Mathematics. — *Skew Correlation between Three and More Variables*, I. By Prof. M. J. VAN UVEN. (Communicated by Prof. A. A. NIJLAND).

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I. *Skew Correlation between three variables.*

In order to furnish a model for the treatment of skew correlation between an arbitrary number (n) of variables, we shall first establish the method of treating the case of three variables. We continue the method we followed formerly in treating the case of two variables, exposed in the paper "Over het bewerken van scheeve correlatie" ("On Treating Skew Correlation") [1], recently completed by the paper: "Scheeve Correlatie tusschen twee veranderlijken" ("Skew Correlation between Two Variables") [2].

These papers (the former being distributed over three articles) will be designated by the abbreviations S. C. I, a, b, c, S. C. II.

The three variables may be called x_1, x_2, x_3. For the variable x_a $(a = 1, 2, 3)$ ν_a values, $\xi_a(1), \xi_a(2), \ldots \xi_a(k_a), \ldots \xi_a(\nu_a)$ [3], are recorded. As a rule the interval between two class-centres is constant: $\xi_a(k_a) - \xi_a(k_a - 1) = c_a$ $(a = 1, 2, 3)$.

The frequency of the set $\xi_1(k_1), \xi_2(k_2), \xi_3(k_3)$ may be denoted by $Y(k_1, k_2, k_3)$. For the total number N of the observed sets ξ_1, ξ_2, ξ_3 we have

$$N = \sum_{i_1=1}^{\nu_1} \sum_{i_2=1}^{\nu_2} \sum_{i_3=1}^{\nu_3} Y(i_1, i_2, i_3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 1$$

Thus the relative frequency (a posteriori probability) of the set $\xi_1(k_1), \xi_2(k_2), \xi_3(k_3)$ is

$$y(k_1, k_2, k_3) = \frac{Y(k_1, k_2, k_3)}{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 2$$

What is properly meant by recording $\xi_a(k_a)$ for x_a, is that x_a is

[1]) Versl. K. A. v. W. **34**, p. 787 en p. 965; **35**, p. 129. (Proceed. K. Ak. v. Wet. Amsterdam: Vol. **28**, p. 797 and p. 919; Vol. **29**, p. 580).

[2]) Versl. K. A. v. W. (Proceed. K. Ak. v. Wet. Amsterdam, Vol. **32**, p. 408) (with summary in English).

[3]) Using, also further on, the brackets ( ) in denoting the class-numbers, we shall, in the following text, designate a functional connexion by $\{\ \}$, e.g. $t\{x\}$.

found between $\xi_a(k_a) - \frac{c_a}{2}$ and $\xi_a(k_a) + \frac{c_a}{2}$. Putting

$$\xi_a(k_a) + \frac{c_a}{2} = x_a(k_a) \quad , \quad (a = 1, 2, 3) \quad . \quad . \quad . \quad . \quad . \quad . \quad 3$$

so that $x_a(k_a)$ indicates the upper limit of the class k_a, we may describe the three-dimensional frequency distribution by the statement:

For $Y(k_1, k_2, k_3)$ sets x_1, x_2, x_3 is found: $x_a(k_a - 1) < x_a < x_a(k_a)$, $a = 1,2,3$. **A**

If the correlation between x_1, x_2, x_3 is itself linear, then the probability of the set ξ_1, ξ_2, ξ_3 is expressed by an infinitesimal probability formula, which we may construct as follows:

We compute the mean $\bar{\xi}_a$ of all observed values ξ_a, and the deviations $u_a = \xi_a - \bar{\xi}_a$ from that mean.

Then the probability that such a set of deviations is found between u_1, u_2, u_3 and $u_1 + du_1, u_2 + du_2, u_3 + du_3$ has theoretically the infinitesimal value:

$$dW = Ce^{-f} du_1 \, . \, du_2 \, . \, du_3.$$

Here the symbol f represents a positive-definite homogeneous quadratic form in the u_a:

$$f \equiv h_1^2 u_1^2 + 2\lambda_{12} h_1 h_2 u_1 u_2 + 2\lambda_{13} h_1 h_3 u_1 u_3 + h_2^2 u_2^2 + 2\lambda_{23} h_2 h_3 u_2 u_3 + \\ + h_3^2 u_3^2 \equiv \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \lambda_{\alpha\beta} h_\alpha h_\beta u_\alpha u_\beta ,$$

where

$$\lambda_{\alpha\alpha} = 1 \quad , \quad \lambda_{\alpha\beta} = \lambda_{\beta\alpha} \quad , \quad \lambda_{\alpha\beta}^2 < 1.$$

Putting

$$|\lambda_{\alpha\beta}| = \begin{vmatrix} \lambda_{11} , & \lambda_{12} , & \lambda_{13} \\ \lambda_{21} , & \lambda_{22} , & \lambda_{23} \\ \lambda_{31} , & \lambda_{32} , & \lambda_{33} \end{vmatrix} = \begin{vmatrix} 1 , & \lambda_{12} , & \lambda_{31} \\ \lambda_{12} , & 1 , & \lambda_{23} \\ \lambda_{31} , & \lambda_{23} , & 1 \end{vmatrix} = \Lambda,$$

we find for the constant factor C:

$$C = \frac{h_1 h_2 h_3 \sqrt{\Lambda}}{\sqrt{\pi^3}} .$$

Before analysing this three-dimensional probability formula, we shall introduce the *unimodular* variables t_1, t_2, t_3 by the relations:

$$t_a = h_a u_a \quad , \quad (a = 1, 2, 3)$$

Hence the infinitesimal probability formula is expressed in these unimodular variables as follows:

$$dW = \frac{\sqrt{\Lambda}}{\sqrt{\pi^3}} e^{-f} dt_1 \, . \, dt_2 \, . \, dt_3 , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 4$$

where

$$f \equiv t_1^2 + 2\lambda_{12} t_1 t_2 + 2\lambda_{31} t_1 t_3 + t_2^2 + 2\lambda_{23} t_2 t_3 + t_3^2 \equiv \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \lambda_{\alpha\beta} t_\alpha t_\beta , \quad 5$$

with

$$\lambda_{\alpha\alpha} = 1 \quad , \quad \lambda_{\alpha\beta} = \lambda_{\beta\alpha} \quad , \quad \lambda^2_{\alpha\beta} < 1 \quad , \quad \Lambda = |\lambda_{\alpha\beta}| \qquad . \ . \ . \ \mathbf{6}$$

For the study of the probability formula **4** we shall provisionally suppose t_1, t_2, t_3 to be the original variables.

Putting

$$\lambda_{23} = \cos\varphi_1 \quad , \quad \lambda_{31} = \cos\varphi_2 \quad , \quad \lambda_{12} = \cos\varphi_3 \, , \qquad . \ . \ . \ \mathbf{7}$$

we may illustrate the form f geometrically by considering a skew system of (rectilinear) coordinates t_1, t_2, t_3, whereby the axes t_2 and t_3 include the angle φ_1, the axes t_3 and t_1 the angle φ_2, the axes t_1 and t_2 the angle φ_3 (fig. 1). The axes $OQ_1 (= t_1)$, $OQ_2 (= t_2)$, $OQ_3 (= t_3)$ may (eventually prolonged) cut the sphere of radius unity with centre O ("unity-sphere") at the points Φ_1, Φ_2, Φ_3.

