Mathematics. — *The Introduction of Coordinates into Projective Geometry*. By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE.)

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The construction of Projective Geometry from a system of axioms is generally concluded by the introduction of coordinates, after which it appears that the defined geometry is identical with an analytical geometry, a geometry of coordinates.

Since the investigations of VON STAUDT different methods for this have been indicated. They come to this that the points, for the moment those of a straight line, are brought into a one-one correspondence with the numbers of a system of numbers in which operations exist that satisfy special conditions.

The properties of this system of numbers depend on the phase to which we suppose projective geometry to have been developed.

If we have introduced a complete system of axioms, such as are necessary for the construction of the "ordinary" projective geometry, the system of numbers will have all the properties of the system of the real numbers or of that of the real and the complex numbers.

For the geometry in the wider sense which we get through only postulating axioms of *Verknüpfung* but which for the rest is completely developed in the sense that the *fundamental theorem* holds good — the projective correspondence of two lines to each other is defined by three conjugated pairs of points — the corresponding system of numbers will lack all properties of order and continuity the same as the defined geometry. The operations addition and multiplication will satisfy the usual requirements. The system of numbers need not be identical with that of the real numbers but it can coincide e.g. with the set of the rational or the algebraic numbers, or it can consist of a finite number of numbers.

The modern works on projective geometry [1]) often follow this method and define the notion of coordinates before axioms of order or continuity have been introduced. In this paper too this point of view is taken.

The ways in which coordinates are introduced, are different. VON STAUDT develops an algebra of *Würfe*; SCHUR calculates with *prospectivities*, VEBLEN and YOUNG give a more direct algebra of points.

In what follows a method of introducing coordinates is sketched that rests on operations which are defined for the points of a *conic*. Accordingly

[1]) See e.g. O. VEBLEN and J. W. YOUNG, *Projective Geometry*, Volume I (1910), p. 141 sqq.

it supposes the definition and the most important properties of the conics to be given — in so far as they follow from the assumed axioms, hence with the exclusion of those of order or continuity. For the proofs of these properties we can best refer to VEBLEN and YOUNG [1]) as there only use is made of the axioms of *"alignment"* and *"extension"* and of the fundamental theorem. Of the theorems that are necessary for what follows, we mention especially the theorem of PASCAL and the theorems on projectivities on conics.

The method indicated here gives an application of the former theorem and has, perhaps, the advantage of a certain graphicalness. Before developing it we shall first point out a disadvantage. It is entirely dependent on the fundamental theorem as the theory of the conics rests on this.

By other methods we also arrive at the notion of coordinates when we omit the fundamental theorem (or the validity of the theorem of PAPPUS, which amounts to the same). In this case the system of numbers only lacks the commutativity of multiplication. (Non-Pascal geometry.) Such an extension, an introduction of coordinates in a still earlier phase of the axiomatic development, is therefore excluded here.

We choose a conic K. For the points of K we shall define a few operations. We choose two different points on K that we call resp. ∞ and o and at ∞ we draw the tangent S to K. By the *sum* $a+b$ of two points a and b of K we understand the point K that we get by the following construction. Join $a\ b$, cut the join by S and join the point of intersection to O. We call the second point of intersection of K and this join $a+b$ (Fig. 1).

If by the join of a point of K and itself we understand the tangent, the operation is *possible and one-valued* for all pairs of points of K except for the pair that consists of two points coinciding in ∞.

The addition is *commutative:* $a+b=b+a$.

This is at once evident from the construction.

The addition is *associative:* $(a+b)+c=a+(b+c)$.

Proof. Cf. Fig. 2, where a, b, c, $(a+b)$ and $(b+c)$ are indicated. We must prove that the joins of $(a+b)$ and c and of a and $(b+c)$ cut each other on S.

With a view to this consider the hexagon $a - b - c - (a+b) - O - (b+c)$. The side a, b cuts the side $(a+b)$, O on S; in the same way b, c and O, $(b+c)$ cut each other on S. According to the theorem of PASCAL also the other two opposite sides of the hexagon cut each other on S.

