Hydrodynamics. — *On the application of statistical mechanics to the theory of turbulent fluid motion.* III. [1]) By J. M. BURGERS. (Mededeeling N°. 12 uit het laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft). (Communicated by Prof. P. EHRENFEST).

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8. *Deduction of a new form for the dissipation condition.*

In Part II we have devoted much attention to a relation that was deduced from the dissipation condition and led to certain results concerning the resistance coefficient and the distribution of the velocity of the mean motion. As the distribution obtained differed rather from that observed experimentally, we tried to penetrate into the physical meaning of this condition; in doing so we struck upon the circumstance that, as the mean motion in our equations was defined as a mean with respect to time, a variation effected in the intensity of the relative motion, applied during a small interval, brought about a variation of the mean motion over the whole of the time considered. The latter variation in its turn influenced the transmission of energy from the mean motion to the relative motion — again during the whole of the time considered, a result which seemed rather difficult to understand from a physical point of view.

As has been mentioned at the end of § 6, this difficulty — though, as will be shown in § 11, it is more an apparent than a real one — can be obviated to a certain extent by defining the mean motion as a mean with respect to x. In the normal state of turbulent motion we may safely expect that this will lead to the same result as the use of mean values with respect to time, and indeed in various investigations relating to turbulent motion space means are used instead of or along with time means. The reason that in our former considerations we adhered to mean values defined with respect to time, was simply that in the case of a stationary mean motion this seemed to be the most natural procedure.

In order to develop a system of formulae on the basis of mean values defined with respect to x, we start afresh from the real motion. Quantities relating to the latter shall be distinguished by the index r: u_r, v_r, ζ_r, p_r. Then we define the mean velocity U and the mean pressure P by the equations:

$$U=\frac{1}{L}\int u_r\,dx\,, \qquad \frac{\partial P}{\partial x}=\frac{1}{L}\int\frac{\partial p_r}{\partial x}\,dx \quad . \quad . \quad . \quad . \quad (51^a)$$

[1]) Parts I, II have appeared in these Proceedings **32**, p. 414, 643, 1929.

The integrations with respect to x are extended from $x = x_0$ to $x = x_0 + L$, where L is a length which may be taken arbitrarily great.

Neglecting quantities of the order L^{-1}, we further have:

$$\frac{1}{L}\int v_r \, dx = 0, \qquad \frac{1}{L}\int \zeta_r \, dx = \frac{dU}{dy}, \text{ etc.} \quad . \quad . \quad . \quad (51^b)$$

Moreover in order to ensure the stationary character of the mean motion, and to be able to fix the value of the mean pressure gradient, we shall suppose:

$$\frac{1}{L}\int \frac{\partial u_r}{\partial t} \, dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51^c)$$

Then the relative motion is defined by means of the formulae:

$$u = u_r - U, \qquad v = v_r, \qquad \zeta = \zeta_r - \frac{dU}{dy}, \qquad p = p_r - P. \quad (52)$$

Now from the equation for the x-component of the real motion (expressed in non-dimensional variables):

$$\frac{\partial u_r}{\partial t} + \frac{\partial}{\partial x} u_r^2 + \frac{\partial}{\partial y} u_r v_r = -\frac{\partial p_r}{\partial x} + R^{-1} \triangle u_r \quad . \quad . \quad . \quad (53)$$

we deduce (by integrating it with respect to x and dividing it by L):

$$R^{-1}\frac{d^2 U}{dy^2} = -\frac{P_I - P_{II}}{L} + \frac{1}{L}\int dx \frac{\partial}{\partial y} u_r v_r .$$

In the second term on the right hand side of this equation we write $u_r v_r = Uv + uv$; then the term $\frac{1}{L}\int dx \frac{\partial}{\partial y}(Uv) = \frac{d}{dy}\left(\frac{U}{L}\int v \, dx\right)$ may be omitted on account of (51^b); in this way we get:

$$R^{-1}\frac{d^2 U}{dy^2} = -\frac{P_I - P_{II}}{L} + \frac{1}{L}\frac{d}{dy}\int dx \, uv . \quad . \quad . \quad . \quad . \quad (54)$$

We consider again the case of the motion between two fixed parallel walls. As we may deduce from the equation for the y-component of the real motion, $\frac{P_I - P_{II}}{L}$ can be treated as independent of y; hence from (54) we obtain:

$$R^{-1}\frac{dU}{dy} = -\frac{(P_I - P_{II})\, y}{L} + \frac{1}{L}\int dx \, uv \quad . \quad . \quad . \quad . \quad (54^a)$$