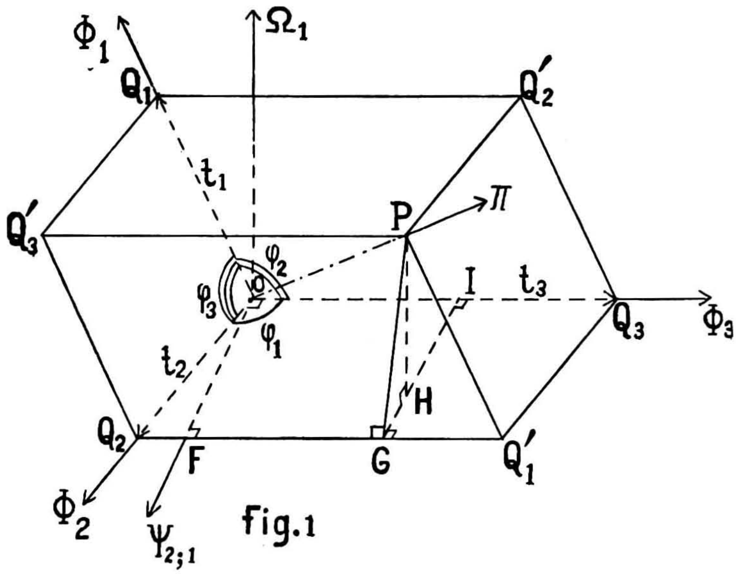

fig.1

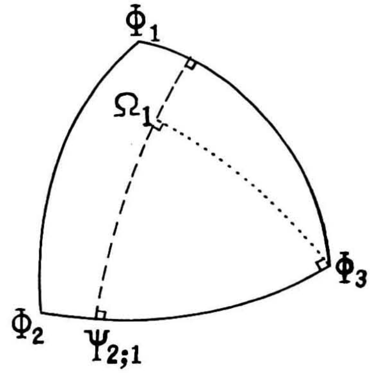

fig.2

Then on this unity-sphere we have a triangle $\Phi_1\ \Phi_2\ \Phi_3\ [(\Phi)]$ the sides of which are $\varphi_1, \varphi_2, \varphi_3$. In our sketches we have taken all three sides $\varphi_1, \varphi_2, \varphi_3$ obtuse (fig. 2).

P being a point with the (skew) coordinates t_1, t_2, t_3, the square of the radius vector $OP = r$, carrying from the origin O to that point, amounts to

$$OP^2 = r^2 = f \qquad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ \mathbf{8}$$

In order to integrate easily the probability differential, we shall write the quadratic form f as a sum of three squares. The geometrical meaning of this is, that we decompose $OP = r$ along three rectangular axes.

So we shall decompose OP

1⁰. along $O\Phi_3$,

2^0. along $O\Psi_{2;1}$ [1]) within the plane $\Phi_2 O\Phi_3$ perpendicular to $O\Phi_3$, $O\Psi_{2;1}$ being directed to that side of $O\Phi_3$ where $O\Phi_2$ lies,

3^0. along $O\Omega_1$ perpendicular to the plane $\Phi_2 O\Phi_3$, directed to the same side as $O\Phi_1$.

Now we find for the component Z_3 along $O\Phi_3$:

$$Z_3 = \text{proj. } OP \text{ on } O\Phi_3 = (\text{proj. } OQ_1 + \text{proj. } Q_1Q'_3 + \text{proj. } Q'_3P) \text{ on } O\Phi_3 =$$
$$= t_1 \cos\varphi_2 + t_2 \cos\varphi_1 + t_3.$$

To compute the second component ($\zeta_{2;1}$), we drop the perpendicular PH from P on $\Phi_2O\Phi_3$ and (within the plane $Q_2Q'_1PQ'_3$) the perpendicular PG on $Q_2Q'_1$; then $\angle HGP$ is the solid angle between the planes $\Phi_2O\Phi_3$ and $Q_2Q'_1PQ'_3$, hence the supplement of the solid angle at the edge $O\Phi_3$, thus the supplement of the angle Φ_3 of the spherical triangle (Φ), whence $GH = GP \, . \cos(\pi - \Phi_3) = - GP \, . \cos\Phi_3$.

Further we have $GP = Q'_1P . \sin GQ'_1P = Q'_1P . \sin(\pi - \varphi_2) = Q'_1P . \sin\varphi_2$.

So we find for the projection GH of Q'_1P on $\overrightarrow{\Psi_{2;1}O}$: $GH = - Q'_1P \, . \sin\varphi_2 \cos\Phi_3 = - t_1 \sin\varphi_2 \cos\Phi_3$; therefore the projection $GH (= - HG)$ of Q'_1P on $\overrightarrow{O\Psi_{2;1}}$ is: $+ t_1 \sin\varphi_2 \cos\Phi_3$.

Hence the component $\zeta_{2;1}$ of OP along $O\Psi_{2;1}$ amounts to:

$$\zeta_{2;1} = \text{proj. } OP \text{ on } O\Psi_{2;1} = (\text{proj. } OQ'_1 + \text{proj. } Q'_1P) \text{ on } O\Psi_{2;1} =$$
$$= OF + t_1 \sin\varphi_2 \cos\Phi_3 = t_2 \cos\left(\varphi_1 - \frac{\pi}{2}\right) + t_1 \sin\varphi_2 \cos\Phi_3$$

or

$$\zeta_{2;1} = t_1 \sin\varphi_2 \cos\Phi_3 + t_2 \sin\varphi_1.$$

Finally we obtain for the component z_1 along $O\Omega_1$:

$$z_1 = \text{proj. } OP \text{ on } O\Omega_1 = HP = GP \sin(\pi - \Phi_3) = t_1 \sin\varphi_2 \sin\Phi_3.$$

So we have:

$$\left.\begin{aligned} z_1 &= \sin\varphi_2 \sin\Phi_3 \, . \, t_1, \\ \zeta_{2;1} &= \sin\varphi_2 \cos\Phi_3 \, . \, t_1 + \sin\varphi_1 \, . \, t_2, \\ Z_3 &= \cos\varphi_2 \, . \, t_1 + \cos\varphi_1 \, . \, t_2 + t_3. \end{aligned}\right\} \quad . \; . \; . \quad 9 \left\{\begin{array}{l} (1) \\ (2;\ 1) \\ (3;\ 21) \end{array}\right.$$

In fig. 1 Z_3, $\zeta_{2;1}$, z_1 are represented by OI, IH, HP respectively.

The perpendicular $O\Omega_1$ on $\Phi_2O\Phi_3$ meets the unity-sphere at either of the poles Ω_1 of $\Phi_2\Phi_3$, and particularly at that pole, which lies with Φ_1 on the same side of $\Phi_2\Phi_3$.

Constructing in a similar way the pole Ω_2 of $\Phi_3\Phi_1$ and the pole Ω_3 of $\Phi_1\Phi_2$, the points Ω_1, Ω_2, Ω_3 form the opposite triangle of that triangle which is usually called the polar triangle of $\Phi_1\Phi_2\Phi_3$. Nevertheless we shall further on denote that very triangle $\Omega_1\Omega_2\Omega_3$ by "the polar triangle of $\Phi_1\Phi_2\Phi_3$"

[1]) The sign; between the subscripts points out, that the arrangement of these subscripts is relevant. Subscripts not separated by the sign; are permutable.

