Hydrodynamics. - On the application of statistical mechanics to the theory of turbulent fluid motion. II. ${ }^{1}$ ). By J. M. Burgers. (Mededeeling $\mathrm{N}^{0}$. 12 uit het laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft). (Communicated by Prof. P. Ehrenfest).

## (Communicated at the meeting of May 25, 1929).

4. Examination of the function $\varphi$. - Flow between fixed parallel walls.

The object of this second part is mainly to consider more in detail some of the suppositions made in Part I, especially as an examination of the properties that must be assigned to the function $\varphi$ led to results, which for one part seemed promising, but at the other side pointed to a formula for the distribution of the velocity $U$ of the mean flow, rather differing from that which is found experimentally.

The function $\varphi_{k}$ was defined in equation (20); it depends only on the $y$-coordinate of the point $k$. By eliminating $t_{k}$ between (24) and (22) we get the integral equation for $\varphi_{k}$ :

$$
\begin{equation*}
\varphi_{k}=2 C-R^{-1}-2 A \sum_{\xi} t_{k}(\xi) \mathrm{e}^{-\beta \sum_{l}\left\{R-z_{l}(\xi)-\eta_{l} t_{l}(\xi)\right\}} \tag{25}
\end{equation*}
$$

In studying this equation we have to demand in the first place that the summation extended over the $\xi$-space shall be convergent. This makes it necessary that the exponential function becomes zero, when the $\xi$ 's go to infinity in any arbitrary direction. ${ }^{2}$ ) The functions $z_{l}(\xi)$ are essentially positive; the functions $t_{l}(\xi)$ may be negative as well as positive (comp. eq. $11^{a}, 11^{b}$ ); hence it is necessary that $\beta$ shall be positive, and secondly that the form:

$$
\begin{equation*}
\varepsilon^{-2} \Phi \equiv \sum_{l}\left\{R^{-2} z_{l}(\xi)-\varphi_{l} t_{l}(\xi)\right\} \tag{26}
\end{equation*}
$$

or written in full:

$$
\begin{equation*}
\varepsilon^{-2} \Phi \equiv \sum_{l}\left\{\frac{\left(4 \xi_{l}-\xi_{l+1}-\xi_{l+i}-\xi_{l-1}-\xi_{l-i}\right)^{2}}{R^{2} \varepsilon^{4}}-\frac{p_{l}\left(\xi_{l+1}-\xi_{l-1}\right)\left(\xi_{l+i}-\xi_{l-i}\right)}{4 \varepsilon^{2}}\right\} . \tag{a}
\end{equation*}
$$

[^0](in which $\psi$ is independent of the $\xi$ s) shall be a positive definite quadratic function of the $N$ variables $\xi_{1} \ldots \xi_{N}$. Clearly this imposes a certain condition on the function $q$. If f.i. we assume that $\psi$ is positive, then it will be seen that when $\Phi$ should be negative for a certain direction of the $\xi$-space, which of course is possible only for a direction giving positive values of $t_{l}$ in the greater part of the field, the right hand side of equation (25) would become negative infinite, and no solution could be obtained. If $q$ had too large negative values in some part of the field (it cannot be negative everywhere), then difficulties of the same kind may arise for directions in the $\xi$-space giving negative values of $t_{l}$ in that part of the field.

Before investigating the condition satisfied by $\varphi$, however, I should prefer first to deduce the corresponding formulae for the case of the flow between two fixed parallei walls, as this case affords a better possibility for a comparison with experimental results.

In the case of the flow between fixed walls we shall denote the distance of the walls by $h$; the mean velocity of the flow over a cross section of the channel by $V_{0}$; the pressure gradient $(-d p / d x)$ by $J$, and the frictional force per unit area of the walls by $S$. Then: $2 S=J h$. We again shall use nondimensional variables by dividing all lengths by $h$, all velocities by $V_{0}$, etc.; further we put $R=\varrho V_{0} h / \mu$ and $C=$ $S / \varrho V_{0}{ }^{2}=J h / 2 \varrho V_{0}{ }^{2}$. The origin of the system of coordinates will be placed midway between the walls, so that the latter are situated resp. at $y=-\frac{1}{2}$ and $y=+\frac{1}{2}$. The equation for the mean motion now becomes: ${ }^{1}$ )

$$
\begin{equation*}
\frac{1}{R} \frac{d U}{d y}=-2 C y+\overline{u v} \tag{27}
\end{equation*}
$$

(on account of the symmetry both $d U / d y$ and $\overline{u v}$ are zero in the axis of the channel). As $U=0$ at both walls we have the relation:

$$
\begin{equation*}
-\int d y y \frac{d U}{d y}=\int d y U=1 . \quad . \quad . \quad . \quad . \tag{28}
\end{equation*}
$$

from which we deduce:

$$
\begin{equation*}
C=6 \int d y y \bar{u} v+6 R^{-1}=\frac{6}{L} \iint_{0} d x d y y \bar{u} v+6 R^{-1} \tag{29}
\end{equation*}
$$

The limits of the integration with respect to $y$ are $-\frac{1}{2}$, $+\frac{1}{2}$ (unless purposely specified otherwise); those with respect to $x: x_{0}, x_{0}+L$.