We multiply this equation by y and integrate it over the breadth of the channel (i. e. from $y = -\frac{1}{2}$ to $y = +\frac{1}{2}$); then we get:

$$\frac{P_I - P_{II}}{L} = \frac{12}{L}\iint dx \, dy \, y \, uv + 12 R^{-1}.$$

Substitution of this result into the former equation leads to:

$$\frac{dU}{dy} = -12\,y - \frac{12\,R\,y}{L}\iint dx\,dy\,y\,uv + \frac{R}{L}\int dx\,uv \quad . \quad . \quad (55)$$

Now the equation for the rate of increase of the energy of the real motion can be written (with unsignificant neglections):

$$E = \int (p_r\,u_r)_I\,dy - \int (p_r\,u_r)_{II}\,dy - R^{-1}\iint dx\,dy\,\zeta_r^2 \quad . \quad . \quad . \quad (56)$$

As $\int u_r\,dy = 1$, further $\zeta_r = \frac{dU}{dy} + \zeta$, and $\frac{1}{L}\int \zeta\,dx = 0$, we may transform this expression into:

$$E = (P_I - P_{II}) - R^{-1}\,L\int dy\left(\frac{dU}{dy}\right)^2 - R^{-1}\iint dx\,dy\,\zeta^2 \quad . \quad (56^a)$$

in which terms that do not become of the order L have been neglected.

Substituting now the values of $(P_I - P_{II})$ and of dU/dy as given above, we obtain:

$$E = \left.\begin{aligned} & -\frac{R}{L}\int dy\left(\int dx\,uv\right)^2 + \frac{12R}{L}\left(\iint dx\,dy\,y\,uv\right)^2 + \\ & + 12\iint dx\,dy\,y\,uv - R^{-1}\iint dx\,dy\,\zeta^2 \end{aligned}\right\} \quad . \quad (57)$$

Until now we have restricted ourselves to the consideration of the motion at a given instant of time. Henceforth we shall introduce the supposition that *the mean value of E with respect to time must be zero.* This will give us the new form of the dissipation condition.

In order to put this supposition into a form which can serve as a basis for a statistical treatment, we use the same scheme as has been applied in § 2, Part I (l.c. p. 417 seqq.). We represent the various types of relative motion, occurring in the sequence constituting the normal state of turbulent motion, by means of points in the ξ-space. The numbers of the points in the cells of equal volume ω in which the ξ-space is divided, are denoted by $n_1 = \nu_1\,M$, $n_2 = \nu_2\,M, \ldots$, so that $\Sigma\,\nu = 1$ (comp. eq. 9). Then, making use of (10), the dissipation condition can be written:

$$\Sigma\,\nu\,E = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

In order to compare this new form with that given in § 4, equation (31), we again introduce the abbrevations (3), and express integrations over the x, y-plane by summations with respect to the index $k = k_1 + ik_2$. When we put:

$$F^* \equiv -R^{-1}\,\Sigma\,\nu\,E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58^a)$$

we obtain:

$$\left.\begin{aligned}\varepsilon^{-2} F^{\star} \equiv \frac{\varepsilon}{L} \sum_{\xi} \nu \sum_{k_2} (\sum_{k_1} t_k)^2 - \frac{12\,\varepsilon^2}{L} \sum_{\xi} \nu (\sum_{k} y_k\, t_k)^2 + \\ + \frac{12}{R} \sum_{\xi} \sum_{k} \nu\, y_k\, t_k + \frac{1}{R^2} \sum_{\xi} \sum_{k} \nu\, z_k = 0\end{aligned}\right\} \quad . \quad (59)$$

It will be seen that the first two terms of this expression differ from the corresponding terms occurring in (31). In the normal state we may expect that for the majority of the various relative motions presenting themselves in course of time, mean values with respect to x will be equal to the corresponding mean values with respect to time, so that:

$$\frac{\varepsilon}{L} \sum_{k_1} t_k \doteq \sum_{\xi} \nu\, t_k = \bar{t}_k \; .$$

Then (59) will be equal to (31).

The variation of $F^{\star}$ which is caused by an arbitrary variation of one of the ν's, however, is different in the two cases. On account of the new formula (58^a) we obtain:

$$\delta F^{\star} = - R^{-1}\, E\, \delta \nu.$$

Hence when we retain the probability hypothesis formulated in § 2, the "most probable distribution" of representative points now becomes:

$$\nu = A\, e^{\beta E/L} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (60)$$

Here for the parameter we have written β/L instead of β, in order to simplify some of the formulae occurring in further deductions. (It will be seen that E/L measures the mean rate of increase of energy per unit length of the channel).