The addition has further the properties

$$a+0=0+a=a$$
$$a+\infty=\infty+a=\infty \quad (a \neq \infty).$$

The operation is unambiguously reversible; there is always one point x for which $a+x=b$, provided a and b do not coincide in ∞. We indicate

[1]) VEBLEN and YOUNG, l.c. p. 109 sqq.

this element by $x = b - a$ and in this way we have defined „subtraction". As a special case to any point there corresponds an opposite.

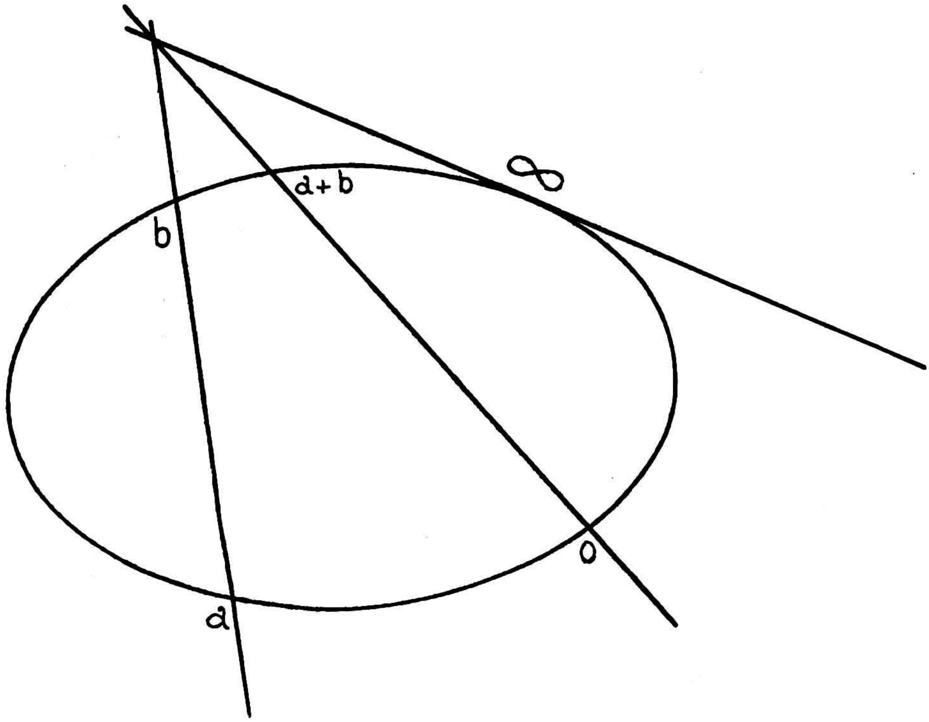

Fig. 1.

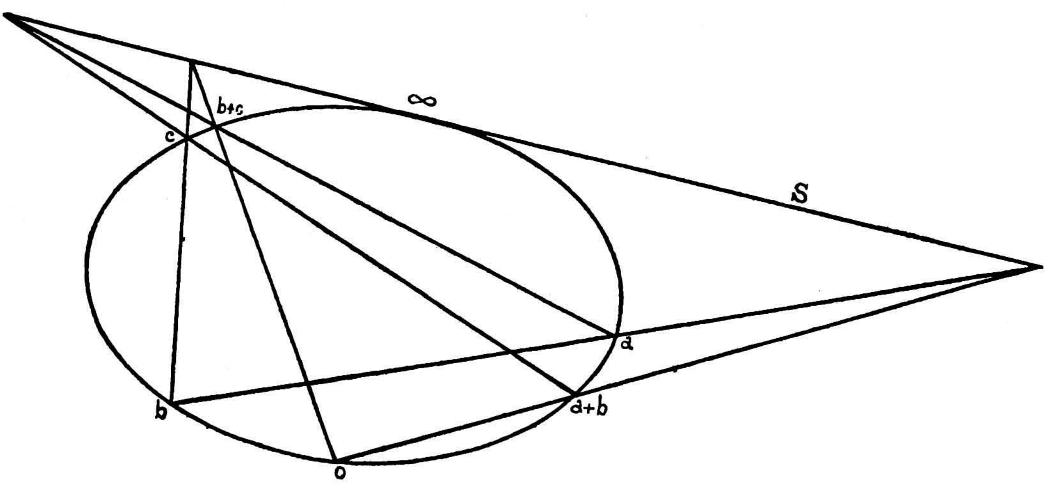

Fig. 2.