The dissipation condition also in this case has the form given in equation (2), Part I. Eliminating again $d \tilde{U} / d y$ and $C$, and using the abbreviations (3), we get:

[^1]\[

$$
\begin{align*}
F \equiv \iint d x d y(\bar{t})^{2}-\frac{12}{L}( & \left.\iint d x d y y t\right)^{2}+ \\
& +\frac{12}{R} \iint d x d y y \bar{t}+\frac{1}{R^{2}} \iint d x d y z=0 \tag{30}
\end{align*}
$$
\]

When now we introduce once more the system of representative points in the $\xi$-space, we can express this condition in the following form, analogous to (13):

$$
\left.\begin{array}{rl}
\varepsilon^{-2} F=\sum_{k}\left(\sum_{\zeta} v t_{k}\right)^{2}-\frac{12 \varepsilon^{2}}{L}\left(\sum_{k} \sum_{\zeta} v y_{k} t_{k}\right)^{2} & +\frac{12}{R} \sum_{k} \sum_{\xi} v y_{k} t_{k}+1  \tag{31}\\
& +\frac{1}{R^{2}} \sum_{k} \sum_{\xi} v z_{k}=0
\end{array}\right\}
$$

This formula enables us to calculate the variation of $F$ produced by an arbitrary variation of one of the $r$ 's. In order to make our formulae correspond as much as possible to those of § 3, we put:

$$
\begin{equation*}
\frac{24 \varepsilon^{2}}{L} y_{k} \sum_{k^{\prime}} \sum_{\xi^{\prime}} v y_{k^{\prime}} t_{k^{\prime}}\left(\xi^{\prime}\right)-2 \sum_{\xi^{\prime}} v^{\prime} t_{k}\left(\xi^{\prime}\right)-12 y_{k} R^{-1}=q_{k}^{\prime} . \tag{32}
\end{equation*}
$$

Then:

$$
\begin{equation*}
\varepsilon^{-2} \delta F=\sum_{k}^{\prime}\left(R^{-2} z_{k}-\varphi_{k} t_{k}\right) \delta \nu=\xi^{-2} \Phi \delta \nu^{\prime} \tag{33}
\end{equation*}
$$

when we use the same formula (26) as before. This again leads to (21) as the expression for the "most probable distribution". Hence the same condition has to be imposed on $\Phi$, and consequently on $q$, as in the former case.

From (32) the following expressions can be deduced for the new function $\varphi$ :

$$
\begin{align*}
\varphi=24 y \int d y y \bar{t}-2 \bar{t}-12 y R^{-1}=-y(4 & \left.C-12 R^{-1}\right)-2 \bar{t}= \\
& =R^{-1}\left(2 \frac{d U}{d y}+12 y\right) \tag{34}
\end{align*}
$$

At the wall $y=-\frac{1}{2}$, where $t=0$, the function $q$ has the value:

$$
\begin{equation*}
\varphi\left(-\frac{1}{2}\right)=2 C-6 R^{-1} \tag{a}
\end{equation*}
$$

at the other wall it has the opposite value, whereas in the axis of the channel:

$$
\begin{equation*}
\varphi(0)=0 . \tag{b}
\end{equation*}
$$

Finally: ${ }^{1}$ )

$$
\begin{equation*}
-\int_{-1 / 2}^{0} d y y p=\frac{1}{2} R^{-1} \tag{c}
\end{equation*}
$$

${ }^{1}$ ) The corresponding relation in the former case is: $\int_{0}^{1 / 2} 4 d y=\frac{1}{2} R^{-1}$.

It seems reasonable to suppose that in this case (as well as in the former one), $p$ is a monotonous function of $y$ in either half of the channel breadth. This of the course implies the same character for the part of the frictional force due to the presence of the relative motion, i.e. for $t=-u v$, and for $d U / d y$. There has never been an indication to the contrary, and also from the theoretical point of view there seems to be no reason why it should be otherwise, unless it be supposed that types of relative motion with a very definite "wave pattern" over the breadth of the channel should be preponderant. This seems improbable, however, at least in the case of smooth walls.

We now return to the condition to be fulfilled by $\Phi$ and "re-translate" this condition into the language of the continuous field, by putting in (26):

$$
z=\zeta^{2}=(\triangle \psi)^{2} \quad, \quad t=-u v=\frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y}
$$

and substituting integration with respect to $x$ and to $y$ for the summation over the points of the lattice. In this way we get:

$$
\begin{equation*}
\Phi=\iint d x d y\left\{R^{-2}(\triangle \psi)^{2}-\varphi(y) \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y}\right\}>0 . \tag{35}
\end{equation*}
$$

for all possible functions $\psi$, satisfying the boundary conditions $\psi=\frac{\partial \psi}{\partial y}=0$ at both walls.

Now from investigations by Lorentz and by OrR it can be deduced that there exist types of motion, wholly enclosed within a strip of breadth $D$ and satisfying the boundary conditions at the borders of the strip, for which :

$$
\begin{align*}
& \int d x \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \text { for no value of } y \text { is negative }  \tag{a}\\
& \iint d x d y(\triangle \psi)^{2}=\frac{A}{D^{2}} \iint d x d y \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} . \tag{b}
\end{align*}
$$

where $A$ is a certain coefficient, the lower limit of which is, according to OrR, 177. ${ }^{1}$ ) We shall suppose that such a motion is present in the strip $-\frac{1}{2} \leqslant y \leqslant-\frac{1}{2}+D$, where $D<\frac{1}{2}$; outside of this strip $\psi$ shall have the value zero. Then, if $\varphi_{\text {min }}$ denotes the smallest value of $\varphi$ occurring in this strip, we have:
$\iint d x d y \varphi \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y}>\varphi_{\min } \iint d x d y \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y}$, or $>\varphi_{\min } \frac{D^{2}}{A} \iint d x d y(\triangle \psi)^{2}$.

