Mathematics. — *Determination of the Elementary Numbers of the Quadratic Surface by means of Representations of Systems of Quadratic Surfaces on the Points of a Linear Space.* By J. W. A. VAN KOL. (Communicated by Prof. HENDRIK DE VRIES).

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§ 1. In the same way as SCHUBERT, Kalkül der abzählenden Geometrie, § 22, where the elementary numbers of the quadratic surface by means of degenerations have been determined, we indicate by μ the condition that the quadratic surface pass through a given point, by ϱ that it touch a given plane, by ν that it touch a given straight line, by φ a quadratic surface that is degenerate in a double degree-plane containing a single rank-conic which is at the same time a single class-conic and by ψ a quadratic surface which is degenerate in two single degree-planes of which the intersection is a double rank-line containing two single class-points.

In what follows three numbers are supposed to be known, viz. $\mu^9 = 1$, $\mu^8\varrho = 3$ and $\mu^8\nu = 2$.

§ 2. *The system* μ^7.

Let us suppose the two-dimensional set of the quadratic surfaces F^2 that pass through seven given points $A_1, \ldots, A_7$ to be represented projectively on the points of a plane α. This representation does not possess any singular elements. A line of α is the image of a pencil of surfaces F^2. The systems of the surfaces F^2 that touch a given plane, resp. a given line, are represented on a cubic, resp. a conic. Consequently $\mu^7\varrho^2 = 9$, $\mu^7\nu\varrho = 6$ and $\mu^7\nu^2 = 4$.

§ 3. *The system* μ^6.

We suppose the three-dimensional set of the quadratic surfaces F^2 that pass through six given points $A_1, \ldots, A_6$ to be projectively represented on the points of a linear three-dimensional space R_3.

The planes $A_1 A_2 A_3$ and $A_4 A_5 A_6$ form a pair of planes that is a part of ∞^2 degenerations ψ that are represented in one point H_{23}. Accordingly our representation has ten cardinal points H_{ik} (i, $k = 2, \ldots, 6$; $i \neq k$).

A plane in R_3 is the image of a net of surfaces F^2 and a line in R_3 is the image of a pencil of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic surface F_α that has single points in H_{ik}. We prove the latter by cutting F_α by a line a that passes e.g. through H_{23} and is, accordingly, the image of a pencil of quadratic surfaces F^2 to which the pair of planes $A_1 A_2 A_3$, $A_4 A_5 A_6$ belongs. This pencil contains two non-degenerate surfaces F^2 that touch α. Consequently F^2 is cut by a outside H_{23} in two points.

The system of the surfaces F^2 that touch two given planes α and β, is represented on the curve of intersection $k_{\alpha\beta}$ of F_α and F_β, which is of the ninth order and has single points in H_{ik}.

$k_{\alpha\beta}$ cuts F_γ outside the cardinal points in 17 more points. Hence: $\mu^6\varrho^3 = 17$.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic cone K_l. The image surface is quadratic because a pencil of quadratic surfaces contains two individuals that touch a given straight line. Let T_l be the image of the surface F^2 that contains l. A line r through T_l is the image of a pencil of surfaces F^2 to which belongs the surface F^2 that contains l. Now as a rule this pencil does not contain any surface that touches l. Accordingly r has no point outside T_l in common with the image surface. The image surface is, therefore, a quadratic cone with vertex in T_l.

The system of the surfaces F^2 that touch two given lines l and m, is represented on the biquadratic curve of intersection k_{lm} of the surfaces F_l and F_m. Neither k_{lm} nor F_l pass through H_{ik}.

The intersection of $k_{\alpha\beta}$ and F_l, of k_{lm} and F_α and of k_{lm} and F_n gives the numbers $\mu^6\nu\varrho^2 = 18$, $\mu^6\nu^2\varrho = 12$ and $\mu^6\nu^3 = 8$.

§ 4. *The system* μ^5.

We suppose the four-dimensional system of the quadratic surfaces F^2 that pass through five given points $A_1, \ldots, A_5$, to be projectively represented on the points of a linear four-dimensional space R_4.