Expressing the angles of the spherical triangle (Φ) in the sides ω_a of its polar triangle (Ω) by means of $\Phi_a = \pi - \omega_a$ $(a = 1, 2, 3)$, we obtain:

$$\left.\begin{aligned} z_1 &= \sin\varphi_2 \sin\omega_3 \,.\, t_1, \\ \zeta_{2;1} &= -\sin\varphi_2 \cos\omega_3 \,.\, t_1 + \sin\varphi_1 \,.\, t_2, \\ Z_3 &= \cos\varphi_2 \,.\, t_1 + \cos\varphi_1 \,.\, t_2 + t_3. \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad 9^{\text{bis}}$$

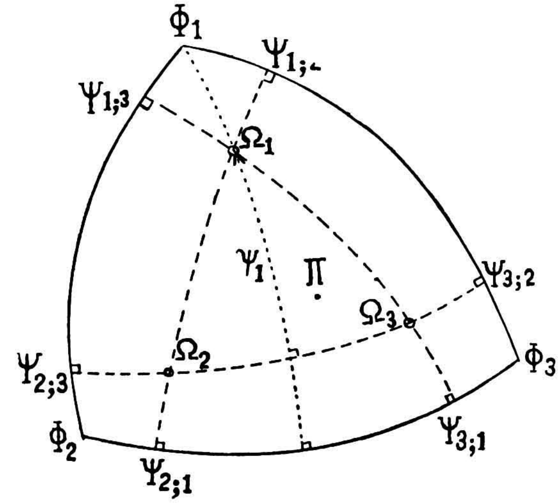

fig.3

Prolonging the sides of triangle (Ω) (which are acute in our sketches), $\Omega_2\Omega_3$ meets φ_2 at $\Psi_{3;2}$, φ_3 at $\Psi_{2;3}$; $\Omega_3\Omega_1$ meets φ_3 at $\Psi_{1;3}$, φ_1 at $\Psi_{3;1}$; $\Omega_1\Omega_2$ meets φ_1 at $\Psi_{2;1}$, φ_2 at $\Psi_{1;2}$ (fig. 3).

Each of the six triplets

$$\Omega_1\,\Psi_{2;1}\,\Phi_3,\quad \Omega_2\,\Psi_{1;2}\,\Phi_3,\quad \Omega_1\,\Psi_{3;1}\,\Phi_2,\quad \Omega_2\,\Psi_{3;2}\,\Phi_1,\quad \Omega_3\,\Psi_{1;3}\,\Phi_2,\quad \Omega_3\,\Psi_{2;3}\,\Phi_1$$

determines a rectangular system of coordinates. The components of $OP = r$ in these 6 systems are

$$z_1\,\zeta_{2;1}\,Z_3,\quad z_2\,\zeta_{1;2}\,Z_3,\quad z_1\,\zeta_{3;1}\,Z_2,\quad z_2\,\zeta_{3;2}\,Z_1,\quad z_3\,\zeta_{1;3}\,Z_2,\quad z_3\,\zeta_{2;3}\,Z_1.$$

The point Π where OP cuts the unity-sphere, is the common image point of these 6 triplets.

As $\sin\varphi_2 \sin\Phi_3$ equals the sine of the altitude of (Φ) issuing from Φ_1, this latter being the supplement of the altitude ψ_1 of (Ω) issuing from Ω_1, we have

$$\left.\begin{aligned} \sin\varphi_2 \sin\Phi_3 &= \sin\varphi_2 \sin\omega_3 = \sin\psi_1, \\ \sin\varphi_3 \sin\Phi_1 &= \sin\varphi_3 \sin\omega_1 = \sin\psi_2, \\ \sin\varphi_1 \sin\Phi_2 &= \sin\varphi_1 \sin\omega_2 = \sin\psi_3, \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad 10$$

Hence we may write for z_a :

$$z_a = \sin \psi_a \, . \, t_a \qquad \ldots \ldots \ldots \qquad 11$$

We now have:

$$\left.\begin{aligned} \sqrt{\Lambda} &= \sqrt{1 - \lambda_{23}^2 - \lambda_{31}^2 - \lambda_{12}^2 + 2\,\lambda_{23}\,\lambda_{31}\,\lambda_{12}} = \\ &= \sqrt{1 - \cos^2\varphi_1 - \cos^2\varphi_2 - \cos^2\varphi_3 + 2\cos\varphi_1\cos\varphi_2\cos\varphi_3} = \\ &= \sin\varphi_a \, . \, \sin\psi_a \, , \quad (a = 1, 2, 3) \end{aligned}\right\} \quad 12$$

$$\left.\begin{aligned} \cos\omega_1 &= -\cos\Phi_1 = \frac{\cos\varphi_2\cos\varphi_3 - \cos\varphi_1}{\sin\varphi_2\sin\varphi_3} = \frac{\lambda_{31}\,\lambda_{12} - \lambda_{23}}{\sqrt{(1-\lambda_{31}^2)(1-\lambda_{12}^2)}} = \\ &= \frac{\Lambda_{23}}{\sqrt{\Lambda_{22}\,\Lambda_{33}}}\ ^{1)} \quad , \quad \sin\omega_1 = \frac{\sqrt{\Lambda}}{\sqrt{\Lambda_{22}\,\Lambda_{33}}}. \end{aligned}\right\} \quad 13$$

Putting

$$\gamma_{23} = \cos\omega_1 \quad , \quad \gamma_{31} = \cos\omega_2 \quad , \quad \gamma_{12} = \cos\omega_3 \quad \ldots \quad 7\text{bis}$$

and

$$\Gamma = |\,\gamma_{\alpha\beta}\,| = \begin{vmatrix} \gamma_{11}, & \gamma_{12}, & \gamma_{13} \\ \gamma_{21}, & \gamma_{22}, & \gamma_{23} \\ \gamma_{31}, & \gamma_{32}, & \gamma_{33} \end{vmatrix} \quad \ldots \ldots \quad 6\text{bis}$$

with

$$\gamma_{\alpha\alpha} = 1, \qquad \gamma_{\alpha\beta} = \gamma_{\beta\alpha}, \qquad \gamma_{\alpha\beta}^2 < 1,$$

we have, as a counterpart of **12**,

$$\left.\begin{aligned} \sqrt{\Gamma} &= \sqrt{1 - \cos^2\omega_1 - \cos^2\omega_2 - \cos^2\omega_3 + 2\cos\omega_1\cos\omega_2\cos\omega_3} = \\ &= \sin\omega_a\sin\psi_a \quad (a = 1, 2, 3), \end{aligned}\right\} \quad 12\text{bis}$$

and, as a counterpart of (**13**),

$$\left.\begin{aligned} \cos\varphi_1 &= -\cos\Omega_1 = \frac{\cos\omega_2\cos\omega_3 - \cos\omega_1}{\sin\omega_2\sin\omega_3} = \frac{\gamma_{31}\,\gamma_{12} - \gamma_{23}}{\sqrt{(1-\gamma_{31}^2)(1-\gamma_{12}^2)}} = \\ &= \frac{\Gamma_{23}}{\sqrt{\Gamma_{22}\,\Gamma_{33}}}, \qquad \sin\varphi_1 = \frac{\sqrt{\Gamma}}{\sqrt{\Gamma_{22}\,\Gamma_{33}}}, \end{aligned}\right\} \quad 13\text{bis}$$

whence the *mutual* relations between γ_{ab} and λ_{ab}

$$\gamma_{ab} = \frac{\Lambda_{ab}}{\sqrt{\Lambda_{aa}\,\Lambda_{bb}}} \quad , \quad \lambda_{ab} = \frac{\Gamma_{ab}}{\sqrt{\Gamma_{aa}\,\Gamma_{bb}}} \quad \ldots \ldots \quad 13\text{ter}$$

The magnitude γ_{ab} is the *total* coefficient of correlation between t_a and t_b. Moreover:

$$\Lambda^{3/2} = \sin\varphi_1\sin\varphi_2\sin\varphi_3 \, . \, \sin\psi_1\sin\psi_2\sin\psi_3 =$$
$$= \sin^2\varphi_1\sin^2\varphi_2\sin^2\varphi_3 \, . \, \sin\omega_1\sin\omega_2\sin\omega_3,$$
$$\Gamma^{3/2} = \sin\varphi_1\sin\varphi_2\sin\varphi_3 \, . \, \sin^2\omega_1\sin^2\omega_2\sin^2\omega_3,$$

thus

$$\Lambda^{1/2} \, . \, \Gamma^{1/2} = \sin\varphi_1\sin\varphi_2\sin\varphi_3 \, . \, \sin\omega_1\sin\omega_2\sin\omega_3, \quad \ldots \quad 14$$

and

$$\sqrt{\Lambda} = \frac{\Gamma}{\sin\omega_1\sin\omega_2\sin\omega_3} \quad , \quad \sqrt{\Gamma} = \frac{\Lambda}{\sin\varphi_1\sin\varphi_2\sin\varphi_3} \quad \ldots \quad 15$$

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[1]) Λ_{ab} denotes the minor (algebraic complement) of λ_{ab} in the determinant Λ.

