Mathematics. — *A Representation of a Complex of Biquadratic Twisted Curves of the First Kind on Point Space.* By J. W. A. VAN KOL. (Communicated by Prof. HENDRIK DE VRIES).

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§ 1. The complex of the biquadratic twisted curves of the first kind k^4 that pass through five given points $A_1, \ldots, A_5$, cut a given line a_1 twice and a given line a_2 once, may be represented on the points of space in the following way. We choose a quadratic cone K^2 and a line c. We suppose a projective correspondence to be established between the points K of a_1 and the tangent planes $\varkappa$ of K^2 and another one between the points C of a_2 and the planes γ through c. To a curve k^4 that cuts a_1 in K_1 and K_2, and a_2 in C, we associate the point of intersection of the planes $\varkappa_1$, $\varkappa_2$ and γ associated to K_1, K_2 and C.

§ 2. *The vertex T of K^2 is a cardinal point; in T are represented the ∞^2 curves k^4 that pass through the point A of a_2 which is associated to the plane cT.*

c is a singular line; the ∞^2 curves k^4 that are represented on c, cut a_1 in pairs of points of a quadratic involution I.

We have still to investigate whether it is possible that of a group of eight associated points five lie in $A_1, \ldots, A_5$, two on a_1 and one on a_2; for the consequence would be the appearance of a cardinal point. This, however, is not the case. If in a group of eight associated points five lie in $A_1, \ldots, A_5$ and two on a_1, the eighth lies on the twisted cubic that passes through $A_1, \ldots, A_5$ and has a_1 as chord. As a rule this curve does not cut a_2.

§ 3. There are ∞^1 curves k^4 that are singular for the representation, viz. the curves k^4 that pass through A and cut a_1 in a pair of points of I. They are represented on the rays of the pencil in the plane cT that has T as centre and they lie on the quadratic surface through A, $A_1, \ldots, A_5$ that contains a_1.

§ 4. Our complex contains the following systems of ∞^1 degenerate curves k^4:

The twisted cubic through $A_1, \ldots, A_5$ that has a_1 as chord, is completed by its chords that cut a_2, to a system of degenerate curves k^4 that is represented on a line through T.

Each of the five twisted cubics that pass through $A_1, \dots, A_5$ and cut a_1 and a_2, is completed by its chords that cut a_1, to a system of degenerate curves k^4 that is represented on a line cutting c and touching K^2.

The twisted cubics that pass through $A_1, \dots, A_5$ and cut a_1, are completed by their chords which cut a_1 and a_2, to a system of degenerate curves k^4 that is represented on a biquadratic curve c^4 that passes through T and cuts c three times and which is, accordingly, of the second kind. We prove this by cutting c^4 by a tangent plane $\varkappa$ of K^2. The number of points of intersection outside T is equal to the number of curves of the system that pass through the point K of a_1 that is associated to $\varkappa$. Now through K there passes one twisted cubic that passes at the same time through $A_1, \dots, A_5$ and is completed by its two chords that cut a_1 and a_2 to degenerate curves k^4. Further there is one twisted cubic that passes through $A_1, \dots, A_5$, cuts a_1 outside K, and has a ray of the plane pencil (K, a_2) as chord. This is a consequence of the property that the twisted cubics which pass through five given points, produce a polar correspondence in an arbitrary plane, so that the three points of intersection of any of the curves with the plane form a polar triangle [1]). The aforesaid twisted cubic forms a degenerate curve k^4 through K with the ray of the plane pencil (K, a_2) which it has as chord. Accordingly $\varkappa$ cuts c^4 outside T in three points. From the above mentioned property it also follows that c^4 has a singular point in T and that a plane through c cuts c^4 outside c in one point, so that c^4 cuts c three times.

The twisted cubics that pass through $A_i, \dots, A_m$ and cut a_1 once and a_2 twice, are completed by their chords through A_n to a system of degenerate curves k^4 that is represented on a twisted cubic which passes through T and has c as chord. This is proved in a similar way as above.

A twisted cubic that passes through $A_i, \dots, A_m$ and cuts a_1 twice, cuts a ray of the plane pencil (A_n, a_2) only then twice when this ray lies with $A_i, \dots, A_m$ and a_1 on one quadratic surface. If S_1 and S_2 are the points where the plane $A_n a_2$ is cut outside A_n by the twisted cubic that passes through $A_1, \dots, A_5$ and has a_1 as chord, the quadratic surface through $A_i, \dots, A_m$ and a_1 and $A_n S_1$ (resp. $A_n S_2$) contains ∞^1 twisted cubics that pass through $A_i, \dots, A_m$ and cut a_1 and $A_n S_1$ (resp. $A_n S_2$) twice and that are completed by $A_n S_1$ (resp. $A_n S_2$) to a system of degenerate curves k^4 which is represented on a line cutting c.