[^2]Substituting this into (35) we find:

$$
\varphi_{\min }<\frac{A}{R^{2} D^{2}}
$$

On account of our supposition about the monotonous character of $\varphi$, the smallest value in such a strip is to be found at the border $y=-\frac{1}{2}+D$; hence, writing temporarily $y^{\prime}$ for $\frac{1}{2}+y$, we get:

$$
\begin{equation*}
\varphi\left(y^{\prime}\right)<\frac{A}{R^{2} y^{\prime 2}} \tag{36}
\end{equation*}
$$

As on the other hand the maximum value of $\varphi$ is given by $\left(34^{a}\right)$, we must conclude that $\varphi$ everywhere lies below the curve given in the accompanying diagram fig. 1. Calling $\delta$ the value of $y^{\prime}$ for which the


Fig. 1.
limiting value (36) becomes equal to $2 C$, we easily deduce from fig. 1:

$$
-\int_{-1 / 2}^{0} d y y \varphi<\frac{1}{2} \int_{0}^{1 / v} d y^{\prime} \varphi<C \delta+\frac{1}{2} \int_{j}^{\infty} d y^{\prime} \frac{A}{R^{2} y^{\prime 2}}, \text { or }<\frac{\sqrt{2 A C}}{R} .
$$

Comparing this with (34c), we obtain:

$$
\begin{equation*}
C>\frac{1}{8 A} \tag{37}
\end{equation*}
$$

Taking $A=177$ (the lower limit given by OrR), we find:

$$
C>0,00070
$$

This result is interesting as it has been (and still is) a matter of discussion, whether in the case of absolutely smooth walls the resistance coefficient $C=S / \varrho V_{0}{ }^{2}$ will decrease to zero for infinite values of the Reynolds' number or not. ${ }^{1}$ ) It is generally accepted that this coefficient is approximately independent of the form of the section of the channel, provided that the Reynolds' number is defined in all cases by means

[^3]of the so-called hydraulic radius, $r_{H}$, which is equal to twice the area of the section, divided by its perimeter. Now the lowest values for $C$ that have been observed until now for a tube of circular section, seem to be: ${ }^{1}$ )


The measurements made by Stanton and Pannell and those by JАков and Erк are considered to be the most accurate; their results can be represented by the formula:

$$
C=0,00090+0,0763\left(2 \varrho V_{0} t_{H} / \mu\right)^{-0.35} .
$$

The limiting value is of the same order of magnitude as that given by (37). It has to be reminded, however, that our method of estimating was still rather rough; a more precise method might raise the theoretical limit somewhat (unless it might prove that OrR's number 177 is not applicable - comp. note ${ }^{1}$ ), p. 646, but then at least $C>0,00043$ ).

The inequality (36) can be used also to deduce an approximate formula for the distribution of the velocity of the mean motion over the breadth of the channel. Combining it with (34) we find:

$$
\frac{d U}{d y}+6 y=\frac{R \varphi}{2}<\frac{A}{2 R y^{\prime 2}}
$$

Hence in the central part of the channel, where $y^{\prime}$ is not very small compared to unity, we may write approximately:

$$
\frac{d U}{d y}=-6 y
$$

from which we obtain:

$$
\begin{equation*}
u=\frac{5}{4}-3 y^{2} \tag{38}
\end{equation*}
$$

This gives for the velocity in the axis 1,25 times the mean velocity. In reality it is much less, about 1,1 times the mean velocity. In the case of pure laminar motion we have:

$$
\frac{d U}{d y}=-12 y, \quad U=\frac{3}{2}-6 y^{2}
$$

our value of $d U / d y$ is just half of that existing in the laminar motion.

[^4]The same relation is obtained in the case of the flow between two walls, moving with respect to each other, and as far as I can see the case of the motion through a tube with circular section does not promise a different result, so that the discrepancy from the eiperimental data is even greater here. ${ }^{1}$ )
5. Review of the assumptions made in § 2; physical interpretation of the condition $\Phi>0$.

In view of the discrepancy mentioned at the end of the foregoing $\S$, it seems worth while to consider again the principal suppositions that have been made.

The basis of our assumptions was that, apart from the boundary conditions and the equation of continuity, the dissipation condition is the only equation governing the turbulent motion.

We have pictured in the $\xi$-space the assembly of all possible fields, satisfying the boundary conditions and the equation of continuity. Every imaginable type of motion, Lorentz's vortices, ORR's solutions, all kinds of solutions constructed by various authors have their representative points in this space, just as well as wholly arbitrary fields. The various intensities of one and the same type of field are represented by points, lying at various distances on the same radius vector through the origin.

Then we have sought for a principle for selecting a set of $M$ points, satisfying (13) or in the other case considered (31), that might serve as an appropriate basis for calculating the necessary mean values. As this principle we have chosen a probability hypothesis, and it is perhaps not superfluous to remember that we have not spoken about the probability of any special type of relative motion, or of a certain distribution of vortices, etc.; what we have counted was on the contrary the number of various sequences in which the set of $M$ individual fields of flow could be arranged. It is only at the end of the calculations that we come to formula (21), which gives a measure for the statistical frequency of any special field.