Λ and Γ being of the 3rd order, we have also (annexe to **7** and **7bis**)

$$\left.\begin{array}{l}\sin\varphi_1=\sqrt{1-\lambda_{23}^2}=\sqrt{\Lambda_{11}},\quad \sin\varphi_2=\sqrt{1-\lambda_{31}^2}=\sqrt{\Lambda_{22}},\\ \qquad\qquad \sin\varphi_3=\sqrt{1-\lambda_{12}^2}=\sqrt{\Lambda_{33}},\end{array}\right\} \quad \mathbf{16}$$

$$\left.\begin{array}{l}\sin\omega_1=\sqrt{1-\gamma_{23}^2}=\sqrt{\Gamma_{11}},\quad \sin\omega_2=\sqrt{1-\gamma_{31}^2}=\sqrt{\Gamma_{22}},\\ \qquad\qquad \sin\omega_3=\sqrt{1-\gamma_{12}^2}=\sqrt{\Gamma_{33}},\end{array}\right\} \quad \mathbf{16^{bis}}$$

or, summarized,

$$\sin\varphi_a=\sqrt{\Lambda_{aa}}\quad,\quad \sin\omega_a=\sqrt{\Gamma_{aa}}\quad.\quad (a=1,2,3)\ .\ .\ .\ \mathbf{16^{ter}}$$

We can now put the equations **9bis** — by means of **7**, **11**, **13** (**13ter**), **16** — into the form:

$$\left.\begin{array}{ll} z_1=\left(\dfrac{\Lambda}{\Lambda_{11}}\right)^{1/2}.\,t_1\,, & (1)\\ \zeta_{2;1}=\dfrac{-\Lambda_{21}\,.\,t_1+\Lambda_{11}\,.\,t_2}{\sqrt{\Lambda_{11}}}\,, & (2;\,1)\\ Z_3=\lambda_{31}\,.\,t_1+\lambda_{23}\,.\,t_2+t_3\,. & (3;\,21)\end{array}\right\}\ .\ .\ .\ .\ \mathbf{9^{ter}}$$

By summing up the squares we easily regain the expression **5** for f.

We may observe, that z_1 is a (linear) function of t_1, $\zeta_{2;1}$ of t_1 and t_2, Z_3 of t_1, t_2 and t_3.

Moreover:

$$\frac{\partial(z_1,\ \zeta_{2;1},\ Z_3)}{\partial(t_1,\ t_2,\ t_3)}=\frac{dz_1}{dt_1}\cdot\frac{\partial\zeta_{2;1}}{\partial t_2}\cdot\frac{\partial Z_3}{\partial t_3}=\left(\frac{\Lambda}{\Lambda_{11}}\right)^{1/2}\times\Lambda_{11}^{1/2}\times 1=\sqrt{\Lambda}\,,$$

whence, by passing from the variables t_1, t_2, t_3 to the variables z_1, $\zeta_{2;1}$, Z_3:

$$\sqrt{\Lambda}\,.\,dt_1\,.\,dt_2\,.\,dt_3\rightleftarrows dz_1\,.\,d\zeta_{2;1}\,.\,dZ_3\ .\ .\ .\ .\ .\ .\ \mathbf{17}$$

Evidently $\sqrt{\Lambda}\,.\,dt_1\,.\,dt_2\,.\,dt_3$ represents the element of volume dV expressed in the skew coordinates t_1, t_2, t_3:

$$dV=\sqrt{\Lambda}\,.\,dt_1\,.\,dt_2\,.\,dt_3\rightleftarrows dz_1\,.\,d\zeta_{2;1}\,.\,dZ_3\ .\ .\ .\ \mathbf{17^{bis}}$$

So we may put the infinitesimal probability dW into the form:

$$dW=\frac{1}{\sqrt{\pi^3}}\,e^{-(z_1^2+\zeta_{2;1}^2+Z_3^2)}\,dz_1\,.\,d\zeta_{2;1}\,.\,dZ_3\ .\ .\ .\ .\ \mathbf{18}$$

Putting in general

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{P}e^{-p^2}\,dp=\Theta\{P\},\ ^{1)}\ .\ .\ .\ .\ .\ .\ .\ \mathbf{19}$$

and further:

$$\Theta\{z_1\}=s_1\quad,\quad \Theta\{\zeta_{2;1}\}=\sigma_{2;1}\quad,\quad \Theta\{Z_3\}=S_3\,,\ .\ .\ .\ \mathbf{20}$$

we obtain besides for dW the formula:

$$dW=d\Theta\{z_1\}\,.\,d\Theta\{\zeta_{2;1}\}\,.\,d\Theta\{Z_3\}=ds_1\,.\,d\sigma_{2;1}\,.\,dS_3\,,\ .\ .\ \mathbf{21}$$

and likewise 5 analogous expressions.

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[1]) Cf. the footnote [3]) on page 793.

In order to isolate two of the variables, e.g. t_1 and t_2, we must keep t_1 and t_2 constant (with the ranges dt_1 and dt_2). Integrating now dW over t_3 (from $-\infty$ to $+\infty$) we obtain the probability of the set t_1, t_2 (with the ranges dt_1, dt_2), t_3 being arbitrary.

Now the integration over t_3 (with t_1 and t_2 constant) may be replaced by that over Z_3 (from $-\infty$ to $+\infty$).

On account of $\int\limits_{Z_3=-\infty}^{+\infty} d\Theta\{Z_3\} = \Theta\{+\infty\} = +1$, we get:

Probability of the set t_1, t_2 (ranges dt_1, dt_2), t_3 being arbitrary:

$$d_{(3)}W = d\Theta\{z_1\} \,.\, d\Theta\{\zeta_{2:1}\} = \frac{1}{\pi} e^{-(z_1^2+\zeta_{2:1}^2)} \, dz_1 \,.\, d\zeta_{2:1} \quad . \quad . \quad . \quad 22$$

We might have obtained this same infinitesimal probability, if we had started with the division $f = z_2^2 + \zeta_{1;2}^2 + Z_3^2$; hence this other formula for $d_{(3)}W$:

$$d_{(3)}W = d\Theta\{z_2\} \,.\, d\Theta\{\zeta_{1:2}\} = \frac{1}{\pi} e^{-(z_2^2+\zeta_{1:2}^2)} \, dz_2 \,.\, d\zeta_{1:2} \quad . \quad . \quad 22^{\text{bis}}$$