Formula (60) has a more simple structure than the one obtained formerly (equation 21), in as much as it is not an implicit equation: for every given function ψ (i.e. for every given point of the ξ-space) the value of E is at once wholly determined by (57). When we put:

$$Z = \sum e^{\beta E/L} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (61)$$

(where the summation is extended over the whole of the ξ-space), we have:

$$A = Z^{-1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (61^a)$$

The value of β is determined by the condition (58); it is easily to be seen that this condition may be written:

$$\frac{\partial Z}{\partial \beta} = 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (62)$$

When β has been found all mean values can be calculated; so for instance we get for the resistance coefficient (compare the definition of C in § 4):

$$C = \frac{P_I - P_{II}}{2\,L} = \frac{6}{L} \, \frac{\sum \left(e^{\beta E/L} \iint dx\, dy\, y\, uv \right)}{Z} + \frac{6}{R} \quad . \; . \quad (63)$$

It may be of interest to remark that equation (60) can be obtained also by FOWLER's method of calculating averages by means of the introduction of a partition function. [1]) This avoids the use of the conception of a "most probable distribution". We then have to reason as follows: We have assumed that the course of the turbulent motion during a given lapse of time T can be described "microscopically" by giving the states of the field at M moments, separated by equal intervals T/M. In such a sequence of instantaneous states the various fields of motion represented by the points of the ξ-space (or rather by the centra of the cells in which this space is divided) occur resp. $n_1, n_2, n_3 \dots$ times, where $\Sigma\, n = M$ (summation extended over all cells of the ξ-space). For a "macroscopic" observer the set of numbers $n_1, n_2, n_3 \dots$ only is of importance; not the various ways in which the individual fields may be arranged. When each possible sequence is counted for one, then there are

$$W = \frac{M!}{n_1!\, n_2!\, n_3! \dots}$$

differently arranged sequences corresponding to a given set of numbers $n_1, n_2, n_3, \dots$ (comp. eq. 14). Hence in the assembly of all possible sequences the average value of any one of the n's, say of n_j, is given by:

$$\bar{n}_j = \frac{\Sigma^\star\, W\, n_j}{\Sigma^\star\, W},$$

where the summation $\Sigma^\star$ is to be extended over all values of the n's, consistent with the relations:

$$\sum_j n_j = M, \qquad \sum_j n_j\, E_j = 0.$$

Making use now of FOWLER's methods we can easily show that the average value $\bar{n}_j$ is equal to the value given by the system of equations (60)—(62). Summarizing in a few lines: we introduce the partition function:

$$f(s) = \sum_j s^{E_j},$$

s being a complex variable; then we have:

$$\Sigma^\star\, W = \frac{1}{2\pi i}\int_\gamma \frac{ds}{s}\,[f(s)]^M, \quad \Sigma^\star\, W n_j = \frac{M}{2\pi i}\int_\gamma \frac{ds}{s}\, s^{E_j}\,[f(s)]^{M-1},$$

where γ denotes a closed contour circulating counter-clockwise round $s = 0$. Application of the method of "steepest descent" finally gives:

$$\bar{n}_j = M\,\frac{\vartheta^{E_j}}{f(\vartheta)}.$$

[1]) Comp. C. G. DARWIN and R. H. FOWLER, Phil. Mag. (6) **44**, p. 450, 1922; R. H. FOWLER, Statistical Mechanics (Cambridge 1929), p. 22 seqq.

ϑ being the root of $df/ds=0$. When s is replaced by $e^{\beta/L}$, $\bar{n}_j$ by $M\nu_j$, we immediately see that this result is equivalent to the one formerly obtained.

All mean values may be defined in the same way, as for any quantity X we may write, starting from equation (10):

$$\bar{X}=\Sigma\frac{\bar{n}_j}{M}X_j=\sum_j\left(\frac{\Sigma^\star W\,n_j/M}{\Sigma^\star W}\right)X_j=\frac{\Sigma^\star W(\sum_j n_j\,X_j/M)}{\Sigma^\star W}.$$

This formula, however, is practical only when the quantity $\sum_j n_j\,X_j$ can be calculated easily.