This formula (21) possesses some properties which make it appear rather appropriate for the description of the turbulent motion. In the first place it contains no undetermined constants. As the exponent of $e$ is a quadratic function of the variables $\xi_{1} \ldots \xi_{N}$ (comp. formula $26^{a}$ ), the integration with respect to any of the $\xi$ 's can be effected in an elementary way; the great number of the variables (which in the limit ought to be made infinite) makes this procedure impractical, however, especially on account of the appearance of the function $\varphi$. Though a method for solving this difficulty has not yet been found, still we see from the integral equation either for $t$ or for $\varphi$, that the mean amplitude of the

[^5]relative motion is wholly determined - a consequence of the circumstance that the dissipation condition in its forms (5) or (30) is not homogeneous with respect to this amplitude.

Formula (21) automatically yields the so-called "laminar layers" along the walls of the channel. In consequence of the boundary conditions all types of motion that present appreciable values of $u$ and $v$ in the neighbourhood of the walls necessarily bring with them very great values of $z$; hence as $\varphi$ cannot surpass the value $2 C$, the term $\sum_{k} R^{-2} z_{k}$ in the exponent will become preponderant, making the value of $\nu$ for such fields become very small.

Finally, assuming (provisionally) that $\varphi$ is a monotonous function of $\boldsymbol{y}$ in every half of the channel breadth, which consequently always has the same sign as $d U / d y$, we deduce from the formula (21) that everywhere those types of fields have the greatest chance of occurring, that give values for $t=-u v$ of the same sign as $d U / d y-j u s t$ as it must be in order to account for the observed great value of the resistance. Hence the necessary "correlation" between $u$ and $v$ comes in automatically (through the intermediary of the dissipation condition), notwithstanding the fact that we have not made use of the otherwise very important theories about the origin of this correlation, as have been worked out by Taylor, Prandtl and others. Their deductions, however, introduce the conception of a "mean free path" of the elements of the fluid, which in itself is an unknown quantity.

The fact that the correlation in our results comes in automatically is due to the circumstance that the increase of the energy of the relative motion at any instant is mainly determined by the formula:

$$
\begin{equation*}
E=\iint d x d y\left\{-u v \frac{d U}{d y}-R^{-1} \zeta^{2}\right\} . \tag{39}
\end{equation*}
$$

which is positive for fields having the right correlation (combined with not too great values for $\zeta^{2}$ ), whereas it is negative for fields with the wrong correlation. ${ }^{1}$ )

It is not, however, this expression which occurs in the exponent of

[^6]formula (21), but the function $\Phi$, or as we may write it for purpose of comparison, making use of (34):
\[

$$
\begin{equation*}
-R \Phi=\iint_{0} d x d y\left\{-u v\left(2 \frac{d U}{d y}+12 y\right)-R^{-1} \zeta^{2}\right\} \tag{40}
\end{equation*}
$$

\]

We have demanded that $-\Phi$ always should be negative; this of course may be very well compatible with a positive value of (39).

In this connection it is of importance to remark that the expression (40) has a meaning wholly apart from the introduction of the $\xi$-space or of the probability hypothesis. In order to show this, we start from the equation:

$$
\begin{equation*}
\bar{E}=\iint d x d y\left\{-\overline{u v} \frac{d U}{d y}-R^{-1} \overline{\zeta^{2}}\right\} . \tag{41}
\end{equation*}
$$

which determines the rate of increase of the mean energy of the relative motion. In the normal turbulent state the mean energy of the relative motion has a constant value; so then $\bar{E}=0$ and (41) becomes identical with equation (2), Part I.

Now the turbulent motion when viewed ,,microscopically", i.e. at a series of instants with sufficiently small intervals between them, must be considered as a sequence of widely varying types of fields of flow, and the quantities $\overline{u v}, \zeta^{2}$ are obtained as a mean over the values of $u v, \zeta^{2}$ presented by every individual member of the sequence. The order of the various individual fields in the sequence is of no importance in our considerations; it seems legitimate, however, to suppose that in a long interval of time, most types occur repeatedly, at least with a certain degree of approximation, and that a mean amplitude can be assigned to each of them (as in fact has been assumed in all our deductions and is expressed by formula (21)). Let us compare this normal state with one, in which the intensity of one of the members of the sequence has been changed, f.i. by first increasing it for a short interval of time and then diminishing it, in such a way that the mean values of linear quantities are not altered, whereas those of quantities of the second degree are increased in a given constant proportion. We shall suppose moreover that this variation is executed every time this type of motion appears. When the velocity components and the vorticity of this special type of flow are proportional resp. to $u^{\prime}, v^{\prime}, \zeta^{\prime}$, then in the varied sequence the values of $\bar{u}, \bar{v}, \bar{\zeta}$ will again be zero, whereas the values of $\overline{u v}, \overline{\zeta^{2}}$, etc. in any point of the field will change with amounts proportional to the values of $u^{\prime} v^{\prime}, \zeta^{\prime 2}$, etc. at that point. Hence we may write:

$$
\begin{equation*}
\delta \overline{u v}=u^{\prime} v^{\prime} \delta \alpha, \quad \delta \overline{\zeta^{2}}=\zeta^{\prime 2} \delta a . \tag{42}
\end{equation*}
$$

where $\delta a$ is a positive number, depending on the degree of the intensification and the interval of time during which it is applied, but independent of $x$ and $y$ and of the time.