With any plane through $A_4 A_5$ the plane $A_1 A_2 A_3$ forms a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^3 degenerations ψ is represented on a cardinal line h_{45}. Evidently the representation has ten cardinal lines h_{ik} ($i, k = 1, \ldots, 5$; $i \neq k$). Each of the cardinal lines, e.g. h_{46}, is cut by three of the others, i.c. h_{12}, h_{23} and h_{13}.

A linear i-dimensional space ($i = 1, 2, 3$) in R_4 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic space Ω_α of which the lines h_{ik} are single lines. We prove the latter by cutting Ω_α by a line that cuts a line h_{ik}.

The system of the surfaces F^2 that touch two given planes α and β, is represented on the surface of intersection $F_{\alpha\beta}$ of Ω_α and Ω_β, which is of the ninth degree and of which the lines h_{ik} are single lines.

$F_{\alpha\beta}$ is cut by Ω_γ along the lines h_{ik} and a curve $k_{\alpha\beta\gamma}$ of the order 17, which is the image curve of the system of the surfaces F^2 that touch three given planes α, β and γ. $k_{\alpha\beta\gamma}$ cuts each of the lines h_{ik} in three points. For we can show that each of the planes $A_i\ A_k\ A_l$ is a part of three degenerate surfaces F^2 that touch α, β and γ.

$k_{\alpha\beta\gamma}$ is cut by Ω_δ outside cardinal points in $3.17 - 10.3 = 21$ points. Hence: $\mu^5 \varrho^4 = 21$.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic space Ω_l with a double line d_l. d_l is the image of the pencil of surfaces F^2 that contain l. Ω_l touches the lines h_{ik} .

The system of the surfaces F^2 that touch two given lines l and m, is represented on the surface of intersection F_{lm} of Ω_l and Ω_m, which is of the fourth degree and has four double points viz. in the points of intersection of d_l and Ω_m and in those of d_m and Ω_l.

The system of the surfaces F^2 that touch three given lines l, m and n, is represented on the curve of intersection k_{lmn} of Ω_l, Ω_m and Ω_n, which is of the eighth order.

Intersection of $k_{\alpha\beta\gamma}$ and Ω_l, of $F_{\alpha\beta}$ and F_{lm}, of k_{mn} and Ω_α and of F_{lm} and F_{no} gives the numbers: $\mu^5 \nu \varrho^3 = 34$, $\mu^5 \nu^2 \varrho^2 = 36$, $\mu^5 \nu^3 \varrho = 24$, and $\mu^5 \nu^4 = 16$.

§ 5. *The system* μ^4.

We suppose the five-dimensional system of the quadratic surfaces F^2 that pass through four given points $A_1, \ldots, A_4$, to be projectively represented on the points of a linear five-dimensional space R_5.

The plane $A_1\ A_2\ A_3$ forms with any plane through A_4 a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^4 degenerations ψ is represented on a cardinal plane σ_4. Our representation has four cardinal planes σ_i $(i = 1, \ldots, 4)$ of which any two have a point in common.

In the same way any plane through $A_1\ A_2$ forms with any plane through $A_3\ A_4$ a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^4 degenerations ψ is represented on a cardinal surface $\sigma_3{}^2$ which is quadratic because through two arbitrary points there pass two pairs of planes of which one plane passes through $A_1 A_2$, the other through $A_3 A_4$. Any line of one scroll of $\sigma_3{}^2$ is the image of a system of ∞^3 degenerations ψ which have a plane through $A_1\ A_2$ in common, and any line of the other scroll of $\sigma_3{}^2$ is the image of a system of ∞^3 degenerations ψ that have a plane through $A_3 A_4$ in common. The linear, three-dimensional space which contains $\sigma_3{}^2$, is the image of the system of the ∞^3 surfaces F^2 that contain the lines $A_1\ A_2$ and $A_3\ A_4$. Evidently there are three quadratic cardinal surfaces $\sigma_i{}^2$ $(i = 1, 2, 3)$. Any of the planes σ_i has a line in common with any of the surfaces $\sigma_i{}^2$. Two surfaces $\sigma_i{}^2$ have two points in common each of which is a point of intersection of two planes σ_i.