The magnitudes $z_1, z_2, \zeta_{2:1}, \zeta_{1;2}$ being independent of z_3, we may express both the differentials $d_{(3)}W$ in terms of z_1 and z_2; so we obtain:

$$d_{(3)}W = \frac{1}{\pi} e^{-(z_1^2+\zeta_{2:1}^2)} \frac{\partial \zeta_{2:1}}{\partial z_2} \,.\, dz_1 \,.\, dz_2 = \frac{1}{\pi} e^{-(z_2^2+\zeta_{1:2}^2)} \frac{\partial \zeta_{1:2}}{\partial z_1} \,.\, dz_1 \,.\, dz_2 \,, \quad 22^{\text{ter}}$$

whence

$$e^{-(z_1^2+\zeta_{2:1}^2)} \frac{\partial \zeta_{2:1}}{\partial z_2} = e^{-(z_2^2+\zeta_{1:2}^2)} \frac{\partial \zeta_{1:2}}{\partial z_1} \,, \quad . \quad . \quad . \quad . \quad . \quad . \quad 23$$

and generally:

$$e^{-(z_a^2+\zeta_{b:a}^2)} \frac{\partial \zeta_{b:a}}{\partial z_b} = e^{-(z_b^2+\zeta_{a:b}^2)} \frac{\partial \zeta_{a:b}}{\partial z_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 23^{\text{bis}}$$

In order to isolate *one* of the variables, e. g. t_1, we must keep t_1 constant (with the range dt_1), t_2 and t_3 being arbitrary. Then we obtain the probability of the value t_1 (with the range dt_1), t_2 and t_3 being arbitrary. Replacing the integration over t_2 (with t_1 constant) by that over $\zeta_{2:1}$, and taking account of $\int\limits_{\zeta_{2:1}=-\infty}^{+\infty} d\,\Theta\{\zeta_{2:1}\} = \Theta\{+\infty\} = +1$, we arrive at:

Probability of the value t_1 (range dt_1), t_2 and t_3 being arbitrary:

$$d_{(23)}W = d\Theta\{z_1\} = \frac{1}{\sqrt{\pi}} e^{-z_1^2} \, dz_1 \quad . \quad . \quad . \quad . \quad . \quad 24$$

If a three-dimensional frequency distribution, given by the empirical data:

"For $Y(k_1, k_2, k_3)$ individuals is found $t_a(k_a - 1) < t_a < t_a(k_a)$, $a = 1, 2, 3$" 25 shall be in accordance with the probability formula **4** (**5**, **6**), it must be possible to construct — by means of **24**, **22**, **21** — three functions z_1, $\zeta_{2;1}$, Z_3, which are connected with the variables t_1, t_2, t_3 by the relations **9ter**. The coefficients of the relations **9ter** having been determined, the constants λ_{ab} on the one hand and the coefficients of correlation γ_{ab} on the other, can be calculated.

The construction of the function z_1 out of **24** is performed by equalizing the theoretical probability of: [$t_1 < t_1 (k_1)$, t_2 and t_3 arbitrary], resulting from **24**, to the empirical value of this probability, deduced from **25**.

For this probability $s_1 (k_1)$ of: [$t_1 < t_1 (k_1)$, t_2 and t_3 arbitrary] we find: theoretically:

$$s_1 (k_1) = \int\limits_{t_1=-\infty}^{t_1(k_1)} d_{(23)} W = \Theta \{ z_1 (k_1) \}, \quad \text{where} \quad z_1 (k_1) = \left(\frac{A}{A_{11}}\right)^{1/2} . \, t_1 (k_1),$$

empirically:

$$s_1 (k_1) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{\nu_2} \sum_{i_3=1}^{\nu_3} y (i_1, i_2, i_3) = \frac{\sum\limits_{i_1=1}^{k_1} \sum\limits_{i_2=1}^{\nu_2} \sum\limits_{i_3=1}^{\nu_3} Y (i_1, i_2, i_3)}{N}.$$

Hence we obtain (putting successively $k_1 = 1, 2, \ldots, \nu_1 - 1$) $\nu_1 - 1$ pairs z_1, t_1 [1]).

If the frequency distribution $\{ Y;\ t_1, t_2, t_3 \}$ really corresponds to the formula **4** (**5**, **6**), it must appear, that the values $z_1 (k_1)$ resulting from

$$\Theta \{ z_1 (k_1) \} = s_1 (k_1) = \frac{\sum\limits_{i_1=1}^{k_1} \sum\limits_{i_2=1}^{\nu_2} \sum\limits_{i_3=1}^{\nu_3} Y (i_1, i_2, i_3)}{N} \quad \left[z_1 (k_1) = \left(\frac{A}{A_{11}}\right)^{1/2} . \, t_1 (k_1) \right], \quad \mathbf{B(1)}$$

are proportional to the associated values $t_1 (k_1)$.

The function $\zeta_{2;1}$ might be constructed, if the empirical treatment enabled us to give an infinitesimal range to the variable t_1. Then we could determine the empirical probability of $t_1 = t_1 (j_1)$ with the range dt_1, t_2 being $< t_2 (k_2)$, t_3 being arbitrary.

Putting

$$d_{j_1} \Theta \{ z_1 \} = \Theta \{ z_1 (j_1) \} - \Theta \{ z_1 (j_1) - dz_1 \},$$

the theoretical expression of this probability is

$$d_{j_1} \Theta \{ z_1 \} . \, \Theta \{ \zeta_{2;1} (j_1, k_2) \}, \quad \text{where} \quad \zeta_{2;1} (j_1, k_2) = \frac{- A_{21} . \, t_1 (j_1) + A_{11} . \, t_2 (k_2)}{\sqrt{A_{11}}}.$$

The least range however we actually can take for t_1 is the class-interval $\triangle t_1 = t_1 (k_1) - t_1 (k_1 - 1)$, so that the corresponding z_1 lies between $z_1 (k_1 - 1)$ and $z_1 (k_1)$. Thus we must operate with a *finite* difference:

$$\triangle_{k_1} \Theta \{ z_1 \} = \Theta \{ z_1 (k_1) \} - \Theta \{ z_1 (k_1 - 1) \}.$$

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[1]) The values of x_1 corresponding to $z_1 = -\infty$ and $z_1 = +\infty$ are essentially undetermined; they need not coincide with the extreme class-limits (see S.C. I a, Dutch text p. 793, English text p. 803).

Then it is necessary to take for the function $\zeta_{2;1}(j_1, k_2)$ a value, computed by substituting for $t_1(j_1)$ a certain *mean* value between $t_1(k_1-1)$ and $t_1(k_1)$. Denoting this mean value by $t_1(k_1-\frac{1}{2})$, we have:
Probability of $[t_1(k_1-1) < t_1 < t_1(k_1), t_2 < t_2(k_2), t_3$ arbitrary$]$:

$$\Delta_{k_1}\, \Theta\{z_1\} \,.\, \Theta\{\zeta_{2;1}(k_1-\tfrac{1}{2}, k_2)\}.$$

Considering now the probability of $[t_2 < t_2(k_2), t_3$ arbitrary, *being given* that $t_1(k_1-1) < t_1 < t_1(k_1)]$, we find for its *theoretical* value:

$$\sigma_{2;1}(k_1-\tfrac{1}{2}, k_2) = \Theta\{\zeta_{2;1}(k_1-\tfrac{1}{2}, k_2)\},$$

where
$$\zeta_{2;1}(k_1-\tfrac{1}{2}, k_2) = \frac{-\Delta_{21}\,.\,t_1(k_1-\frac{1}{2}) + \Delta_{11}\,.\,t_2(k_2)}{\sqrt{\Delta_{11}}}.$$