The new system of values of $\overline{u v}, \overline{\zeta^{2}}$ will in general not be compatible with the original mean motion. With the aid of the equations of Reynolds and Lorentz, however, we can calculate a new mean motion, assuming thereby that the total amount of fluid, crossing a section of the channel, remains constant. Then at the same time the Reynolds' number retains its value.

When the new mean motion has been found, we can determine the value of $\bar{E}$ for the varied system. We shall write $\delta \bar{E}$ for it, corresponding to $\delta a$. If now it should appear that $\delta E / \delta a>0$, then this would mean that in our varied sequence - with one member intensified - the mean energy of the relative motion tends to increase. Of course we cannot determine in which way the increase of mean energy is distributed over the various members of the sequence, and so we cannot prove rigorously that our system in this case is unstable. Still it would seem natural to accept as a criterion for the stable character of normal turbulent motion that the intensification of any individual member of the sequence should bring about a decrease of the mean energy, and that on the contrary the weakening of any member should cause an increase of mean energy.

We may regard this matter from another side. It is always possible to find types of flow for which $\delta E / \delta u<0$; hence if there were other types with $\delta E / \delta \alpha>0$, we might construct sequences in which definite members were intensified in a given proportion to each other, but in an arbitrary degree (whereas no member was weakened), that would satisfy the condition $E=0$. This should mean that the dissipation condition would not put a limit to the mean energy of the relative motion, as it could be increased indefinitely, if only a certain proportion was observed. In view of our starting point which accepted the dissipation condition as the only condition governing the turbulent motion, this would seem to be rather improbable.

Hence we might suppose that for all types of motion present (or imaginable) in the sequence the condition:

$$
\begin{equation*}
\frac{\delta E}{\delta a}<0 \tag{43}
\end{equation*}
$$

ought to be fulfilled.
There are various ways of calculating the quantity $\delta E / \delta \alpha$. In the following lines we shall start from the equation of energy for the whole motion, instead of using (41); this has no influence on the result, as the energy of the mean motion is independent of the time (in consequence of its definition as a mean with respect to time); so nor this energy itself, neither its variation does appear in $\bar{E}$, which measures the rate of change of the energy.

As the variation $\delta u v$ given by (42) will be a function of $x$ as well as of $y$, we may no longer suppose that the mean motion, determined
by the varied sequence, is everywhere parallel to the axis of $x$. Hence we have to start from the general equations:

$$
\left.\begin{array}{l}
U \frac{\partial U l}{\partial x}+V \frac{\partial U}{\partial y}=-\frac{\partial P}{\partial x}+R^{-1} \triangle U-\frac{\partial}{\partial x} \overline{u^{2}}-\frac{\partial}{\partial y} \overline{u v}  \tag{44}\\
u \frac{\partial V}{\partial x}+V \frac{\partial V}{\partial y}=-\frac{\partial P}{\partial y}+R^{-1} \triangle V-\frac{\partial}{\partial x} \overline{u v}-\frac{\partial}{\partial y} \overline{v^{2}}
\end{array}\right\}
$$

whereas the energy equation becomes:

$$
\begin{equation*}
\bar{E} \equiv \int(P U)_{I} d y-\int(P U)_{I I} d y-R^{-1} \iint d x d y\left\{\left(\frac{\partial V}{\partial x}-\frac{\partial U}{\partial y}\right)^{2}+\bar{\zeta}^{2}\right\} \tag{45}
\end{equation*}
$$

Here $V$ denotes the $y$-component of the mean motion, $P$ the mean pressure at any point. The first and the second integrals in the expression for $\bar{E}$ represent the work done by the mean pressures in the sections I (at $x_{0}$ ) and II (at $x_{0}+L$ ); the third term is the total loss of energy, due to the internal friction. The equation for $\bar{E}$ is not strictly true as some terms have been neglected which partly measure the work done at the sections I, II by the varying pressures, etc. of the relative motion, and partly the kinetic energy which is transported across these sections. We may discard these terms, however, as the amount contributed by them remains nearly constant when $L$ is increased without limit. If equation (45) had been divided by $L$, then these terms would become of the order $L^{-1}$.

In the normal state we have $\partial U / \partial x=0, V=0, \partial \overline{u^{2}} / \partial x=0, \partial \bar{u} \bar{v} / \partial x=0$; further: $P=$ Constant $-J x-\overline{v^{2}}$, where $J$ is the pressure gradient.

Now we apply our variation, then $\delta u^{2}=u^{\prime 2} \delta a$, etc. When we consider a field of motion resembling those described by ORR, which are stretched out over indefinite lengths and are more or less periodic with respect to $x$, we see that integrals of the type $\int \delta \overline{u^{2}} d x$ are to be considered as quantities of the order of $L$, in view of which various other quantities may be neglected. In the case of a field vanishing beyond a certain distance (like Lorentz' vortex) the quantities that are neglected automatically become zero, when the sections I, II are put away far enough.