A linear i-dimensional space $(i = 1, \ldots, 4)$ in R_5 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic four-dimensional space ${}^4\Omega_\alpha$ in which σ_i and σ_i^2 are single.

The system of the surfaces F^2 that touch two given planes α and β, is represented on the three-dimensional space $\Omega_{\alpha\beta}$ of the ninth degree which ${}^4\Omega_\alpha$ and ${}^4\Omega_\beta$ have in common and in which σ_i and σ_i^2 are single.

$\Omega_{\alpha\beta}$ has in common with ${}^4\Omega_\gamma$ the planes σ_i, the surfaces σ_i^2 and a surface $F_{\alpha\beta\gamma}$ of the degree 17 that is the image of the system of the surfaces F^2 that touch three given planes α, β and γ. We can prove that $F_{\alpha\beta\gamma}$ has three lines in common with any of the planes σ_i and three conics with any of the surfaces σ_i^2.

$F_{\alpha\beta\gamma}$ is cut by ${}^4\Omega_\delta$ along straight lines lying in the planes σ_i, conics lying in the surfaces σ_i^2 and a curve $k_{\alpha\beta\gamma\delta}$ of the order 21 that is the image of the system of the surfaces F^2 that touch four given planes α, β, γ and δ. $k_{\alpha\beta\gamma\delta}$ cuts each of the planes σ_i in three points and each of the surfaces σ_i^2 in ten points.

$k_{\alpha\beta\gamma\delta}$ is cut by ${}^4\Omega_\varepsilon$ outside cardinal points in $21 \,.\, 3 - 4 \,.\, 3 - 3 \,.\, 10 = 21$ points. Hence: $\mu^4\varrho^5 = 21$. This number is equal to the dual number $\mu^5\varrho^4$ already found. In the following §§ we shall derive only those numbers of which the dual numbers have not yet been determined.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic four-dimensional space ${}^4\Omega_l$ with a double plane δ_l that is the image of the system of the surfaces F^2 that contain l. ${}^4\Omega_l$ touches any of the planes σ_i along a line and any of the surfaces σ_i^2 along a conic.

The system of the surfaces F^2 that touch two given lines l and m, is represented on the biquadratic three-dimensional space Ω_{lm} that ${}^4\Omega_l$ and ${}^4\Omega_m$ have in common. Ω_{lm} contains two double conics, viz. the intersection of δ_l and ${}^4\Omega_m$, which is the image curve of the system of the surfaces F^2 that contain l and touch m, and the intersection of δ_m and ${}^4\Omega_l$, which is the image curve of the system of the surfaces F^2 that contain m and touch l. Ω_{lm} has one quadruple point of contact with any of the planes σ_i and two qnadruple points of contact with any of the surfaces σ_i^2.

The system of the surfaces F^2 that touch three given lines l, m and n, is represented on the surface of the eighth degree F_{lmn} that ${}^4\Omega_l$, ${}^4\Omega_m$ and ${}^4\Omega_n$ have in common. F_{lmn} contains twelve double points.

The system of the surfaces F^2 that touch four given lines l, m, n and o, is represented on the curve of the order sixteen k_{lmno} which ${}^4\Omega_l$, ${}^4\Omega_m$, ${}^4\Omega_n$ and ${}^4\Omega_o$ have in common.

Intersection of $k_{\alpha\beta\gamma\delta}$ and ${}^4\Omega_l$, of $F_{\alpha\beta\gamma}$ and Ω_{lm}, of $\Omega_{\alpha\beta}$ and F_{lmn}, of ${}^4\Omega_\alpha$ and k_{lmno} and of ${}^4\Omega_p$ and k_{lmno} gives the numbers:

$\mu^4\nu\varrho^4 = 42$, $\mu^4\nu^2\varrho^3 = 68$, $\mu^4\nu^3\varrho^2 = 72$, $\mu^4\nu^4\varrho = 48$ and $\mu^4\nu^5 = 32$.

§ 6. *The system* μ^3.

We suppose the six-dimensional system of the quadratic surfaces F^2 that pass through three given points A_1, A_2 and A_3, to be projectively represented on the points of a linear six-dimensional space R_6.