The twisted cubics that pass through $A_i, \dots, A_m$, cut a_1 and a_2, and have a ray of the plane pencil (A_n, a_1) as chord, are completed by these chords to a system of degenerate curves k^4 that is represented on a curve of the order six that passes through T and cuts c five times.

The transversal through A_n of a_1 and a_2 is completed by the twisted

[1]) Vg. R. STURM, Die Lehre von den geometrischen Verwandtschaften, 4, 103.

cubics that have this transversal as chord, pass through $A_i, \dots, A_m$ and cut a_1 to a system of degenerate curves k^4 that is represented on a line which cuts c and touches K^2.

The line $A_i A_k$ is completed by the twisted cubics that pass through A_l, A_m, A_n, cut $A_i A_k$ and a_1 twice and a_2 once, and which, accordingly, lie on the quadratic surface that passes through A_l, A_m, A_n and contains $A_i A_k$ and a_1, to two systems of degenerate curves k^4 that are represented on two lines which cut c.

The conic that passes through A_i, A_k, A_l and cuts a_1 and a_2, is completed by the conics that pass through A_m, A_n and cut the said conic twice and a_1 once, to a system of degenerate curves k^4 that is represented on a line which cuts c and touches K^2.

The conics that pass through A_i, A_k, A_l and cut a_1, are completed by the conics that pass through A_m, A_n, cut a_1 and a_2 and cut one of the aforesaid conics twice, to a system of degenerate curves k^4 that is represented on a conic which passes through T, cuts c, and lies in a tangent plane of K^2.

§ 5. K^2 is the image surface of the system of the curves k^4 that touch a_1.

The curves k^4 that pass through a given point P, lie on the quadratic surface ω^2 that passes through $A_1, \dots, A_5$ and P and contains a_1. Let a_2 cut ω^2 in P_1 and P_2. The curves k^4 on ω^2 that pass through P and P_1 as well as those that pass through P and P_2 cut a_1 in pairs of points of a quadratic involution. Consequently the system of the curves k^4 that pass through a given point P, is represented on two lines a_P and a'_P, that cut c.

a_P and a'_P together cut K^2 in four points; hence:

There are four curves k^4 that pass through a sixth given point and touch a_1.

This number can also be deduced directly. For each of the above mentioned involutions on a_1 has two double points.

§ 6. Let k_b be the image curve of the system Σ_1 of the curves k^4 that have a given chord b. By making use of the property that the biquadratic curves of the first kind that pass through six given points and cut a given line twice, lie on the quadratic surface that is defined by these elements, we find that through a given point of a_1 or a_2 there pass two resp. one curve of Σ_1. Hence a tangent plane K^2 cuts k_b in all in three points. Accordingly k_b is a twisted cubic that passes through T and has c as chord.

k_b cuts K^2 outside T in four points. Hence:

There are four curves k^4 that touch a_1 and cut a third given line twice.

§ 7. Let us call O_l the image surface of the system Σ_2 of the curves k^4 that cut a given line l. In order to determine the degree we cut O_l by a line that cuts c and touches K^2. The number of points of intersection outside c is equal to the number of curves of Σ_2 that pass through a given point of a_1 and a given point of a_2. This number is equal to two as the biquadratic curves of the first kind that pass through seven given points and cut a given line, form a biquadratic surface with double points in the given points [1]). From this property it also follows that c is a quadruple line of O_l. Consequently O_l is a surface of the sixth degree that has a double point in T and on which c is a quadruple line.

By investigating in how many points O_l is cut outside c and T by the pair of lines a_P, a'_P and by k_b, we find the following numbers, of which the former also directly follows from a property indicated in § 5:

There are four curves k^4 that pass through a given point P and cut a given line l.

There are eight curves k^4 that have a given chord b and cut a given line l.

§ 8. Two surfaces O_l and O_m cut each other along the line c, which must be counted sixteen times, and a curve k_{lm} of the order twenty, which is obviously the image curve of the system of the curves k^4 that cut two given lines l and m. k_{lm} has a quadruple point in T and cuts c in sixteen points.

Intersection of k_{lm} with K^2 and O_n gives:

There are 32 curves k^4 that touch a_1 and cut two given lines l and m.

There are 48 curves k^4 that cut three given lines l, m and n.

§ 9. The system Σ_3 of the curves k^4 that cut a_2 twice and, accordingly, each have two image points on a line through T, is represented on a plane α_{a_2} that passes through T. For the curves of Σ_3 cut a_1 in pairs of points of the quadratic involution on a_1 produced by the pencil of quadratic surfaces that pass through $A_1, \ldots, A_5$ and contain a_2.