Having regard to the relations which are fulfilled in the normal state, and observing that $\delta U$ and $\delta V$ must obey the equation of continuity. we obtain:

$$
\left.\begin{array}{ll}
U \frac{\partial}{\partial x} \delta U+\frac{d U}{d y} \delta V-R^{-1} \triangle(\delta U)+\frac{\partial}{\partial x} \delta P & =-\delta \alpha\left(\frac{\partial u^{\prime 2}}{\partial x}+\frac{\partial u^{\prime} v^{\prime}}{\partial y}\right) \\
U \frac{\partial}{\partial x} \delta V- & R^{-1} \triangle(\delta V)+\frac{\partial}{\partial y} \delta P \tag{b}
\end{array}\right)=-\delta \alpha\left(\frac{\partial u^{\prime} v^{\prime}}{\partial x}+\frac{\partial v^{\prime 2}}{\partial y}\right)
$$

and further:

$$
\left.\begin{array}{rl}
\delta \bar{E}=\int d y(U \delta P & +P \delta U)_{I}-\int d y(U \delta P+P \delta U)_{I I}-  \tag{47}\\
& -R^{-1} \iint d x d y\left\{2 \frac{d U}{d y}\left(\frac{\partial}{\partial y} \delta U-\frac{\partial}{\partial x} \delta V\right)+\zeta^{\prime 2} \delta a\right\}
\end{array}\right\}
$$

The latter equation can be simplified by making use of the condition that $\int U d y$ must remain constant and by neglecting terms which do not become of the order of $L$; in this way we get:

$$
\begin{equation*}
\delta \bar{E}=\int d y U\left(\delta P_{I}-\delta P_{I I}\right)-R^{-1} \iint d x d y\left\{2 \frac{d U}{d y} \frac{\partial}{\partial y} \delta U+\zeta^{\prime 2} \delta \alpha\right\} \tag{48}
\end{equation*}
$$

Now from equation ( $46^{a}$ ), by integrating it with respect to $x$ and again neglecting terms which do not become of the order of $L$, we deduce:

$$
\begin{equation*}
\delta P_{I}-\delta P_{I I}=-R^{-1} \int d x \frac{\partial^{2}}{\partial y^{2}} \delta U+\delta a \int d x \frac{\partial u^{\prime} v^{\prime}}{\partial y} \tag{49}
\end{equation*}
$$

As equation $\left(46^{b}\right)$ shows that $\partial \delta P / \partial y$ does not become of the order of $L$, we may to the order of approximation accepted, consider $\delta P_{I}-\delta P_{I I}$ as independent of $y$. Then the first term of the expression (48) becomes simply $\delta P_{I}-\delta P_{I I}$.

By multiplying equation (49) by ( $\left(\frac{3}{2}-6 y^{2}\right)$ and integrating it over the breadth of the channel (applying partial integration), we obtain:

$$
\begin{equation*}
\delta P_{I}-\delta P_{I I}=12 \delta \alpha \iint_{1} d x d y y u^{\prime} v^{\prime} \tag{a}
\end{equation*}
$$

On the other hand the first term of the second member of the expression (48) may be transformed (again making use of partial integrations and of (49)) as follows:
$-R^{-1} \iint d x d y 2 \frac{d U}{d y} \frac{\partial}{\partial y} \delta U=-2\left(\delta P_{I}-\delta P_{I I}\right)-2 \delta \alpha \iint d x d y \frac{d U}{d y} u^{\prime} v^{\prime}$
Hence finally we obtain:

$$
\begin{equation*}
\frac{\delta \bar{E}}{\delta a}=\iint d x d y\left\{-u^{\prime} v^{\prime}\left(2 \frac{d U}{d y}+12 y\right)-R^{-1} \zeta^{\prime 2}\right\} \tag{50}
\end{equation*}
$$

in which the same expression appears as in (40).
So the condition for the positive definite character of the function $\Phi$ appears to be identical with the condition (43).

The above calculations can be extended also to the case of the flow through a zylindrical pipe with an arbitrary form of section, where the relative motion is three-dimensional.

## 6. Concluding remarks.

As has been mentioned at the end of $\S 4$ the observed distribution
of the velocity of the mean motion does not correspond to that given by equation (38). The value of $d U / d y$ appears to be much smaller, even so much that the condition $\delta \bar{E} / \delta \alpha$ seems not to be fulfilled for some types of relative motion, possessing values of $u^{\prime} v^{\prime}$ of the opposite sign of that which corresponds to the usual correlation. So we have to suppose that an abnormally frequent occurrence of such types of motion is prevented by some other cause, which is not revealed in our deductions.

When we come back once more to our starting point: that the dissipation condition in the case of the turbulent motion of a viscous fluid plays the same role as the condition of constant energy in the case of a conservative system, it is necessary to point out one great difference between the two conditions. The condition of constant energy in a conservative system is an exact and an absolute one, which is valid at every moment. The dissipation condition, on the other hand, is not a condition governing the instantaneous state of the turbulent motion; it expresses a relation which is fulfilled approximatively when we consider the history of our system, during a great interval of time. This becomes especially clear in view of the deductions concerning the quantity $\delta \bar{E} / \delta a$, where we considered a variation of the system, which could establish itself only in a very long time. The dissipation condition seems to be too "elastic" to make feel its influence immediately when any deviation from the "normal state" occurs.

There is no doubt, of course, that the dissipation condition has to be fulfilled. The question is, however, which is the variational equation that governs the exponent of the distribution function?

