqq \varrho, \leqq \sigma$ in $\overline{Q}$ is equal to that of the polynomials of the degrees ϱ, σ in Q. In the same way the number of mod. Q_0 linearly independent forms of the degrees ϱ, σ is equal to the number of mod. $\overline{Q}$ linearly independent polynomials of the degrees $\leqq \varrho, \leqq \sigma$ in R. As soon as ϱ, σ exceed a certain limit, the former number is constant viz. equal to the characteristic function $a_{00}\ (Q_0)$; consequently the same thing holds for the latter number:

There are only a finite number of mod. Q linearly independent polynomials in R and this number is equal to $a_{00}\ (Q_0)$.

We can also express this in the following way:

The domain of restclasses R/Q is a modulus of finite rank[1]) *relative to Γ.*

§ 29. The same result can also be reached in a different way without making use of H-ideals and characteristic functions. At the same time we shall find a remarkable relation between lengths and degrees.

Let in the first place P be a zero-dimensional prime ideal and let $\{\xi_1, \ldots, \xi_n;\ \eta_1, \ldots, \eta_m\}$ be its general zero. In this case all ξ_1, η are alge-

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[1]) Rank = Maximum number of elements linearly independent with respect to Γ.

braic relative to Γ, hence the field $\Gamma(\xi, \eta)$ has a finite degree relative te Γ. Let this degree be γ. In consequence of the isomorphism

$$\Gamma(\xi, \eta) \cong R/P$$ [1]

also R/P has a finite degree relative to Γ.

Now let Q be a primary ideal corresponding to P. According to § 24 there exists a series of composition:

$$(Q_0 = R);\ Q_1 = P;\ Q_2; \dots ; Q_l = Q.$$

As P (being a zero-dimensional prime ideal) has no true divisors the series of composition has the properties 1—7 of § 25, and according to § 26 these implie

$$Q_i/Q_{i+1} \cong R/P$$

Hence also:

$$\text{Rank } Q_i/Q_{i+1} = \text{Rank } R/P$$

Summation from $i = 0$ to $i = l - 1$ gives

$$\text{Rank } Q_0/Q_l = l.\ \text{Rank } R/P,$$
$$\text{Rank } R/Q = l.\ \text{Rank } R/P,$$

by which it is proved that the rank of R/Q is finite. The rank of R/Q will be called the *degree* of Q. At the same time we have found the following relation:

The degree of a zero-dimensional primary ideal is equal to this length multiplied by the degree of the corresponding prime ideal. [2]

§ 30. In order to extend these theorems to more-dimensional ideals, we shall first prove a few auxiliary theorems.

LEMMA. *Let* Q *be a primary ideal in* $\Gamma[x_1, \dots, x_n;\ y_1, \dots, y_n]$, P *the the corresponding prime ideal* $\{\xi_1, \dots, \xi_n, \dots\}$ *a general zero of* P. *If now* ξ_1 *is transcendent (variable) and* t *a variable which we adjoin to* Γ, *then the ideal* $(Q, x_1 - t)$ *in* $\Gamma(t)[x, y]$ *is also primary, and* $(P, x_1 - t)$ *is the corresponding prime ideal. The length of* $(Q, x_1 - t)$ *is equal to that of* Q.

PROOF. In the first place it is easily seen that a polynomial $f(x, y)$ which is independent of t, can only be $\equiv 0\ (Q, x_1 - t)$, if it is $\equiv 0\ (Q)$. To see this we need only write the congruence $f \equiv 0\ (Q, x_1 - t)$ in full as an equation between polynomials x, y and t, make the fractions disappear by multiplying both members by the denominator $n(t)$, put $t = x_1$ and divide the congruence $n(x_1) \cdot f(x, y) \equiv 0\ (Q)$ arising in this way by $n(x_1)$, which is allowed, because we have $n(\xi_1) \neq 0$ and hence $n(x_1) \not\equiv 0\ (P)$.

In order to prove that $(Q, x_1 - t)$ is primary and that $(P, x_1 - t)$ is the corresponding prime ideal, we must show that:

1. $ab \equiv 0\ (Q, x_1 - t)$ and $a \not\equiv 0\ (Q, x_1 - t)$ imply $b \equiv 0\ (P, x_1 - t)$.
2. $b \equiv 0\ (P, x_1 - t)$ implies $b^\rho \equiv 0\ (Q, x_1 - t)$, and conversely.

[1] Cf. „Nullstellentheorie" loc. cit. § 3, 2.

[2] Cf. H. GRELL, loc. cit. § 3, Satz 2.

We can suppose the polynomials a, b that are rational in t, to be whole and rational in t. Further we may everywhere replace t mod. (x_1-t) by x_1. If this is done, we have only polynomials in the x and y and all our congruences $a\,b \equiv (Q, x_1-t)$, etc. may, therefore, also be read as: $a\,b \equiv 0\,(Q)$, etc. Thus 1. and 2. only express that Q is primary and that P is the corresponding prime ideal.