The *empirical* value of this very probability is found to be

$$\sigma_{2;1}(k_1-\tfrac{1}{2}, k_2) = \frac{\sum_{i_2=1}^{k_2}\sum_{i_3=1}^{\nu_3} y(k_1, i_2, i_3)}{\sum_{i_2=1}^{\nu_2}\sum_{i_3=1}^{\nu_3} y(k_1, i_2, i_3)} = \frac{\sum_{i_2=1}^{k_2}\sum_{i_3=1}^{\nu_3} Y(k_1, i_2, i_3)}{\sum_{i_2=1}^{\nu_2}\sum_{i_3=1}^{\nu_3} Y(k_1, i_2, i_3)}.$$

So we find, equalizing both expressions for the:
Probability of $[t_2 < t_2(k_2), t_3$ arbitr., *being given*: $t_1(k_1-1) < t_1 < t_1(k_1)]$:

$$\left.\begin{aligned} &\Theta\{\zeta_{2;1}(k_1-\tfrac{1}{2}, k_2)\} = \sigma_{2;1}(k_1-\tfrac{1}{2}, k_2) = \frac{\sum_{i_2=1}^{k_2}\sum_{i_3=1}^{\nu_3} Y(k_1, i_2, i_3)}{\sum_{i_2=1}^{\nu_2}\sum_{i_3=1}^{\nu_3} Y(k_1, i_2, i_3)}, \\ &\text{where}\quad \zeta_{2;1}(k_1-\tfrac{1}{2}, k_2) = \frac{-\Delta_{21}\,.\,t_1(k_1-\frac{1}{2}) + \Delta_{11}\,.\,t_2(k_2)}{\sqrt{\Delta_{11}}}. \end{aligned}\right\}\quad \mathbf{B}(2;1)$$

If we succeed in determining exactly the mean values $t_1(k_1-\frac{1}{2})$ $(k_1 = 2, \ldots, \nu_1-1)$, it must appear — provided that the given frequency distribution be in accordance with the probability formula **4 (5, 6)** —, that the values of $\zeta_{2;1}(k_1-\frac{1}{2}, k_2)$ computed from **B**(2;1) are linearly dependent from the corresponding values $t_1(k_1-\frac{1}{2}), t_2(k_2)$. In this case we can calculate the required values of λ_{ab} and γ_{ab} from the coefficients of the linear functions $z_1 = a_1\, t_1,\ \zeta_{2;1} = \alpha_1\, t_1 + \alpha_2\, t_2$. However we must, before making this calculation, ascertain whether the variables t_a are really unimodular. To test this we have to consider the function Z_3.

To construct the function Z_3 empirically, we should be able to give — also in the empirical treatment — an infinitesimal range, not only to t_1, but also to t_2. Then it would be possible to determine the empirical probability of; $[t_1 = t_1(j_1)$ (range dt_1), $t_2 = t_2(j_2)$ (range dt_2), $t_3 < t_3(k_3)]$.
Putting

$$d_{j_1 j_2}\, \Theta\{\zeta_{2;1}\} = \Theta\{\zeta_{2;1}(j_1, j_2)\} - \Theta\{\zeta_{2;1}(j_1, j_2) - d\zeta_{2;1}\},$$

this probability is theoretically expressed by

$$d_{j_1}\, \Theta\{z_1\} \,.\, d_{j_1 j_2}\, \Theta\{\zeta_{2;1}\} \,.\, \Theta\{Z_3(j_1, j_2, k_3)\}$$

where
$$Z_3(j_1, j_2, k_3) = \lambda_{31}\,.\,t_1(j_1) + \lambda_{23}\,.\,t_2(j_2) + t_3(k_3).$$

Actually we must operate with the *finite* ranges $\triangle t_1 = t_1(k_1) - t_1(k_1-1)$, $\triangle t_2 = t_2(k_2) - t_2(k_2-1)$ and with the corresponding finite ranges $\triangle z_1$, $\triangle \zeta_{2;1}$.

So we have (besides $\triangle_{k_1} \Theta\{z_1\}$) to consider:

$$\triangle_{k_1 k_2} \Theta\{\zeta_{2;1}\} = \Theta\{\zeta_{2;1}(k_1 - \tfrac{1}{2}, k_2)\} - \Theta\{\zeta_{2;1}(k_1 - \tfrac{1}{2}, k_2 - 1)\}.$$

We must therefore take a value of the function $Z_3(j_1, j_2, k_3)$, which is computed by substituting for t_1 the value $t_1(k_1 - \tfrac{1}{2})$ (mentioned already above), and for t_2 a mean value $t_2(k_2 - \tfrac{1}{2})$ between $t_2(k_2-1)$ and $t_2(k_2)$. So we obtain for the

Probability of:

$$[t_1(k_1 - 1) < t_1 < t_1(k_1), \quad t_2(k_2 - 1) < t_2 < t_2(k_2), \quad t_3 < t_3(k_3)]:$$

$$\triangle_{k_1} \Theta\{z_1\} \,.\, \triangle_{k_1 k_2} \Theta\{\zeta_{2;1}\} \,.\, \Theta\{Z_3(k_1 - \tfrac{1}{2}, k_2 - \tfrac{1}{2}, k_3)\}.$$

Hence the probability of [$t_3 < t_3(k_3)$, *being given*: $t_1(k_1-1) < t_1 < t_1(k_1)$, $t_2(k_2-1) < t_2 < t_2(k_2)$] has for its *theoretical* value:

$$S_3(k_1 - \tfrac{1}{2}, k_2 - \tfrac{1}{2}, k_3) = \Theta\{Z_3(k_1 - \tfrac{1}{2}, k_2 - \tfrac{1}{2}, k_3)\},$$

where $\quad Z_3(k_1 - \tfrac{1}{2}, k_2 - \tfrac{1}{2}, k_3) = \lambda_{31} \,.\, t_1(k_1 - \tfrac{1}{2}) + \lambda_{23} \,.\, t_2(k_2 - \tfrac{1}{2}) + t_3(k_3)$.

Its *empirical* value is found to be

$$S_3(k_1 - \tfrac{1}{2}, k_2 - \tfrac{1}{2}, k_3) = \frac{\sum\limits_{i_3=1}^{k_3} y(k_1, k_2, i_3)}{\sum\limits_{i_3=1}^{\nu_3} y(k_1, k_2, i_3)} = \frac{\sum\limits_{i_3=1}^{k_3} Y(k_1, k_2, i_3)}{\sum\limits_{i_3=1}^{\nu_3} Y(k_1, k_2, i_3)}.$$

By equalizing both expressions for S_3, we have for the

Probability of:

[$t_3 < t_3(k_3)$, *being given*: $t_1(k_1-1) < t_1 < t_1(k_1)$, $t_2(k_2-1) < t_2 < t_2(k_2)$]:

$$\left.\begin{array}{l} \Theta\{Z_3(k_1 - \tfrac{1}{2}, k_2 - \tfrac{1}{2}, k_3)\} = S_3(k_1 - \tfrac{1}{2}, k_2 - \tfrac{1}{2}, k_3) = \dfrac{\sum\limits_{i_3=1}^{k_3} Y(k_1, k_2, i_3)}{\sum\limits_{i_3=1}^{\nu_3} Y(k_1, k_2, i_3)}, \\ \text{where } Z_3(k_1 - \tfrac{1}{2}, k_2 - \tfrac{1}{2}, k_3) = \lambda_{31} \,.\, t_1(k_1 - \tfrac{1}{2}) + \lambda_{23} \,.\, t_2(k_2 - \tfrac{1}{2}) + t_3(k_3) \end{array}\right\} \mathbf{B}(3;21)$$

If we succeed in determining exactly the mean values $t_1(k_1 - \tfrac{1}{2})$ and $t_2(k_2 - \tfrac{1}{2})$, then it must appear — provided that the given frequency distribution be in accordance with the probability formula **4** (**5**, **6**) — that the values of $Z_3(k_1 - \tfrac{1}{2}, k_2 - \tfrac{1}{2}, k_3)$ computed from **B**(**3**; **21**) are linearly connected with the corresponding values $t_1(k_1 - \tfrac{1}{2})$, $t_2(k_2 - \tfrac{1}{2})$, $t_3(k_3)$.