When it could be supposed that the expression (39) for the increase of the instantaneous energy exerted some influence on the statistical distribution, and that the exponent occurring in (21) consisted of a linear combination of both (39) and (40), in such a way that we could write for the function $p$ in (21), (35) etc.:

$$
\varphi=R^{-1}\left(\frac{2+\lambda}{1+\lambda} \frac{d U}{d y}+\frac{12 y}{1+\lambda}\right)
$$

then a distribution of the velocity might be obtained, corresponding somewhat better with that observed experimentally. So $\lambda=3$ would give : $U=1,1-1,2 y^{2}$; the lower limit for $C$ then becomes: $1 / 5 A=0,0011$.

I have not succeeded in finding an equation which seems to lead to such a formula. There is, however, another way which promises some help : in order to define the mean motion we may make use of mean values with respect to $x$ in stead of mean values with respect to time. In this way the difficulties mentioned are obviated to some extent, and a form of the dissipation condition is obtained, which leads to a variational equation, differing from the one used until now. In Part III of this paper we hope come back to this point. ${ }^{1}$ )
${ }^{1}$ ) This remark has been added in the proof.

In all calculations we have adhered to the condition that the total flow over a section of the channel remains absolutely constant. This seems to be necessary, as on account of the equation of continuity the total flow must be the same for all sections, so that a change at one section would necessarily bring with it a change over the whole length of the channel, which can be taken arbitrarily great. Only when elastic phenomena are taken into account, this condition might be violated.

## 7. Appendix to § 2, Part I.

In § 2 for the case of an ideal (frictionless) fluid, moving uninfluenced by exterior forces, a demonstration was sketched of the theorem:

$$
\begin{equation*}
\sum_{k} \frac{\partial \dot{\xi}_{k}}{\partial \xi_{k}}=0 \tag{16}
\end{equation*}
$$

It is possible to give a proof of this theorem, starting from equation (18), as it becomes after the substitution of (17), etc.:

$$
\begin{equation*}
\frac{\partial \Psi}{\partial t}=\iint d x^{\prime} d y^{\prime} G\left(x, y ; x^{\prime}, y^{\prime}\right)\left\{\frac{\partial \Psi}{\partial y^{\prime}} \frac{\partial \triangle \Psi}{\partial x^{\prime}}-\frac{\partial \Psi}{\partial x^{\prime}} \frac{\partial \triangle \Psi}{\partial y^{\prime}}\right\} \tag{18*}
\end{equation*}
$$

but without translating this equation into discontinuous terms. ${ }^{1}$ ) However, as has been mentioned already, it is necessary then to specify the area over which the stream function $\Psi$ has to be increased. We will do this by accepting the following formula for $\delta \Psi$ :

$$
\delta \Psi=\alpha e^{-x\left\{\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}\right\}},
$$

in which $x$ is supposed to be so great that $x^{-1 / 2}$ is very small with regard to distances over which $\Psi$ changes appreciably, and also with regard to the distance of the point $x, y$ from the nearest wall. We now define $\delta \xi_{k}$ (if $k$ is the index number of the point $x, y$ ) by:

$$
\delta \xi_{k}=\iint d x^{\prime} d y^{\prime} \delta \Psi=\pi \alpha x^{-1}
$$

In equation ( $18^{\star}$ ) we write:

$$
G=-\frac{1}{2 \pi} \lg \sqrt{\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}}+G^{\star}\left(x, y ; x^{\prime}, y^{\prime}\right)
$$

Then by direct calculation we get:

$$
-\frac{1}{2 \pi} \int d x^{\prime} d y^{\prime} \lg \sqrt{\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}} \cdot \delta\left\{\frac{\partial \Psi}{\partial y^{\prime}} \frac{\partial \Delta \Psi}{\partial x^{\prime}}-\frac{\partial \Psi}{\partial x^{\prime}} \frac{\partial \triangle \Psi}{\partial y^{\prime}}\right\}=0
$$

[^7]Using partial integrations we obtain on the other hand:

$$
\begin{aligned}
& \iint d x^{\prime} d y^{\prime} G^{\star} \delta\{\ldots\}= \\
& =-\iint d x^{\prime} d y^{\prime}\left\{\frac{\partial}{\partial y^{\prime}}\left(G^{\star} \frac{\partial \Delta \Psi}{\partial x^{\prime}}\right)+\ldots\right\} \delta \Psi= \\
& =-\left\{\frac{\partial}{\partial y^{\prime}}\left(G^{\star} \frac{\partial \Delta \Psi}{\partial x^{\prime}}\right)+\frac{\partial}{\partial x^{\prime}} \Delta\left(G^{\star} \frac{\partial \Psi}{\partial y^{\prime}}\right)-\frac{\partial}{\partial x^{\prime}}\left(G^{\star} \frac{\partial \triangle \Psi}{\partial y^{\prime}}\right)-\right. \\
& \left.-\frac{\partial}{\partial y^{\prime}} \triangle\left(G^{\star} \frac{\partial \Psi}{\partial x^{\prime}}\right)\right\} \delta \xi_{k} \\
& =\begin{array}{c}
x^{\prime}=x \\
y^{\prime}=y \\
y
\end{array}
\end{aligned}
$$