Joins of opposite points all pass through the point of intersection (A) of the tangents at ∞ and O (Fig. 3).

We shall now give a definition of the *product* of two points of K. With a view to this we choose another point 1 on K different from ∞ and O and we join O and ∞ (p).

The construction is about the same as that for addition. Join a b, cut this line and p, join the point of intersection to 1 ; the second point of intersection of this line and K is the product ab (Fig. 4).

The multiplication is *possible* and *one-valued* for all pairs of points with the exception of the pair O, ∞.

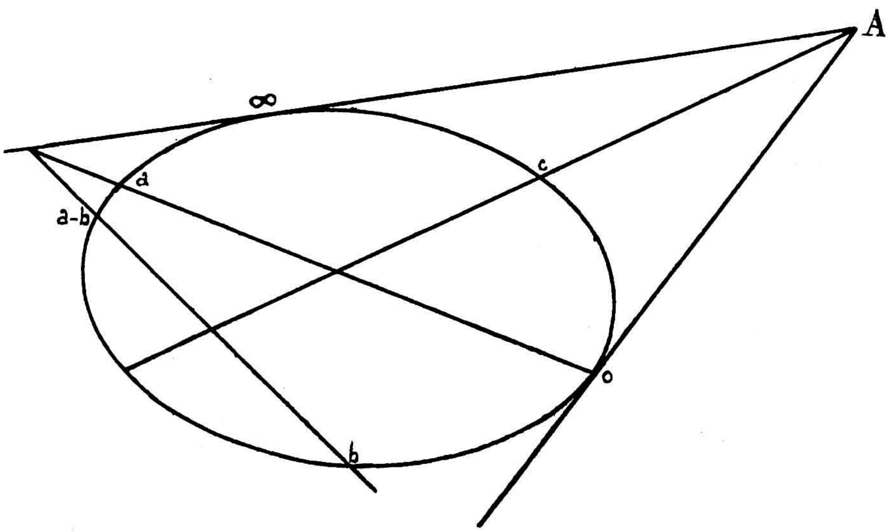

Fig. 3.

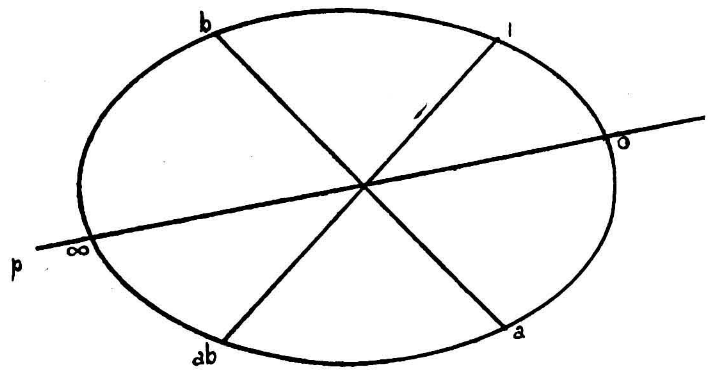

Fig. 4.

The multiplication is *commutative* as appears again directly from the construction : $ab = ba$.

It is also *associative* : $(a\,b)\,c = a\,(b\,c)$. The proof is given just as for the addition by the aid of the theorem of PASCAL. The hexagon to be considered is here

$$a - b - c - a\,b - 1 - b\,c.$$

and p is the line of PASCAL.