If t_1, t_2, t_3 are really unimodular, then the linear relation $Z_3 = A_1 t_1 + A_2 t_2 + A_3 t_3$ must give: $A_3 = 1$, A_1 and A_2 equal to the values λ_{31} and λ_{23} already calculated from the coefficients a_1, α_1, α_2.

In the preceding analysis we have chosen the arrangement $z_1, \zeta_{2;1}, Z_3$; that is to say: we have first left t_2 and t_3 arbitrary, then only t_3 (and at last none of the t_a). We may however just as well leave arbitrary:

first t_1 and t_3, then only t_3. This arrangement furnishes us the new functions z_2 and $\zeta_{1;2}$, Z_3 remaining the same.

z_2 is, as a function of t_2, determined by associating $z_2(k_2)$ with $t_2(k_2)$ according to

$$\Theta\{z_2(k_2)\} = s_2(k_2) = \frac{\sum\limits_{i_1=1}^{\nu_1} \sum\limits_{i_2=1}^{k_2} \sum\limits_{i_3=1}^{\nu_3} Y(i_1, i_2, i_3)}{N} \quad . \quad . \quad . \quad . \quad \mathbf{B}(2)$$

$\zeta_{1;2}$ is, as a function of t_1 and t_2, determined empirically by associating $\zeta_{1;2}(k_1, k_2 - \frac{1}{2})$ with $t_1(k_1)$, $t_2(k_2 - \frac{1}{2})$ in virtue of the relation

$$\Theta\{\zeta_{1;2}(k_1, k_2 - \tfrac{1}{2})\} = \sigma_{1;2}(k_1, k_2 - \tfrac{1}{2}) = \frac{\sum\limits_{i_1=1}^{k_1} \sum\limits_{i_3=1}^{\nu_3} Y(i_1, k_2, i_3)}{\sum\limits_{i_1=1}^{\nu_1} \sum\limits_{i_3=1}^{\nu_3} Y(i_1, k_2, i_3)} \quad . \quad . \quad . \quad . \quad \mathbf{B}(1;2)$$

Provided the mean values asked for be determined in the right way, we shall find, between z_2, $\zeta_{1;2}$, Z_3 on the one hand and t_1, t_2, t_3 on the other, the relations

$$\left.\begin{aligned} z_2 &= \left(\frac{\Delta}{\Delta_{22}}\right)^{1/2} . t_2, \\ \zeta_{1;2} &= \frac{\Delta_{22} . t_1 - \Delta_{12} . t_2}{\sqrt{\Delta_{22}}}, \\ Z_3 &= \lambda_{31} . t_1 + \lambda_{23} . t_2 + t_3, \end{aligned}\right\} \quad . \quad . \quad . \quad \mathbf{9ter} \left\{\begin{aligned} &(2) \\ &(1;2) \\ &(3;12) \end{aligned}\right.$$

which evidently must furnish the same values of λ_{ab} and γ_{ab} as before. Putting

$$z_1^2 + \zeta_{2;1}^2 = q_{2;1}^2 \quad , \quad z_2^2 + \zeta_{1;2}^2 = q_{1;2}^2 \, , \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathbf{26}$$

we must find

$$q_{2;1} = q_{1;2} \, (= q_{12}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathbf{27}$$

since both $q_{2;1}$ and $q_{1;2}$ must represent the projection OH of OP on the plane $\Phi_2 O \Phi_3$.

From **9bis (1)**, **9bis (2; 1)** we derive for the common value q_{12}^2:

$$q_{12}^2 = \sin^2\varphi_2 . t_1^2 - 2\cos\omega_3 \sin\varphi_1 \sin\varphi_2 . t_1 t_2 + \sin^2\varphi_1 . t_2^2.$$

Moreover we find from **9bis (1)**, **9bis (2; 1)** and from the corresponding equations **9bis (2)**, **9bis (1; 2)**:

$$z_1 z_2 - \zeta_{2;1}\zeta_{1;2} = \cos\omega_3 . q_{12}^2 \quad , \quad z_1\zeta_{1;2} + z_2\zeta_{2;1} = \sin\omega_3 . q_{12}^2 \, , \quad . \quad . \quad \mathbf{28}$$

whence

$$\operatorname{tg}\omega_3 = \frac{z_1\zeta_{1;2} + z_2\zeta_{2;1}}{z_1 z_2 - \zeta_{2;1}\zeta_{1;2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathbf{29}$$

The equations **28** and, in particular, the equation **29**, which is independent of the concordance between $q_{1;2}$ and $q_{2;1}$, immediately furnish ω_3, hence also the (total) coefficient of correlation $\gamma_{12} = \cos\omega_3$.

For OP^2 we obtain two expressions, viz.:

$$r^2_{2;1} = z^2_1 + \zeta^2_{2;1} + Z^2_3 = q^2_{2;1} + Z^2_3 \text{ and } r^2_{1;2} = z^2_2 + \zeta^2_{1;2} + Z^2_3 = q^2_{1;2} + Z^2_3 \quad \mathbf{30(2;1)(1;2)}$$

which turn out to be equal, if **27** is satisfied.

If we had operated with the arrangement $z_1, \zeta_{3;1}, Z_2$, we should have obtained for OP^2 a new expression, viz.:

$$r^2_{3,1} = z^2_1 + \zeta^2_{3;1} + Z^2_2 = q^2_{3;1} + Z^2_2, \quad . \quad . \quad . \quad . \quad \mathbf{30(3;1)}$$

the value of which should be equal to the values furnished by **30(2;1)(1;2)**. For the magnitudes $q_{b;a}$, determined by

$$z^2_a + \zeta^2_{b;a} = q^2_{b;a} \quad , \quad (a, b = 1, 2, 3) \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathbf{C}$$

we have therefore together the three controlling equations:

$$q_{a;b} = q_{b;a} \; (= q_{ab}) \quad (a, b = 1, 2, 3) \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathbf{Ia}$$

Putting

$$z_a \, z_b - \zeta_{b;a} \, \zeta_{a;b} = A_{ab} \quad , \quad z_a \, \zeta_{a;b} + z_b \, \zeta_{b;a} = B_{ab}, \quad . \quad . \quad . \quad \mathbf{D}$$

we have, analogous to **29**,

$$\operatorname{tg} \omega_c = \frac{B_{ab}}{A_{ab}}.$$

Leaving it unsettled whether $q_{a:b}$ is equal to $q_{b:a}$ or not, yet we have:

$$\cos \omega_c = \frac{A_{ab}}{\sqrt{(A^2_{ab} + B^2_{ab})}} \quad , \quad \sin \omega_c = \frac{B_{ab}}{\sqrt{(A^2_{ab} + B^2_{ab})}}.$$

Now

$$A^2_{ab} + B^2_{ab} = (z_a \, z_b - \zeta_{b;a} \, \zeta_{a;b})^2 + (z_a \, \zeta_{a:b} + z_b \, \zeta_{b:a})^2 =$$

$$= (z^2_a + \zeta^2_{b;a})(z^2_b + \zeta^2_{a;b}) = q^2_{b;a} \cdot q^2_{a;b}.$$