In order to prove that the length of (Q, x_1-t) is the same as that of Q, we must show that there exists a one-one correspondence between the primary ideals Q' in $\Gamma[x, y]$ for which

$$Q \equiv 0\,(Q') \equiv 0\,(P) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the primary ideals Q'' in $\Gamma(t)\,[x, y]$, for which

$$(Q, x_1-t)\;Q \equiv 0\,(Q'') \equiv 0\,(P, x_1-t). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and that this one-one correspondence leaves intact the relation divisor-multiple.

If Q' is given, we put $Q'' = (Q', x_1-t)$. We know already that in this case Q'' is primary and that (P, x_1-t) is the corresponding prime ideal. It is evident that (2) holds good. From $f(x, y) \equiv 0\,(Q', x_1-t)$ there follows $f(x, y) \equiv 0\,(Q')$ (see above), hence Q' consists of all polynomials in Q'', that are independent of t; accordingly Q' is defined uniquely by Q''. It is evident that $Q'_1 \equiv 0\,(Q'_2)$ implies $Q''_1 \equiv 0\,(Q''_2)$, and conversely. We have still to prove that for Q'' we may choose any given primary ideal that satisfies (2).

Let Q'' be a primary ideal which satisfies (2) and of which, consequently, the corresponding prime ideal is (P, x_1-t). Let Q' be the aggregate of all polynomials in Q'' independent of t. Any polynomial of Q'' after being made whole in t by multiplication by the denominator $n(t)$, may be replaced mod. (x_1-t) by a polynomial that no longer depends on t, accordingly by a polynomial of Q'. Hence $Q'' = (Q', x_1-t)$. It is seen easily that Q' is primary and P the corresponding prime ideal.

§ 31. This lemma may be transformed into a theorem on H-ideals by means of the methods of Part I:

Let Q be a primary H-ideal in $\Gamma[x_0, \ldots, x_n\,; y_0, \ldots, y_n]$, P the corresponding prime ideal, $\{\lambda, \lambda\xi_1, \ldots\}$ a general zero of P. Let ξ_1 be transcendent relative to Γ. If we adjoin a variable t to Γ and if we put

$$Q^{(1)} = (Q, x_1 - t x_0),$$

then $Q_0^{(1)} =$ *the ideal of the polynomials f for which*

$$x_0^{\varrho}\, y_0^{\sigma} f \equiv 0\,(Q^{(1)}),$$

then $Q_0^{(1)}$ is primary and the ideal $P_0^{(1)}$ defined in an analogous way is the corresponding prime ideal. The length of $Q_0^{(1)}$ is equal to that of Q.

PROOF. Let us construct the non-homogeneous ideals $\overline{P}$, $\overline{Q}$, etc. as indicated in § 4. We have (§ 6):

$$\overline{Q^{(1)}} = (\overline{Q}, x_1 - t)$$

and the H-ideal equivalent to $\overline{Q^{(1)}}$ is $Q_0^{(1)}$. The same holds good for $\overline{P^{(1)}}$ and $P_0^{(1)}$. As the transformation to equivalent H-ideals leaves intact the properties prime, primary, corresponding prime and divisibility, we reduce all the properties that are to be proved for $Q_0^{(1)}$, $P_0^{(1)}$ to the same properties for $\overline{Q^{(1)}}$, $\overline{P^{(1)}}$, so that everything is reduced to former lemma.

We are now able to generalize the theorems of § 29 to more dimensions:

§ 32. *The degrees a_{ij} of a primary H-ideal Q are equal to the length of Q multiplied by the corresponding degrees of the corresponding prime ideal.*

PROOF. First of all let the reduced dimension of Q be equal to zero, hence the characteristic function a constant a_{00}, and at the same time the degree. If we then pass to inhomogeneous ideals $\overline{Q}$, $\overline{P}$, according to § 28 the degree of Q is equal to that of $\overline{Q}$ (i.e. to the rank of $\overline{R}/\overline{Q}$), and the degree of P is equal to that of $\overline{P}$; further the correspondence between the primary ideals Q corresponding to P and the primary ideals $\overline{Q}$ corresponding to $\overline{P}$ is a one-one correspondence; consequently the length of Q is equal to that of $\overline{Q}$. If l represents this length, according to § 29 we have:

$$\text{Degree } \overline{Q} = l \,.\, \text{Degree } \overline{P}$$

and, therefore:

$$\text{Degree } Q = l \,.\, \text{Degree } P.$$

We shall now suppose the theorem to be proved for all reduced dimensions $< d$ and the reduced dimension of Q to be equal to d ($d > 0$). Let $\{\lambda, \lambda\xi_1, \ldots, \lambda\xi_n; \mu, \mu\eta_1, \ldots\}$ be a general zero of P. As $d > 0$, one of the ξ_i or η_k must be transcendent; suppose e.g. ξ_1 transcendent. On account of § 20, if t is a variable, we have:

$$a_{ij}\,(Q) = a_{i-1,j}\,(Q, x_1 - tx_0) = a_{i-1,j}\,(Q^{(1)})$$
$$a_{ij}\,(P) = a_{i-1,i}\,(P, x_1 - tx_0) = a_{i-1,j}\,(P^{(1)})$$