The multiplication has besides the following properties :

$$a \,.\, \infty = \infty \,.\, a = \infty \quad (a \neq 0)$$
$$a \,.\, 0 = 0 \,.\, a = 0 \quad (a \neq \infty)$$
$$a \,.\, 1 = 1 \,.\, a = a.$$

The multiplication is unambiguously reversible. If $a \neq o$ and $a \neq \infty$, there is always one point x so that.

$$a\,x = b,$$

for which we write $x = \frac{b}{a}$.

In this way the operation "division" is defined. The construction for the division is indicated in Fig. 5. In order to divide b by a we cut the line b, 1 by p and join the point of intersection to a.

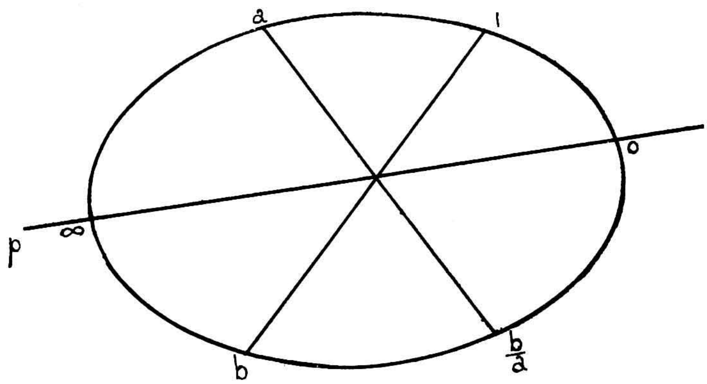

Fig. 5.

This construction can also be performed when a coincides with o or ∞. If the operation of division is extended in this way we have evidently : $\frac{a}{o} = \infty \; (a \neq o)$ and $\frac{a}{\infty} = o \; (a \neq \infty)$. The expressions $\frac{o}{o}$ and $\frac{\infty}{\infty}$ remain indefinite. To any point there corresponds one inverse point. Joins of inverse points all pass through the point of intersection (B) of p and the tangent at 1. Further the point -1, the opposite of 1, is important. By the aid of the theory of poles of the conics it appears that the tangent at -1 passes through B and that, accordingly, -1 is its own inverse. Besides it appears that the rays from 1 to ∞, o, 1 and -1 lie harmonically so that the points ∞, o, 1 and -1 form a harmonical point quadruplet on K.

It appears accordingly that the associativity of the operations of addition and multiplication defined by us are closely connected with the theorem of PASCAL for a conic. It deserves attention that for the rest this theorem does not play any special part in the construction of projective geometry. Where in axiomatics of projective geometry the theorem of PASCAL is mentioned, the special theorem of PASCAL is always meant (for a degenerate conic), which, in order to distinguish it from the general theorem, might more appropriately be indicated by the theorem of PAPPUS. This latter is very important as has already been pointed out : it is closely connected with the commutativity of multiplication in the system of numbers of the coordinates.

When a is a fixed point ($\neq \infty$), the correspondence of the points x' to the points x defined by the equation

$$x' = x + a$$

is a projectivity on the conic. This follows immediately from the construction for the addition, from the definition of projectivity and from the theorem that the points of a conic are projected out of two of them in projective pencils of rays. The points x are projected out of a on s and back again out of o on K.

This projectivity has two double points coinciding in ∞. Inversely it follows from the fundamental theorem that a projectivity which has two double points coinciding in ∞, and where the point o is projected in the point a, is identical with the projectivity that is fixed through $x' = x + a$.