Together with any plane $A_1 A_2 A_3$ forms a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^5 degenerations ψ is represented on a linear three-dimensional cardinal space Σ.

Any plane through $A_1 A_2$ forms with any plane through A_3 a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^5 degenerations ψ is represented on a cubic three-dimensional cardinal space $\Sigma_3{}^3$. $\Sigma_3{}^3$ contains a system S_1 of ∞^1 planes and a system S_2 of ∞^2 lines so that each plane of S_1 and each line of S_2 cut each other. Any plane of S_1 is the image of a system of ∞^4 degenerations ψ that have a plane through $A_1 A_2$ in common and any line of S_2 is the image of a system of ∞^3 degenerations ψ that have a plane through A_3 in common. $\Sigma_3{}^3$ lies in a linear five-dimensional space which is the image of the system of the surfaces F^2 that contain the line $A_1 A_2$. There are three cubic cardinal spaces $\Sigma_i{}^3$ $(i = 1, 2, 3)$. $\Sigma_i{}^3$ has in common with Σ a plane and a straight line, which is the intersection of the planes that $\Sigma_k{}^3$ and $\Sigma_l{}^3$ have in common with Σ. In the first place $\Sigma_i{}^3$ and $\Sigma_k{}^3$ have a quadratic surface in common that contains the lines which $\Sigma_i{}^3$ and $\Sigma_k{}^3$ have in common with Σ and in the second place the line that $\Sigma_l{}^3$ has in common with Σ and that does not lie on the quadratic surface. Accordingly the three cubic cardinal spaces have three lines in common.

The plane $A_1 A_2 A_3$ counted doubly is a part of ∞^5 degenerations φ that are represented in one point S which is the intersection of the common planes of the cardinal spaces $\Sigma_i{}^3$ and Σ. S is a single point of $\Sigma_i{}^3$.

A linear i-dimensional space $(i = 1, \ldots, 5)$ in R_6 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic five-dimensional space ${}^5\Omega_\alpha$ in which Σ and $\Sigma_i{}^3$ are single and that has a double point in S. We prove the latter by cutting ${}^5\Omega_\alpha$ by a line a through S that is the image of a pencil of surfaces F^2 which has a double conic through A_1, A_2 and A_3 as base curve. This pencil contains one surface that touches a given plane. Accordingly a cuts ${}^5\Omega_\alpha$ outside S in one point.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic five-dimensional space ${}^5\Omega_l$ which touches Σ along a plane and each of the spaces $\Sigma_i{}^3$ along a cubic surface, has a single point in S and contains a three-dimensional double space which is the image of the system of the planes F^2 that contain l.

The system $\mu^3 \nu^2 \varrho^3$, i.e. the system of the surfaces F^2 that touch two given lines and three given planes, is represented on a curve k^{68} that is of the order 68 as its order must be equal to the number $\mu^4 \nu^2 \varrho^3$ already found. k^{68} has a 32-fold point in S. For S is the image of the four degenerations φ that are formed by the plane $A_1 A_2 A_3$ counted doubly and the four conics in it that cut the two given lines and touch the three given planes, which degeneration must be counted eight times [1]). We can also show that k^{68} has no point in common with Σ and that it is cut by each of the planes $\Sigma_i{}^3$ in six quadruple points.

k^{68} cuts ${}^5\Omega_l$ outside S and cardinal points in $2.68 - 1.32 = 104$ points. Hence: $\mu^3 \nu^3 \varrho^3 = 104$.

In the same way we can examine the image curves of the systems $\mu^3 \nu^3 \varrho^2$, $\mu^3 \nu^4 \varrho$ and $\mu^3 \nu^5$. By cutting these by ${}^5\Omega_l$ we find the numbers: $\mu^3 \nu^4 \varrho^2 = 112$, $\mu^3 \nu^5 \varrho = 80$ and $\mu^3 \nu^6 = 56$.

§ 7. *The system* μ^2.

We suppose the seven-dimensional system of the quadratic surfaces F^2 that pass through two given points A_1 and A_2, to be projectively represented on the points of a linear seven-dimensional space R_7.