Hence we have — no matter whether $q_{b;a} = q_{a:b}$ is satisfied or not —

$$\left.\begin{aligned} \gamma_{ab} = \cos \omega_c = \frac{A_{ab}}{q_{b;a} \cdot q_{a;b}} \quad , \quad \sqrt{1-\gamma^2_{ab}} = \sin \omega_c = \frac{B_{ab}}{q_{b;a} \cdot q_{a;b}}, \\ \frac{\sqrt{1-\gamma^2_{ab}}}{\gamma_{ab}} = \operatorname{tg} \omega_c = \frac{B_{ab}}{A_{ab}} \quad \left(\frac{\cos \omega_c}{A_{ab}} > 0\right). \end{aligned}\right\} \quad . \quad . \quad \mathbf{E}$$

Putting

$$q_{b;a} \cdot q_{a;b} = Q^2_{ab}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathbf{C^{bis}}$$

we may also write:

$$\left.\begin{aligned} \gamma_{ab} = \cos \omega_c = \frac{A_{ab}}{Q^2_{ab}}, \quad \sqrt{1-\gamma^2_{ab}} = \sin \omega_c = \frac{B_{ab}}{Q^2_{ab}}, \\ \frac{\sqrt{1-\gamma^2_{ab}}}{\gamma_{ab}} = \operatorname{tg} \omega_c = \frac{B_{ab}}{A_{ab}} \quad \left(\frac{\cos \omega_c}{A_{ab}} > 0\right) \end{aligned}\right\} \quad . \quad . \quad . \quad \mathbf{E^{bis}}$$

In the case that $q_{b:a} = q_{a:b} \; (= q_{ab})$ is really satisfied, we have of course

$$Q_{ab} = q_{ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathbf{Ia^{bis}}$$

At present we can put $r^2 = OP^2$ into a form, which is entirely built up of the functions $z_a, \zeta_{b:a}$ $(a. b = 1, 2, 3)$.

From **5**, **7**, **11**, **12bis**, **13bis** ensues:

$$f = r^2 = \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \lambda_{\alpha\beta} t_\alpha t_\beta = t_1^2 + t_2^2 + t_3^2 + 2\cos\varphi_1 . t_2 t_3 + 2\cos\varphi_2 . t_3 t_1 + 2\cos\varphi_3 . t_1 t_2$$

$$= \sum_{\alpha=1}^{3} \frac{z_\alpha^2}{\sin^2\psi_\alpha} + 2\sum_{\alpha=1}^{3} \cos\varphi_\alpha \frac{z_\beta z_\gamma}{\sin\psi_\beta \sin\psi_\gamma} =$$

$$= \frac{\sum_{\alpha=1}^{3} \sin^2\omega_\alpha . z_\alpha^2 + 2\sum_{\alpha=1}^{3} \sin\omega_\beta \sin\omega_\gamma \cos\varphi_\alpha . z_\beta z_\gamma}{\Gamma}$$

or

$$f = \frac{\sum_{\alpha=1}^{3} \sin^2\omega_\alpha . z_\alpha^2 + 2\sum_{\alpha=1}^{3} (\cos\omega_\beta \cos\omega_\gamma - \cos\omega_\alpha) z_\beta z_\gamma}{1 - \cos^2\omega_1 - \cos^2\omega_2 - \cos^2\omega_3 + 2\cos\omega_1 \cos\omega_2 \cos\omega_3} \quad . \quad \mathbf{31}$$

Substituting, in the denominator Γ, for $\cos\omega_1$, $\cos\omega_2$, $\cos\omega_3$ the expressions furnished by **Ebis**, we obtain:

$$\Gamma = 1 - \frac{A_{23}^2}{Q_{23}^4} - \frac{A_{31}^2}{Q_{31}^4} - \frac{A_{12}^2}{Q_{12}^4} + 2\frac{A_{23} A_{31} A_{12}}{Q_{23}^2 Q_{31}^2 Q_{12}^2}$$

$$= \frac{Q_{23}^4 Q_{31}^4 Q_{12}^4 - \sum_{\alpha=1}^{3} A_{\beta\gamma}^2 Q_{\gamma\alpha}^4 Q_{\alpha\beta}^4 + 2 A_{23} A_{31} A_{12} Q_{23}^2 Q_{31}^2 Q_{12}^2}{Q_{23}^4 Q_{31}^4 Q_{12}^4},$$

or, putting:

$$Q_{23}^4 Q_{31}^4 Q_{12}^4 - \sum_{\alpha=1}^{3} A_{\beta\gamma}^2 Q_{\gamma\alpha}^4 Q_{\alpha\beta}^4 + 2 A_{23} A_{31} A_{12} Q_{23}^2 Q_{31}^2 Q_{12}^2 \equiv F\{z, \zeta\}, \quad \mathbf{F}$$

the abbreviated form:

$$\Gamma = \frac{F\{z, \zeta\}}{Q_{23}^4 Q_{31}^4 Q_{12}^4}.$$

Likewise we find for the numerator of r^2:

$$\sum_{\alpha=1}^{3} \frac{B_{\beta\gamma}^2}{Q_{\beta\gamma}^4} . z_\alpha^2 + \sum_{\alpha=1}^{3} \left(\frac{A_{\gamma\alpha}}{Q_{\gamma\alpha}^2} \frac{A_{\alpha\beta}}{Q_{\alpha\beta}^2} - \frac{A_{\beta\gamma}}{Q_{\beta\gamma}^2} \right) z_\beta z_\gamma =$$

$$= \frac{\sum_{\alpha=1}^{3} B_{\beta\gamma}^2 Q_{\gamma\alpha}^4 Q_{\alpha\beta}^4 . z_\alpha^2 + Q_{23}^2 Q_{31}^2 Q_{12}^2 . \sum_{\alpha=1}^{3} (A_{\gamma\alpha} A_{\alpha\beta} Q_{\beta\gamma}^2 - A_{\beta\gamma} Q_{\gamma\alpha}^2 Q_{\alpha\beta}^2) z_\beta z_\gamma}{Q_{23}^4 Q_{31}^4 Q_{12}^4},$$

or, putting:

$$\left.\begin{aligned} &\sum_{\alpha=1}^{3} B_{\beta\gamma}^2 Q_{\gamma\alpha}^4 Q_{\alpha\beta}^4 . z_\alpha^2 + \\ &\quad + 2 Q_{23}^2 Q_{31}^2 Q_{12}^2 \sum_{\alpha=1}^{3} (A_{\gamma\alpha} A_{\alpha\beta} Q_{\beta\gamma}^2 - A_{\beta\gamma} Q_{\gamma\alpha}^2 Q_{\alpha\beta}^2) z_\beta z_\gamma \equiv G\{z, \zeta\}. \end{aligned}\right\} \quad \mathbf{G}$$

the abbreviated form:

$$\text{numerator} = \frac{G\{z, \zeta\}}{Q_{23}^4\, Q_{31}^4\, Q_{12}^4}.$$

The forms F and G are entirely built up of the functions z_a, $\zeta_{a:b}$ $(a, b = 1, 2, 3)$.

Thus we find for $OP^2 = r^2$:

$$r^2 = \frac{G\{z, \zeta\}}{F\{z, \zeta\}} = H\{z, \zeta\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad \mathbf{H}$$

If the conditions **Ia** are fulfilled (whence $Q_{ab} = q_{ab}$, $a, b = 1, 2, 3$), the functions Z_1, Z_2, Z_3 must satisfy

$$Z_1^2 = H\{z, \zeta\} - q_{23}^2, \quad Z_2^2 = H\{z, \zeta\} - q_{31}^2, \quad Z_3^2 = H\{z, \zeta\} - q_{12}^2 \quad \mathbf{Ib}$$

(*To be continued*).