Another point that may be noticed relates to the evaluation of the mean value occurring in (63). The following formal device may be used for this purpose. We write:

$$12\iint dx\,dy\,y\,uv=E'\,L,\qquad R^{-1}\iint dx\,dy\,\zeta^2=E''\,L,$$

$$\int dy\left(\int dx\,uv\right)^2-12\left(\iint dx\,dy\,y\,uv\right)^2=R^{-1}\,E'''\,L^2,$$

so that: $E=L\,(E'-E''-E''')$. Then we put:

$$Z=\Sigma\,e^{\beta' E'-\beta'' E''-\beta''' E'''}\quad\ldots\ldots\ldots\quad(61^a)$$

and replace (62) by the system of equations:

$$\frac{\partial Z}{\partial\beta'}+\frac{\partial Z}{\partial\beta''}+\frac{\partial Z}{\partial\beta'''}=0,\qquad\beta'=\beta''=\beta'''\quad\ldots\ldots\quad(62^a)$$

In that case we have:

$$\frac{12}{L}\frac{\Sigma\left(e^{\beta E/L}\iint dx\,dy\,y\,uv\right)}{Z}=\frac{1}{Z}\frac{\partial Z}{\partial\beta'}=\frac{\partial}{\partial\beta'}(lg\,Z)\quad\ldots\quad(63^a)$$

In the deductions of the following §, however, we shall adhere to the expression (61) and to formula (62), containing one parameter β only, though the extension to the case of more β's is not difficult.

9. *Investigation of equation* (62).

The first question which now presents itself is whether the sum occurring in (61) converges. According to a lemma called after SCHWARZ we have:

$$\int dy\,y^2\,.\int dy\left(\int dx\,uv\right)^2>\left\{\int dy\left(y\int dx\,uv\right)\right\}^2,$$

or:

$$\frac{1}{12}\int dy\left(\int dx\,uv\right)^2>\left(\iint dx\,dy\,y\,uv\right)^2.$$

Hence the terms of E that are of the fourth degree relatively to ψ

are definite negative, and so E certainly will become negative when in the ξ-space we go to infinity in an arbitrary direction. [1])

In order to get some insight into the general character of the distribution of points, determined by (60), we introduce polar coordinates into the ξ-space, and begin with the consideration of a cone of infinitely small aperture $d\chi$ stretching out radially from the origin. The points within such a cone represent similar fields of motion. Hence putting:

$$\varrho^2 = \frac{1}{L}\iint dx\, dy\, \psi^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

we may write:

$$\frac{12}{L}\iint dx\, dy\, y\, uv - \frac{1}{RL}\iint dx\, dy\, \zeta^2 = K\varrho^2 \quad . \quad . \quad . \quad (65^a)$$

$$\frac{R}{L^2}\int dy\left(\int dx\, uv\right)^2 - \frac{12\,R}{L^2}\left(\iint dx\, dy\, y\, uv\right)^2 = H\varrho^4 \quad . \quad . \quad (65^b)$$

Then K and H have the same values for all points within our cone; these values moreover are independent of L and of N. In consequence of the lemma mentioned above H always is positive; K on the contrary will be negative for some directions in the ξ-space, positive for others. The value of E/L now becomes:

$$E/L = K\varrho^2 - H\varrho^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

The distance r of any point of the ξ-space from the origin is given by:

$$r^2 = \Sigma\, \xi_k^2 = \varepsilon^{-2}\iint dx\, dy\, \psi^2 = \varepsilon^{-2} L\varrho^2 = N\varrho^2 \quad . \quad . \quad . \quad (67)$$

(It has to be reminded that the number of points in the lattice, N, can be obtained by dividing the area of the channel by ε^2. As the breadth of the channel is unity, and the length considered is L, we have $N = \varepsilon^{-2} L$). Now the volume of the element of the cone contained between "spherical surfaces" with radii r and $r + dr$ is equal to $r^{N-1}\, dr\, d\chi = N^{N/2}\varrho^{N-1}\, d\varrho\, d\chi$. Consequently equation (62) may be written:

$$\frac{\partial Z}{\partial \beta} \equiv \frac{\partial}{\partial \beta}\int d\chi \int_0^\infty d\varrho\, \varrho^{N-1}\, e^{\beta(K\varrho^2 - H\varrho^4)} = 0 \quad . \quad . \quad . \quad . \quad (68)$$

The constant factor $N^{N/2}$ has been omitted, as this factor has no influence on the distribution. The integration with respect to $d\chi$ has to be extended over all possible directions of the ξ-space.