According to § 22 the characteristic functions of $Q^{(1)}$, $P^{(1)}$ are the same as those of $Q_0^{(1)}$, $P_0^{(1)}$. Hence:

$$a_{ij}\,(Q) = a_{i-1,j}\,(Q_0^{(1)});\; a_{ij}\,(P) = a_{i-1,j}\,(P_0^{(1)})$$

According to § 31 the length of $Q_0^{(1)}$ is equal to that of Q, hence $= l$. According to the supposition of induction we have for the numbers on the right hand sides the relation

$$a_{i-1,j}\,(Q_0^{(1)}) = l \,.\, a_{i-1,j}\,(P_0^{(1)}),$$

which, accordingly, must also be valid for the numbers on the left hand-sides. Thus the theorem is proved for all $a_{ij}(Q)$ with $i > 0$. There remains $a_{0d}(Q)$.

By interchanging x and y a_{0d} can be treated in exactly the same way as formerly a_{d0}, provided one of the η is transcendent. If this is not the case only a finite number of proportions $1 : \eta_1 ; \ldots : \eta_n$ come into consideration for the zeros of Q. If we now choose a linear form $l(y)$ that does not become zero for any of these values, the ideals $(Q, l(y))$ and likewise $(P, l(y))$ become projectively irrelevant, hence

$$a_{0j}(Q) = a_{0,j-1}(Q, l(y)) = 0$$
$$a_{0j}(P) = a_{0,j-1}(P, l(y)) = 0.$$

In this case the assertion reduces to the triviality

$$0 = l \,.\, 0.$$

Thus the theorem is generally proved.

V. *The Geometrical Signification.*

§ 33. We have seen that the degrees of an H-ideal are equal to the sums of the corresponding degrees of the primary components of the highest dimension (§ 20) and that the degrees of these primary components are equal to their lengths multiplied by the degrees of the corresponding prime ideals (§ 32). The degrees of these prime ideals have a simple geometrical signification:

If M is the manifold of P in the projective $P_{n,m}$, then $a_{ij}(P)$ is the number of points of intersection of this manifold with a linear space that is given by i general linear equations in the x and j general linear equations in the y.

The coefficients of these equations must be considered as independent variables $u_1, u_2, \ldots$ and the coordinates of the points of intersection as algebraic functions of these variables, hence as members of a suitably chosen field Ω containing $\Gamma(u_1, u_2, \ldots)$. We suppose that we have to do with the normal case that Ω is a field of the first kind [1]).

PROOF. Let P be first an ideal of the reduced dimension zero; in this case $i = j = 0$, hence it remains to prove that the degree of P is equal to the number of zeros in a suitable chosen field containing Γ. If we number the coordinates in such a way that $x_0 \equiv\!|\!\equiv 0\,(P)$, $y_0 \equiv\!|\!\equiv (P)$, we can introduce non-homogeneous coordinates for all zeros and instead of P consider the corresponding non-homogeneous (zero-dimensional) ideal $\overline{P}$ (§ 4). The degree of $\overline{P}$ is the same as that of P and is equal to the rank of $\overline{R}/\overline{P}$ (§ 28) and accordingly equal to the degree of the field $\Gamma(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m)$. Zeros of $\overline{P}$ are the systems $\{\xi_1^{(\alpha)}, \ldots, \eta_n^{(\alpha)}\}$ conjugated with $\{\xi_1, \ldots, \xi_n, \ldots, \eta_m\}$ in a GALOISian extension Ω of Γ. According to a well known theorem in the theory of GALOIS the number of these conjugated systems is equal to the degree of the field $\Gamma(\xi_1, \ldots, \eta_m)$, hence equal to the degree of P, q. e. d.