Any plane through $A_1 A_2$ forms with any other plane a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^6 degenerations ψ is represented on a four-dimensional cardinal space ${}^4\Sigma^4$ of the fourth degree, as through four arbitrary points there pass four pairs of planes of which one plane passes through $A_1 A_2$. ${}^4\Sigma^4$ contains a system S_1 of ∞^1 linear three-dimensional spaces and a system S_2 of ∞^3 lines so that any space of S_1 cuts any line of S_2. Any space of S_1 is the image of a system of ∞^5 degenerations ψ that have a plane through $A_1 A_2$ in common. Any line of S_2 is the image of a system of ∞^3 degenerations ψ that have a plane in common, which, as a rule, does not pass through A_1 and A_2. ${}^4\Sigma^4$ lies in a linear six-dimensional space which is the image of the system of the surfaces F^2 that contain the line $A_1 A_2$.

Likewise any plane through A_1 forms with any plane through A_2 a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^6 degenerations ψ is represented on a four-dimensional cardinal space ${}^4\Sigma^6$ of the sixth degree, as through four arbitrary points there pass six pairs of planes of which one plane contains A_1 and the other A_2. ${}^4\Sigma^6$ contains two systems S'_1 and S'_2 of ∞^2 planes of which any plane of S'_1 cuts any plane of S'_2. Each plane of S'_1 or S'_2 is the image of a system of ∞^4 degenerations ψ that a plane through A_1 resp. A_2 have in common. ${}^4\Sigma^4$ and ${}^4\Sigma^6$ have two cubic three-dimensional spaces in common, which are the images of the systems of ∞^5 degenerations ψ of which one plane passes through $A_1 A_2$ and the other through A_1 resp. A_2.

Any plane through $A_1 A_2$ counted doubly is a part of ∞^5 degenerations φ. This system of ∞^6 degenerations φ is represented on a line s which the said cubic three-dimensional spaces have in common.

A linear i-dimensional space $(i = 1, \dots, 6)$ in R_7 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic six-dimensional space ${}^6\Omega_\alpha$ in which ${}^4\Sigma^4$ and ${}^4\Sigma^6$ are single and s is a double line.

[1]) Cf. ZEUTHEN, Lehrbuch der abzählenden Geometrie, p. 351.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic six-dimensional space ${}^6\Omega_l$ that touches ${}^4\Sigma^4$ along a biquadratic three-dimensional space and ${}^4\Sigma^6$ along a three-dimensional space of the sixth degree, that has s as single line and has a linear four-dimensional double space which is the image of the system of the surfaces F^2 that contain l.

The system $\mu^2\nu^4\varrho^2$, i.e. the system of the surfaces F^2 that touch four given lines and two given planes, is represented on a curve k^{112} that is of the order 112 because its order is equal to the number $\mu^3\nu^4\varrho^2$ already found.

k^{112} is cut by s in 24 quadruple points. For there are 24 conics of which the planes pass through $A_1 A_2$ and which cut the four given lines and touch the two given planes [1]). Any of these conics together with its plane counted doubly, forms a degeneration φ that must be counted four times [2]) and is, accordingly, represented in a quadruple point of k^{112} on s. We can further show that k^{112} has no point outside s in common with ${}^4\Omega^4$ and that it is cut outside s by ${}^4\Omega^6$ in two sixteen-fold points.

k^{112} cuts ${}^6\Omega_l$ outside s and cardinal points in $2.112 - 1.96 = 128$ points. Hence: $\mu^2\nu^5\varrho^2 = 128$.

In the same way we can examine the image curves of the systems $\mu^2\nu^5\varrho$ and $\mu^2\nu^6$. By cutting these by ${}^6\Omega_l$ we find the numbers: $\mu^2\nu^6\varrho = 104$ and $\mu^2\nu^7 = 80$.

§ 8. *The system* μ.

We suppose the eight-dimensional system of the quadratic surfaces F^2 that pass through a given point A, to be projectively represented on the points of a linear eight-dimensional space R_8.