If now we suppose the two projective point ranges x and x', in which o, x, ∞^2 are resp. associated to a, $x + a$, ∞^2, both to be projected out of a point c of K on p and back again out of 1 on K, there again appear two projective point ranges where (see Fig. 6), in connection with the definition

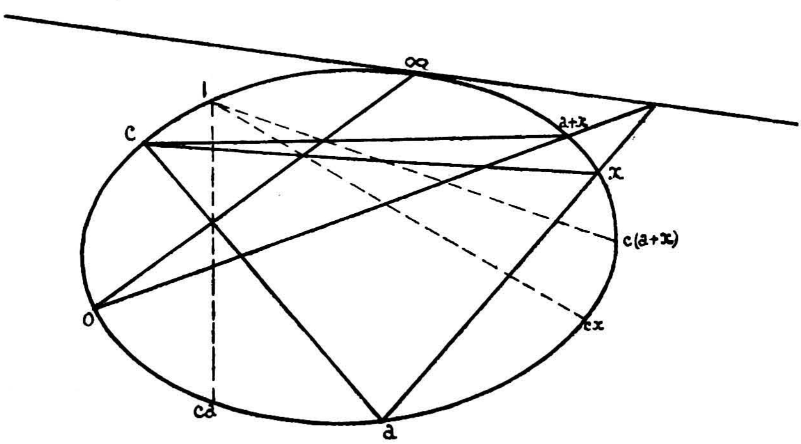

Fig. 6.

for multiplication, to o, cx, ∞^2 there correspond resp. ca, $c\,(x + a)$, ∞^2 so that, seen the remark just made, the equation

$$c\,(x + a) = cx + ca$$

holds good.

In other words the defined addition and multiplication have the *distributive* property [1]).

In this way the fundamental properties of our operations have been

[1]) I owe this simple proof to Prof. VAN DER WOUDE.

proved. These operations with points of a conic, accordingly, satisfy the rules that are valid e.g. for the real numbers. But then our aim — the introduction of coordinates — is reached.

For if l is a straight line, its points can always be projectively associated to those of K. Three arbitrary points A, B, E of l may be associated to ∞, o and 1 of K.

If in the projectivity defined in this way the point P of l corresponds to the point p of K, p is called "the coordinate of P in the system of coordinates A, B, E."

We have already seen that through $x' = x + a$ a projectivity is defined. The same holds good for $x' = a\,x$ $(a \neq o)$ (projection out of a on p and back again out of 1 on K), and for $x' = \frac{1}{x}$ (involutory collineation with B as pole).

Consequently also

$$x' = \frac{ax + b}{cx + d} = \frac{a}{c} + \frac{bc - ad}{c\,(cx + d)}$$
$$(bc - ad \neq 0)$$

defines a projective correspondence between the points of l. Inversely any projectivity on l may be expressed by a broken linear function of the coordinates. For if through such a projectivity the points with coordinates a, b and c are associated to A, B, E, hence to those with coordinates ∞, o, and 1, according to the fundamental theorem it is identical with the projectivity that is defined by the equation

$$x' = \frac{(c - a)\,(x - b)}{(c - b)\,(x - a)}.$$

By the anharmonic ratio of four points (A, B, C, D) of l we shall understand the coordinate of D in the system of coordinates A, B, C. From the fundamental theorem it follows that the equality of anharmonic ratios is the necessary and sufficient condition for the projectivity of point quadruplets. It is at the same time obvious that the anharmonic ratio of four points with coordinates x_1, x_2, x_3, x_4 is equal to

$$\frac{(x_3 - x_1)\,(x_4 - x_2)}{(x_3 - x_2)\,(x_4 - x_1)}.$$

The anharmonic ratio of four harmonical points is equal to -1 according to an above remark.

Now the further development of coordinate geometry proceeds in the same way as in the other methods. By the coordinates of a point P in a plane relative to the fundamental system A, B, C, E we understand the coordinates of the projections P_1 and P_2 (Fig. 7) in resp. the systems of coordinates $A\,C\,E^1$ and $B\,C\,E^2$, where E_1 and E_2 are the projections of the unit point.