If we cut α_{a_2} by the pair of lines a_P, a'_P and by k_b, we find the following numbers, of which the former is again an immediate consequence of a property indicated in § 5:

There is one curve k^4 that cuts a_2 twice and passes through a given point P.

There is one curve k^4 that cuts a_2 as well as a given line b twice.

§ 10. The congruence of the biquadratic curves k'^4 of the first kind that pass through seven given points $A_1, \ldots, A_7$, may be brought into a one-one-correspondence with the points of the plane $\alpha \equiv A_1 A_2 A_3$ by

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[1]) Cf. Prof. JAN DE VRIES, Eine Kongruenz von Raumkurven vierter Ordnung, erster Art. Nieuw Archief v. Wisk. **15**, 229.

associating to any curve k'^4 its fourth point of intersection with α as image point. A_1, A_2 and A_3 are singular points. In A_i are represented the ∞^1 curves k'^4 that touch α in A_i and lie on the quadratic surface $\omega^2{}_i$ that passes through $A_4, \ldots, A_7$ and contains the lines $A_i A_k$ and $A_i A_l$.

A line a through A_i in α is the image of the system of the ∞^1 curves k'^4 that lie on the quadratic surface ω^2 which passes through $A_4, \ldots, A_7$ and contains the lines a and $A_k A_l$. ω^2 as well as $\omega^2{}_i$ contain two curves k'^4 that cut a given line l. Hence the system of the curves k'^4 that cut a given line l, is represented on a biquadratic curve k_l with double points in A_1, A_2 and A_3, which is evidently the intersection of α and the surface formed by the curves k'^4 that cut l. In this way we have proved the property applied in § 7.

ω^2 as well as ω^2_i contain six curves k'^4 that touch a given plane φ. Accordingly the system of the curves k'^4 that touch a given plane φ, is represented on a curve k_φ of the twelfth order with sextuple points in A_1, A_2 and A_3. k_φ cuts k_l outside A_1, A_2 and A_3 in twelve points. *There are, therefore, twelve curves k'^4 that cut a given line and touch a given plane.*

The intersection of two curves k_φ en k_ψ also gives:
There are 36 curves k'^4 that touch two given planes.

§ 11. Through application of the above we can show in a simple way that the system of the curves k^4 that touch a given plane φ, is represented on a surface O_φ of the 18^{th} degree that has a sextuple point in T and on which c is a twelve-fold line.

Intersection of O_φ with (a_P, a'_P), k_b and k_{lm} gives the following numbers, of which the former again follows immediately from § 5:

There are twelve curves k^4 that pass through a given point P and touch a given plane φ.

There are 24 curves k^4 that have a given chord b and touch a given plane φ.

There are 144 curves k^4 that cut two given lines l and m and touch a given plane φ.

§ 12. We can also investigate the representations of different other systems of curves k^4, as e.g. the systems of the curves k^4 that touch two given planes, that cut a given line and touch a given plane, and others.

The numbers that can be deduced from this and the numbers deduced above are the following ones:

$P^7 \nu^2 = 4$	$P^6 T\nu = 4$	$P^5 B^3 = 1$	$P^5 BT\nu = 4$	$P^5 B\nu^4 = 48$
$P^7 \nu\varrho = 12$	$P^6 B^2 = 1$	$P^5 B^2\nu^2 = 8$	$P^5 T\nu^3 = 32$	$P^5 B\nu^3\varrho = 144$
$P^7 \varrho^2 = 36$	$P^6 B\nu^2 = 4$	$P^5 B^2\nu\varrho = 24$	$P^5 T\nu^2\varrho = 96$	$P^5 B\nu^2\varrho^2 = 432$
	$P^6 B\nu\varrho = 12$	$P^5 B^2\varrho^2 = 72$	$P^5 T\nu\varrho^2 = 288$	$P^5 B\nu\varrho^3 = 1296$

Here P represents the condition that a biquadratic curve of the first kind pass through a given point, ν that it cut a given line once, B that it cut a given line twice, T that it touch a given line and ϱ that it touch a given plane.

§ 13. The above enables us to indicate properties of surfaces formed by systems of ∞^1 curves k^4, such as:

The curves k^4 that have a given chord b, form a surface of the eighth degree with quadruple points in $A_1, \ldots, A_5$, on which a_1 and b are double lines and a_2 is a single line.

The curves k^4 that cut two given lines l and m, form a surface of the degree 48 with 24-fold points in $A_1, \ldots, A_5$, on which a_1 is a 16-fold line and a_2, l and m are 8-fold lines. The multiplicity of a_1 is equal to the number of curves k^4 that pass through a given point of a_1 and cut l and m. From the property indicated in § 8 that k_{lm} cuts c in 16 points, it follows that this number is equal to 16. Etc.