if $x$ is sufficiently great.
When we perform the differentiations, and remember that $\triangle G^{\star}=0$, then we may transform this expression and obtain:

$$
\frac{\partial \dot{\xi}_{k}}{\partial \xi_{k}}=\gamma=-2\left\{\left(\frac{\partial^{2} G^{\star}}{\partial x^{\prime 2}}-\frac{\partial^{2} G^{\star}}{\partial y^{\prime 2}}\right) \frac{\partial^{2} \Psi}{\partial x^{\prime} \partial y^{\prime}}-\frac{\partial^{2} G^{\star}}{\partial x^{\prime} \partial y^{\prime}}\left(\frac{\partial^{2} \Psi}{\partial x^{\prime 2}}-\frac{\partial^{2} \Psi}{\partial y^{\prime 2}}\right)\right\}
$$

in which formula after the execution of the differentiations we have to put $x^{\prime}=x, y^{\prime}=y$.

In the case of a straight channel we have:

$$
\left(\frac{\partial^{2} G^{\star}}{\partial x^{\prime} \partial y^{\prime}}\right)_{x^{\prime}=x}=0, \quad\left(\frac{\partial^{2} G^{\star}}{\partial x^{\prime 2}}-\frac{\partial^{2} G^{\star}}{\partial y^{\prime 2}}\right)_{x^{\prime}=x} \text { is independent of } x .
$$

Hence:

Here the integral is not strictly zero, but is reduced to integrals over the boundaries of the field at $x_{0}$ and at $x_{0}+L$.

When the channel is closed, these terms may cancel. In any case this can be proved for the space between two concentric cylinders. By introducing polar coordinates $\gamma$ is transformed into an expression of the form:

$$
\left(\frac{\partial^{2} \Psi}{r \partial r \partial \theta}-\frac{\partial \Psi}{\tau^{2} \partial \theta}\right) f
$$

where $f$ is some function of $r$ only. As $\Psi$ is an univalued function, the integral of this expression over the whole field is zero.


[^0]:    ${ }^{1}$ ) Part I has appeared in these Proceedings 32, p. 414, 1929. The reader is asked to correct an error of print in equation (24): the exponent of $e$ must be read:

    $$
    -\beta \sum_{l}\left\{\frac{z_{l}(\xi)}{R^{2}}-\left(2 C-R^{-1}-2 \bar{t}_{l}\right) t_{l}(\xi)\right\}
    $$

    ${ }^{2}$ ) Every term of the summation relates to one of the cells in which the whole of the s-space was supposed to be divided.

[^1]:    ${ }^{1}$ ) Comp. also: J. M. Burgers, these Proceedings 26, p. 601, 1923. The value of $C$ in that paper, however, is twice the value taken here; moreover the origin of the coordinates had been put in one of the walls.

[^2]:    ${ }^{1}$ ) Comp. H. A. Lorentz. Abhandl. über theoretische Physik, 1, p. 48; W. Mc. F. Orr, Proc. Roy. Irish Acad. 27, p. 124-128, 1907. I must confess that I have controlled the formula (a) only for the case of the LORENTZ' vortex; if this formula should not apply to the function which according to ORR gives the lowest value for $A$, a somewhat higher value of $A$ ought to be accepted in the following calculations, though probably still less than LORENTZ' value 288 for the elliptic vortex.

[^3]:    ${ }^{1}$ ) Comp. L. Hopf, Zeitschr. f. angew. Math. u. Mech. 3, p. 329, 1923; Th. v. Karman, Proc. Ist Intern. Congress for Appl. Mech., Delft, 1924, p. 103; L. Schiller, Physik. Zeitschr. 26, p. 473, 592, 1925.

[^4]:    ${ }^{1}$ ) T. E. Stanton and J. R. Pannell, Phil. Trans. Roy. Soc. London A 214, p. 199, 1914 (comp. Ch. H. Lees, Proc. Roy. Soc. London A 91, p. 46, 1915) ; М. Jakob und S. Erk, Mitt. über Forschungsarbeiten herausgeg. v. V. D. I Heft 267, 1924; H. Bazin, Mém. Acad. d. Sciences (Sav. Etrangers) 32, No. 6, p. 1, 1902. Moore's and Johnston's values are taken from a diagram given by Hopf, l. c., curves 25,26 of fig. 2.

[^5]:    ${ }^{1}$ ) The result for the tube with circular section is not obtained by means of the statistical method, as no generalisation for the three-dimensional case has been made. The deductions of $\S 5$, however, can be extended to this case.

[^6]:    ${ }^{1}$ ) The equation for $E$, when written in full, contains besides those given in the text, a number of other terms, relating to the cross sections of the channel at $x_{0}$ and $x_{0}+L$, which usually are considered as of no importance, and further the integral

    $$
    \iint d x d y\left(u \frac{d \overline{u v}}{d y}+v \frac{d \overline{v^{2}}}{d y}\right)
    $$

    (comp. H. A. Lorentz, l.c. p. 63) of which only the first term is important. This term changes of sign, when the direction of the relative motion is inverted over the whole field, which is not the case with the terms written in (39). Such an inversion of the relative motion has no influence on our formulae, which have been either of the 2nd or of the 4th degree in $u, v, s$.

[^7]:    ${ }^{1}$ ) In this formula, as well as in those which follow, the operator $\Delta$ relates to the variables $x^{\prime}, y^{\prime}$.