Any plane through A forms with any other plane a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^7 degenerations ψ is represented on a five-dimensional cardinal space ${}^5\Sigma^{10}$ of the tenth degree, as through five arbitrary points there pass ten pairs of planes of which one plane passes through A. ${}^5\Sigma^{10}$ contains a system S_1 of ∞^2 linear three-dimensional spaces and a system S_2 of ∞^3 planes of which any space of S_1 cuts any plane of S_2. Any space of S_1 is the image of a system of ∞^5 degenerations ψ that have a plane through A in common. Any plane of S_2 is the image of a system of ∞^4 degenerations ψ that have a plane in common, which, as a rule, does not pass through A.

Any plane through A counted doubly is a part of ∞^5 degenerations φ. This system of ∞^7 degenerations φ is represented on a plane σ that lies in ${}^5\Sigma^{10}$.

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1) Cf. SCHUBERT, Kalkül der abzählenden Geometrie, p. 95, where the number of conics $\mu^2\nu^4\varrho^2 = 24$ is derived.

2) Cf: Note to § 6.

A linear i-dimensional space ($i = 1, \ldots, 7$) in R_8 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic seven-dimensional space ${}^7\Omega_\alpha$ in which ${}^5\Sigma^{10}$ is single and σ is a double plane.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic seven-dimensional space ${}^7\Omega_l$ that touches ${}^5\Sigma^{10}$ along a four-dimensional space of the tenth degree. σ is a single plane in ${}^7\Omega_l$ and ${}^7\Omega_l$ contains a linear five-dimensional double space that is the image of the system of the surfaces F^2 that contain l.

The system $\mu\nu^6\varrho$, i.e. the system of the surfaces F^2 that touch six given lines and a given plane, is represented on a curve k^{104}, for $\mu^2\nu^6\varrho = 104$. k^{104} is cut by σ in 52 double points and does not cut ${}^5\Sigma^{10}$ outside σ. k^{104} cuts ${}^7\Omega_l$ outside σ in $2.104 - 1.104 = 104$ points. Hence: $\mu\nu^7\varrho = 104$.

The system $\mu\nu^7$ is represented on a curve k^{80} that is cut by σ in 34 double points and that does not cut ${}^5\Sigma^{10}$. k^{80} cuts ${}^7\Omega_l$ outside σ in $2.80 - 1.68 = 92$ points. Hence: $\mu\nu^8 = 92$.

§ 9. We now suppose all quadratic surfaces F^2 of space to be projectively represented on the points of a linear nine-dimensional space R_9.

The system of the ∞^8 degenerations ψ is represented on a six-dimensional cardinal space ${}^6\Sigma^{10}$ of the degree ten, as through any six points there pass ten pairs of planes. ${}^6\Sigma^{10}$ contains a system S of ∞^3 linear three-dimensional spaces. Any space of S is the image of a system of ∞^5 degenerations ψ that have a plane in common. As a rule two spaces of S have one point in common.

The system of the ∞^8 degenerations φ is represented on a linear three-dimensional space Σ lying in ${}^6\Sigma^{10}$.

A linear i-dimensional space ($i = 1, \ldots, 8$) in R_9 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic eight-dimensional space ${}^8\Omega_\alpha$ in which ${}^6\Sigma^{10}$ is single and in which Σ is a double space.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic eight-dimensional space ${}^8\Omega_l$ that touches ${}^6\Sigma^{10}$ along a five-dimensional space of the tenth degree. Σ is single in ${}^8\Omega_l$ and ${}^8\Omega$ contains a linear six-dimensional double space which is the image of the system of the surfaces F^2 that contain l.

The system ν^8 is represented on a curve k^{92} that cuts Σ in 92 points, as there are 92 conics that cut eight given lines, and that does not cut ${}^6\Sigma^{10}$ outside Σ. k^{92} cuts ${}^8\Sigma_l$ outside Σ in $2.92 - 1.92 = 92$ points.

Hence: $\nu^9 = 92$.

§ 10. The same as the systems μ^i ($i = 0, \ldots, 7$) of quadratic surfaces, also the dual systems ϱ^i can be projectively represented on the points of a linear space. In stead of the degeneration φ in this case the dual degeneration χ, the quadratic cone, will play a part. By dualising the considerations of the preceding § § we find the numbers that have not yet been derived in those § §.