Let us suppose this theorem to be proved for all ideals of reduced

[1]) I.e. that an equation irreducible in $\Gamma(u_1, u_2, \ldots)$ has no double roots in Ω.

dimension $< d$, and P to have the reduced dimension d. We shall first intersect the manifold M with one single linear form

$$l = \sum_0^n u_k x_k .$$

Let $\{\lambda, \lambda \xi_1, \ldots; \ldots \mu \eta_m\}$ be a general zero of P. Let at least one ξ_i be transcendent (otherwise (P, l) becomes projectively irrelevant as in the latter case of § 32, and our theorem becomes trivial). Then also $\sum u_k \xi_k$ is transcendent relative to $\Gamma(u_1, \ldots, u_n)$. We shall now adjoin $u_1, \ldots, u_n$ to Γ and we shall introduce $\sum_1^n u_k x_k = x_1^*$ as a new coordinate instead of x_1; in this case the general zero of P becomes

$$\{\lambda, \lambda\xi_1^*, \ldots, \lambda\xi_n \; ; \; \mu, \ldots, \mu \eta_m\} \; ; \; \xi_1^* = \sum_1^n u_k \xi_k$$

and we have

$$l = u_0 x_0 + x_1^* = x_1^* - t x_0 \qquad (t = - u_0).$$

All the conditions of the auxiliary theorem of § 31 are now fulfilled. If, therefore, we put

$$P^{(1)} = (P, l) = P, (P_1^* - t x_0)$$

$P_0^{(1)}$ is prime. In the same way as in § 32 we find

$$a_{ij}(P) = a_{i-1, j}(P_0^{(1)}).$$

According to the supposition of induction $a_{i-1, j}(P_0^{(1)})$ is the number of points of intersection of the manifold of $P_0^{(1)}$ with a linear space, given by i—1 general linear forms in the x, and j linear forms in the y. But the manifold of $P_0^{(1)}$ arises itself from that of P by intersection with the general linear form l. Thus the theorem is proved.

§ 34. Let again M be an algebraic manifold of r dimensions defined by a prime ideal P. By the *degrees* $a_{ij}(M)$ $(i + j = r)$ we understand the degrees $a_{ij}(P)$, i.e. the numbers of points of intersection of M with certain linear spaces (§ 23). We now put the question: *what can be said of the degrees of the intersection of M with a spread $f(x, y) = 0$?*

What the answer about must be may be found by considering a surface S in the ordinary projective space P_3. The intersection of S with another surface $f(x) = 0$ decomposes into different irreducible components whose degrees, multiplied by certain multiplicities, are together equal to the product of the degrees of S and f. (This appears e.g. by applying the theorem of BEZOUT to the intersection of the two surfaces with a plane chosen in a most general way).

An analagous theorem holds good in the general case. The intersection of M with the form f decomposes into irreducible manifolds; we shall only consider those which have exactly the dimension r—1 [1]). These

[1]) We can prove that these are the only ones. Cf. O. BLUMENTHAL, Math. Ann. **57**, 1903), p. 356. An algebraic proof of this theorem was communicated to me by W. KRULL.

may be found by seeking the primary components of the reduced dimension $r-1$ of the ideal (P, f). Let $Q_1, \ldots, Q_s$ be these components, $M_1, \ldots, M_s$ their manifolds. By the *multiplicity* of M_α as intersection of M and f we understand the length of the ideal Q; let l be this multiplicity. If further γ, δ are the degrees of f in x und y, *the relation*

$$\sum_\alpha l_\alpha \, . \, a_{ij} (M_\alpha) = \gamma \, . \, a_{i+1,j} (M) + \delta \, . \, a_{i,j+1} (M) \quad . \quad . \quad . \quad . \quad (1)$$

is valid.

In order to prove this relation we replace the degrees of the manifolds by the degrees of the corresponding prime ideals. The relation becomes

$$\sum_\alpha l_\alpha \, . \, a_{ij} (P_\alpha) = \gamma \, . \, a_{i+1,j} (P) + \delta \, . \, a_{i,j+1} (P)$$

or

$$\sum_\alpha a_{ij} (Q_\alpha) = \gamma \, . \, a_{i+1,j} (P) + \delta \, . \, a_{i,j+1} (P)$$

or

$$a_{ij} (P, f) = \gamma \, . \, a_{i+1,j} (P) + \delta \, . \, a_{i,j+1} (P).$$

This is only a special case of theorem 10 (§ 20).

§ 35. We have defined the multiplicities of the partial intersections M as the lengths of certain primary ideals. This definition is only justified by its success: the sum of the products of degree and multiplicity is given by formula (1) and is, therefore, independent of the special situation of M_r and M_{n-r} relative to each other; in other words the definition of multiplicity satisfies the "rule of conservation of number". One would be mistaken by assuming that the notion of length always leads to a definition of multiplicity that satisfies this condition; on the contrary, already in the determination of the points of intersection of an M_2 with another M_2 in P_4 the notion length leads to a definition of multiplicity such that in certain cases where the two surfaces have the degrees 1 and 4 it depends on their relative position whether the sum of the "multiplicities" of their points of intersection is 4 or 5. In these cases we must reject the notion length and try to find another definition of multiplicity.[1])

[1]) Cf. my paper on "Eine Verallgemeinerung des Bezoutschen Satzes", Math. Ann. **99** (1928), p. 497.