Instead of this equation, however, we shall provisionally consider an approximate form, in which the integration with respect to $d\chi$ is not

[1]) The inequality could become an equality only when $\int dx\, uv$ considered as a function of y should be proportional to y. This, however, is impossible, as $uv = 0$ at the walls of the channel, i.e. for $y = -\,^1\!/_2, +\,^1\!/_2$.

executed, and which in consequence relates to one elementary cone only. So putting:

$$Z_1 = \int_0^\infty d\varrho \, e^{N \lg \varrho + \beta (K \varrho^2 - H \varrho^4)}$$

(in which for simplicity we have written N for $N-1$), we will consider the equation:

$$\frac{\partial Z_1}{\partial \beta} = 0.$$

We shall suppose that for the cone considered $K > 0$, as otherwise this equation would admit no solution; then for the evaluation of Z_1 we may apply an approximate method, making use of the circumstance that the exponent of e passes through a maximum for a certain value ϱ_0 of ϱ, which is determined by the equation:

$$\frac{N}{\varrho_0} + 2 \beta K \varrho_0 - 4 \beta H \varrho_0^3 = 0. \quad \ldots \ldots \quad (a)$$

This equation gives us:

$$\varrho_0^2 = \frac{K}{4H} + \frac{K}{4H} \sqrt{1 + \frac{4NH}{\beta K^2}}.$$

As the second derivative of the exponent for $\varrho = \varrho_0$ takes the value:

$$-\frac{N}{\varrho_0^2} + 2 \beta K - 12 \beta K \varrho_0^2 = -4 \sqrt{\beta^2 K^2 + 4 \beta N H},$$

we may write for the exponent:

$$N \lg \varrho_0 + \beta (K \varrho_0^2 - H \varrho_0^4) - 2 \sqrt{\beta^2 K^2 + 4 \beta N H} \, . \, (\varrho - \varrho_0)^2 + \ldots .$$

When we assume provisionally that the coefficient of $(\varrho - \varrho_0)^2$ is large, we may neglect the terms not written out in the above expression (which are of the third and higher degrees in $(\varrho - \varrho_0)$), and we find the following approximate value for Z_1:

$$Z_1 \doteq \sqrt{\frac{\pi}{4}} \frac{e^{N \lg \varrho_0 + \beta (K \varrho_0^2 - H \varrho_0^4)}}{(\beta^2 K^2 + 4 \beta N H)^{1/4}}.$$

Now we apply the condition $\partial Z_1 / \partial \beta = 0$. Having regard to equation (a), we obtain:

$$\frac{\partial}{\partial \beta} \lg Z_1 = K \varrho_0^2 - H \varrho_0^4 - \frac{2 \beta K^2 + 4 N H}{4 (\beta^2 K^2 + 4 \beta N H)} = 0 \quad \ldots \quad (b)$$

As we may increase L indefinitely, N may become as large as we please. Assuming for a moment that β increases simultaneously with N, we shall neglect the third term of equation (b), which gives us:

$$K \varrho_0^2 - H \varrho_0^4 \doteq 0 \quad \ldots \ldots \ldots \quad (c)$$

from which we deduce:

$$\varrho_0^2 = K/H.$$

Then (*a*) gives:

$$\beta = NH/2K^2,$$

which justifies the supposition about the order of magnitude of β. The same result is obtained when the value of $\varrho_0{}^2$ is substituted immediately into equation (*b*); with the abbreviation $NH/\beta K^2 = s$, this equation then takes the form:

$$1 - 2s + \sqrt{1+4s} = \frac{4s}{N}\frac{1+2s}{1+4s},$$

with the solution $s = 2$ for $N \to \infty$.

Equation (*c*) is equivalent to the formula:

$$E_{(\rho=\rho_0)} = 0.$$

The coefficient of $(\varrho-\varrho_0)^2$ in the development of the exponent now assumes the value $3NH/K$; this may be made arbitrarily great by increasing N, which justifies the procedure adopted in the calculation of Z_1. The value of Z_1 finally becomes (when we write again $N-1$ for N):

$$Z_1 \approx \sqrt{\frac{\pi}{3(N-1)}}\left(\frac{K}{H}\right)^{N/2}.$$

This result shows us that small variations in the value of the quotient K/H will have a very large influence on Z_1. Let us suppose that there exists a maximum value for this quotient. As the value calculated for Z_1 is the maximum value that quantity can attain when β is varied, it seems reasonable to expect that in the full integral:

$$Z = \int d\chi \int_0^\infty d\varrho \, e^{(N-1)\lg\rho + \beta(K\rho^2 - H\rho^4)}$$

only those elementary cones will contribute significant amounts for which K/H differs only very little from its maximum value. Then the value of β that makes Z stationary will be approximately:

$$\beta = NH/2K^2 \quad \text{for} \quad K/H = \text{maximum}.$$

So we are led to the supposition that in studying the properties of the "most probable distribution of representative points" and in calculating mean values relating to the normal state of turbulent motion, we have to take regard only of those regions of the ξ-space, that lie around the points defined by

$$E = 0, \qquad K/H = \text{maximum} \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

We readily own that this reasoning is only approximative, and that it is not easy to estimate the extent of the regions mentioned. The increase of the number of dimensions of the ξ-space, which is the consequence of an increase of L and N, makes the study of this problem rather complicate. We have to bear in mind also that all fields which can be deduced from a given one by a shift parallel to the x-axis over an arbitrary multiple of ε (the number of fields obtained in this way is $L\varepsilon^{-1}$), will lead to very nearly the same values of both K and H. [1]) As to such a shift corresponds a certain permutation of the coordinates in the ξ-space, we must expect that there will be $L\varepsilon^{-1}$ different cones for which K/H assumes the same value, f.i. attains its maximum value. Besides it has to be remarked that K and H do not change, when we change the sign of all the coordinates in the ξ-space.