The proof of the theorem that a straight line of the plane is represented by a linear equation and its inverse, the introduction of homogeneous

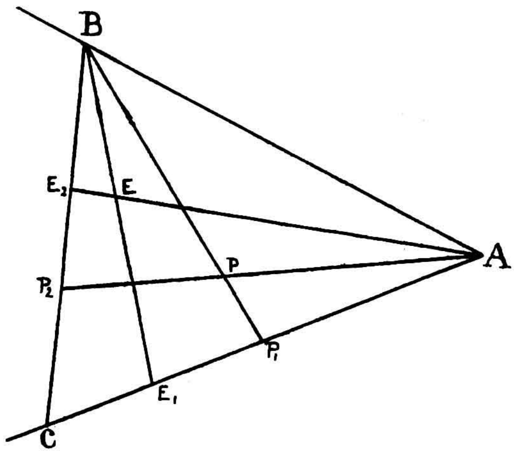

Fig. 7.

coordinates, the extension to space — it is not necessary to enter into it here.

We shall make an exception for the derivation of the equation of a conic in connection with the fact that this curve has played such a prominent part in our introduction of coordinates. Let K be the conic relative to which our operations have been defined (any other conic may be projectively associated to K). Choose the points ∞, A and o as fundamental points of the ternary system of coordinates and the point 1 of K as unit point (Fig. 8). We shall determine the coordinates of a point P of the conic and we indicate this point by t when we consider it as a number out of the system

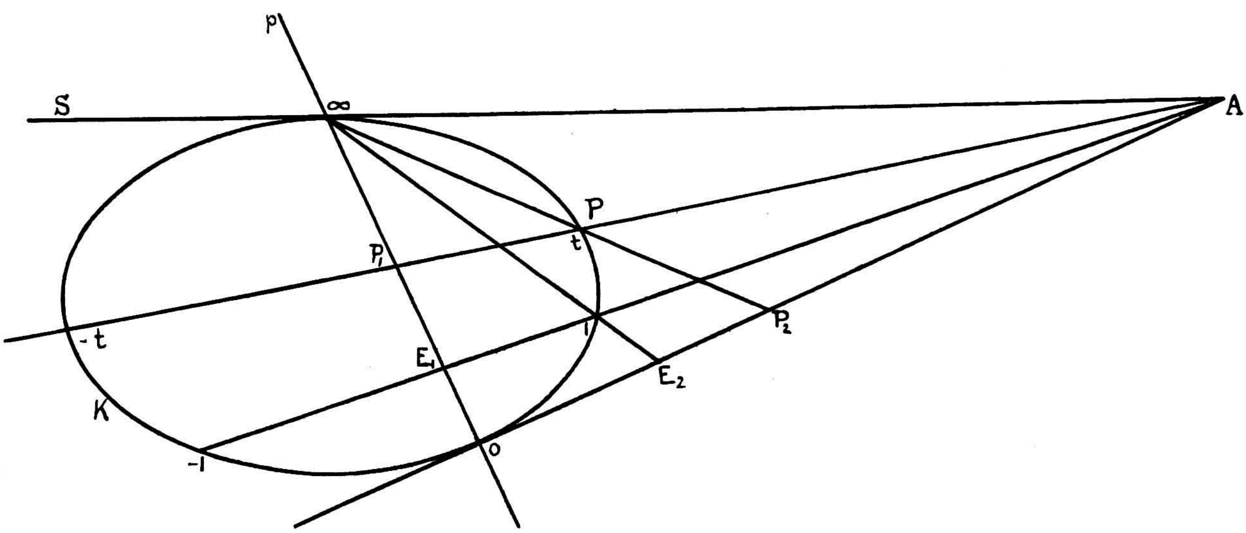

Fig. 8.

of numbers identical with the points of K. If we choose o, ∞ as X-axis, o, A as Y-axis, the x of P is accordingly the coordinate of P_1 on p in the system ∞, o, E_1. To obtain this we must associate p to K so that o and ∞ correspond to themselves and E_1 to 1. We arrive at this correspondence through projection of the points of p on K out of the point -1. P_1 is then projected in Q, which point is, accordingly, identical with the x-coordinate in question. As, however, the line $A\,P$ cuts the conic besides in $-t$, we have $-1 \,.\, x = t \,.\, -t$.

$$x = t^2.$$

The Y-coordinate of P is the coordinate of P_2 relative to A, o, E_2. The Y-axis must, therefore, be projectively associated to K so that o corresponds to itself, A to ∞ and E_2 to 1. We can obtain this by projecting out of ∞ Then to P_2 P is associated, in other words $y = t$.