10. *Further approximations. — Introduction of a special vortex field.* Notwithstanding the difficulties mentioned at the end of the foregoing §, we will go still further with our approximations, and shall assume that in calculating mean values we have to take regard only of a number of equal small elements of volume of the ξ-space, each having its centre in one of the points defined by (69). Then we may even restrict ourselves to the consideration of one such an element of volume, as all of them contribute the same amount. This amounts to saying that we have to determine only that field of relative motion, which makes:

$$E = 0 \quad \ldots\ldots\ldots\ldots \quad (69^a)$$

(that is to say, which fulfills the ordinary dissipation condition), and at the same time makes:

$$K/H = \text{maximum} \quad \ldots\ldots\ldots \quad (69^b)$$

Now here we have arrived at a point of view, which presents some resemblance with that taken in a former paper on the resistance experienced by a fluid in turbulent motion. [2])

In that paper we had tried to get an estimate of the maximum value the resistance coefficient could possibly obtain.

Introducing the notation (comp. l. c. eq. 52):

$$\frac{1}{L}\iint dx\, dy\; y\, uv = \sigma, \qquad \frac{1}{12\,L^2}\int dy \left(\int dx\; uv\right)^2 = (1+\tau)\,\sigma^2,$$
$$\frac{1}{12\,L}\iint dx\, dy\; \zeta^2 = \varkappa\,\sigma,$$

and putting further:

$$\varrho^2 = \frac{1}{L}\iint dx\, dy\; \psi^2 = \lambda\,\sigma,$$

[1]) That the values of K and H in general will not be exactly the same for all fields obtained by the process indicated, is due to the fact that the values of ψ which in one field occur in the points of the end sections (resp. at x_0 and at $x_0 + L$), in another field lie in the interior. The influence of this circumstance will be very small, however.

[2]) J. M. BURGERS, these Proceedings **26**, p. 582, 1923.

we have:

$$K/H = \lambda\,\tau^{-1}(R^{-1} - \varkappa\,R^{-2}).$$

The condition $E = 0$ gives for σ (comp. l. c. eq. 53):

$$\sigma = \tau^{-1}(R^{-1} - \varkappa\,R^{-2}),$$

whereas from (63) we deduce for the resistance coefficient C: [1])

$$C = 6\,(\sigma + R^{-1}).$$

In the paper mentioned we had asked for the maximum value of σ; now we ask for the maximum of K/H. In order to obtain an idea of the order of magnitude of the quantities introduced, we may construct special fields of relative motion and calculate the various integrals for them. As has been shown in that paper, fields which shall give high values for σ have to satisfy the condition that strong vorticity is present only in very thin layers along the walls. This can be obtained by constructing fields, in which the "mean wave length" is small in the region along the walls, and increases towards the central region. Such a field can be built up in the simplest way from an assemblage of elliptic vortices of the type studied by LORENTZ. [2]) These vortices are deduced from circular vortices by a compression in the proportion $\varepsilon = 0{,}475$ in a direction inclined to the x-axis. Two groups of them have to be taken, one lying against the wall $y = -\frac{1}{2}$, consisting of vortices for which uv is mainly negative; the other, lying against the wall $y = +\frac{1}{2}$, is obtained from the former one by a reflexion in the x-axis. The vortices in either group have "thicknesses" D (by which their dimension in the y-direction is denoted) ranging from 1 down to a minimum value D_0. Considering particularly the vortices lying against the wall $y = -\frac{1}{2}$, the number and mean intensity of a subgroup, having thicknesses between D and $D + dD$, is taken such that the contribution of this subgroup in the integral $\frac{1}{L}\int dx\,uv = \overline{uv}$ for a value of $y' = \frac{1}{2} + y$, less than D, is given by:

$$-m\left(D^{-1} - \frac{3}{4}\right)dD\,.\,\varphi\,(y'/D). \qquad (a)$$

where m is a constant, and φ is the function defined by $\varphi(\eta) = \eta^4(1-\eta)^4$. [3])

By means of this formula the value of $\overline{uv}$, due to all vortices together, can be calculated for any value of y; from this calculation was deduced:

$$\tau_0 \cong 2{,}129\,D_0 \qquad \text{(l.c. eq. 57)}$$

As for every individual vortex the relation existed:

$$\iint dx\,dy\,\zeta^2 = \frac{294}{D^2}\left|\iint dx\,dy\,uv\right| \qquad \text{(l.c. eq. 30)}$$

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[1]) L.c. equation (54). In the paper of 1923 the value of C is double of that taken now.

[2]) H. A. LORENTZ, Abhandlungen über theoret. Physik I, p. 48—52.

[3]) Comp. l.c, equation (55). The constant factor m is omitted in the formulae of that paper.

(here the integrals are taken over the area of a single vortex only), the value of $\iint dx\, dy\, \zeta^2$ for the whole system also could be calculated; this gave for $\varkappa$:

$$\varkappa = 131\, D_0^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(l.c. eq. 58)}$$

The quantity λ was not introduced in the former paper. It can be calculated on the same lines as $\varkappa$, as soon as we know the coefficient a in the equation:

$$\iint dx\, dy\, \psi^2 = a\, D^2 \left| \iint dx\, dy\, uv \right|,$$

in which the integrals again relate to a single vortex. It is not difficult to determine this factor: as the elliptic vortices are deduced from circular ones by a compression in the proportion ε, while the velocity component in the direction of the compression was reduced in the same proportion, it follows that the stream function in any point of the elliptic vortex is equal to ε times the stream function in the corresponding point of the circular vortex. We pass over the calculation of the latter quantity, and mention only the result:

$$a = 0{,}144,$$

which leads to:

$$\lambda = 0{,}112.$$

It has to be remarked that this quantity — at least to a first approximation — appears to be independent of D_0.

Now we can introduce the results for τ, $\varkappa$, λ into the expression for K/H.

As the value of D_0 has not yet been fixed, we can ask for that value which makes K/H a maximum. It is readily seen that this is the same value D_0 as makes σ a maximum. Hence in this case the condition (69^b) is identical with the condition of maximum resistance; in other words: we completely fall back on the result of the former paper. We obtain:

$$D_0 = 262\, R^{-1}, \qquad \sigma = 0{,}00090, \qquad K/H = 0{,}00010,$$

and:

$$C = 0{,}0054,$$

which is much higher than the values observed experimentally.

It is interesting to consider the distribution of the velocity of the mean motion obtained in this "model". According to equation (56) of the paper mentioned, we have for values of y, numerically less than $\frac{1}{2} - D_0$:

$$\overline{uv} = \frac{m}{140}\, y,$$

where again we have introduced the factor m, mentioned above in connection with equation (α), and at the same time have substituted y

for $y - \frac{1}{2}$, according to the position of the axes used here. Further it was found that:

$$\sigma = \int_{-1/2}^{+1/2} dy\, y\, \overline{uv} = \frac{m}{1680}\left(1 - \frac{8}{3} D_0\right) = 0{,}00090.$$

Introducing both expressions into equation (55) of this paper, we obtain:

$$\frac{dU}{dy} = -y\left(12 - \frac{8}{3}\frac{m}{140} D_0 R\right) = -4{,}47\, y.$$

This is not much better than the result of § 4, though the curve for U becomes a little bit flatter.

11. *General remarks. Other formulation of the problem.*

The result of the foregoing § does not seem very satisfactory. But we must not forget that we have artificially reduced all integrals over the ξ-space to a single point, and thus have made a caricature of the general equations. So it may be that there is more good in them than would appear from § 10, and it is still possible that formula (63), when correctly worked out, will give a better result. In fact it fulfills in so far the requirements, mentioned in the former paper (l.c. p. 600), as that it leaves room to irregular displacements and deformations of the vortices, etc.

Meanwhile it is of importance to make some remarks about the relation between the two different methods we have used to arrive at a distribution function, especially as it is possible to develop a third system of formulae on a basis which takes a somewhat intermediate position between those two points of view.

It might be argued that the procedure adopted in § 8 of defining the mean motion as a mean with respect to x introduces a similar difficulty as the method of employing time averages did, in so far as the new definition implies that the variation of the relative motion at a certain part of the channel would influence the mean motion over the whole length of the latter. On viewing closer, however, this difficulty is but an apparent one. The deductions of § 8 are based on the instantaneous state of the real motion over the whole field; the various equations served only to obtain a transformation of equation (56), and in this transformation U and P may be considered as purely formal quantities without any further meaning. When the state of the real motion has been given, the value of E can be calculated at once, either from (56) or from (57).