Hence: on a suitably chosen system of coordinates any conic can be represented by the parameter equations

$$x = t^2 \qquad y = t,$$

or by the equation $y^2 = x$. Consequently on an arbitrary system of coordinates a conic is represented by a quadratic equation.

We make a few more remarks on the operations defined on the conic K. For addition as well as for multiplication the points a and b subject to the operation had to be joined and the point of intersection of their join and a straight line (s, resp. p) had to be determined after which there followed projection out of a centre on K (o, resp. 1). For the addition the straight line was a tangent (∞), for the multiplication a chord (∞, o). Between the operations there exists this relation that the fixed chord of the latter operation joins the point of contact of the tangent and the fixed centre both used in the former operation. (Of this special position of the two figures that define the operations, the distributive property is a consequence). If we choose an arbitrary chord p, q (where $p = q$ is not excluded) and an arbitrary centre m, we can evidently relative to these data likewise define an operation between two points of K that is also commutative and associative (Fig. 9).

However it is easily seen that this can be derived from addition and multiplication (resp. the inverse operations). For if we represent the result of the operation applied to a and b by $\overline{a \mid b}$, if a is considered fixed and b variable, the correspondence

$$b' = \overline{a \mid b}$$

is projective where p and q are invariant and m is transformed into a. But this transformation is obtained by the equation

$$\overline{a \mid b} = \frac{ab\,(p+q) - a\,b\,m + p\,q\,m - p\,q\,(a+b)}{ab - (a+b)\,m + (p+q)\,m - p\,q},$$

where a, b are interchangeable with p, q, as might be expected. The special cases addition and multiplication appear when $p = q = \infty$, $m = o$,

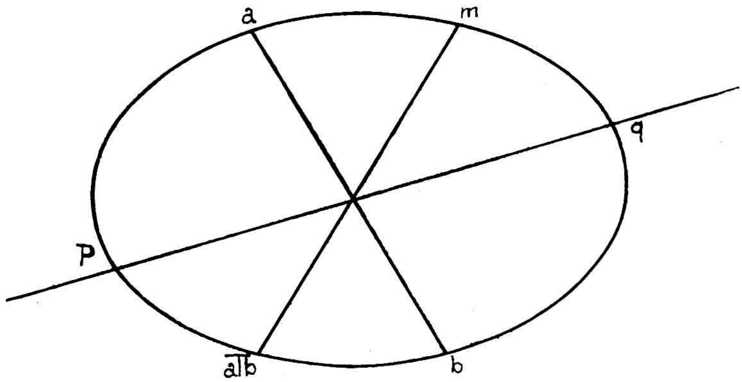

Fig. 9.

resp. $p = \infty$, $q = o$, $m = l$.

If we suppose $p = q = o$, $m = \infty$, we find the operation

$$\overline{a \mid b} = \frac{ab}{a+b} \quad \text{or} \quad \frac{1}{\overline{a \mid b}} = \frac{1}{a} + \frac{1}{b},$$

hence the "*harmonic addition*". It uses the tangent at o and as centre the point ∞. Consequently this operation is not only commutative and associative, but also distributive with the multiplication.

In the equation derived for $\overline{a \mid b}$ we can also consider p, q, a and b as fixed and consider the equation as one that associates a new point to m by joining m to a fixed point, the intersection of p, q and a, b.

In this case the transformation gets the form

$$x' = \frac{(pq - ab)\,x + ab\,(p+q) - pq\,(a+b)}{\{(p+q) - (a+b)\}\,x + ab - pq}$$

It is obviously the general involutory projectivity between the points of the conic. The general projectivity arises by means of a curve of the second class that touches K twice.