The only assumption of a more physical nature that has been made, is the one expressed by equation (51^c), which served to obtain (55). In the deductions of § 8 these equations (51^c), (55) were considered as being valid at every moment; consequently the latter could be used to eliminate dU/dy from (56^a).

An analogous remark may be made in connection with the deductions of Parts I and II. The object of these deductions can be expressed as follows: to obtain a transformation of the integral of the rate of increase of the energy over a long interval of time; in such a transformation time means may be introduced as formal quantities. Thereby, at least in the calculations of § 4, the equation (27) was considered as being valid at every point of the channel (in § 5 a somewhat more general standpoint was taken).

Now instead of accepting either the one or the other supposition, we may take an intermediate point of view, and demand that equation (55) shall be satisfied only on the average over a long interval of time. We may just as well replace equation (27) by its integral over x. The latter way is not wholly so simple, as for a more rigorous treatment it would be necessary to start from equations (44). The consideration of equation (55), however, will be sufficient. We shall adhere to the supposition that the total flow through a section of the channel remains absolutely constant, as this is the kinematical basis of the boundary conditions for the stream function.

When we wish to construct a system of formulae consistent with this idea, we must retain the real motion explicitly in our equations, and eliminate only $P_I - P_{II}$ by means of the condition of constant total flow. We then write (comp. (55) and (56^a)):

$$\frac{v}{L} = \frac{12\,y}{L}\iint dx\,dy\,y\,uv - \frac{1}{L}\int dx\,uv + \frac{1}{R}\frac{dU}{dy} + \frac{12\,y}{R} \quad . \quad . \quad (70^a)$$

$$\frac{E}{L} = \frac{1}{L}\iint dx\,dy\,(12\,y\,uv - R^{-1}\,\zeta^2) - \frac{1}{R}\int dy\left(\frac{dU}{dy}\right)^2 + \frac{12}{R} \quad . \quad (70^b)$$

In these expressions u, v, ζ are derived from the stream function ψ as usual. U is derived from a new stream function for which we shall write Y, and which is a function of y only, so that $U = Y'$. The boundary conditions to be satisfied by these functions are:

$$\begin{cases} y = -\tfrac{1}{2}: & Y = -\tfrac{1}{2}, \quad Y' = \psi = \partial\psi/\partial y = 0 \\ y = +\tfrac{1}{2}: & Y = +\tfrac{1}{2}, \quad Y' = \psi = \partial\psi/\partial y = 0. \end{cases}$$

The ξ-space must now be extended by ε^{-1} new coordinates $\eta_1, \eta_2, \ldots \eta_k, \ldots$, which represent the values Y takes at a cross line of the lattice (i.e. at a row of points, parallel to the y-axis). It will be convenient to call the space determined by the η's: the η-space, and to retain the denomination ξ-space for the one determined by the ξ's only. The whole may be denoted as ξ, η-space. The values of v (which like Y is a function of y only) in the points of a cross line shall be denoted by v_k. Then the following conditions have to be fulfilled:

$$\Sigma\,\nu\,E = 0, \qquad \Sigma\,\nu\,v_k = 0 \;\; (k = 1, 2, \ldots, s^{-1}) \quad . \quad . \quad . \quad (71)$$

where the summation has to be extended over all the cells of the ξ, η-space.

The distribution function now becomes:

$$\nu = A\, e^{\beta (E + \varepsilon \sum_k \gamma_k \nu_k)/L},$$

where the parameters $\beta, \gamma_1, \gamma_2 \ldots$ have to be determined from the conditions (71). Putting:

$$Z = \Sigma\, e^{\beta (E + \varepsilon \sum_k \gamma_k \nu_k)/L} = \Sigma\, e^{\beta (E + \int dy\, \gamma\, \nu)/L} \quad . \quad . \quad . \quad . \quad (72)$$

where the first summation again is extended over all the cells in which the ξ, η-space is divided, we can write these equations:

$$\partial Z / \partial \beta = 0; \qquad \delta Z = 0 \quad \text{for an arbitrary variation of } \gamma(y) \, . \quad (73)$$

We shall not go into a detailed discussion of these equations, and will mention only a few points. Collecting together the various terms in the exponent of e we have:

$$\left. \begin{aligned} \frac{E + \int dy\, \gamma\, v}{L} &= \frac{1}{L} \iint dx\, dy \left\{ -uv \left(-12 y \int dy\, \gamma\, y + \gamma - 12 y \right) - R^{-1} \zeta^2 \right\} - \\ &\quad - \frac{1}{R} \int dy \left(U'^2 - \gamma\, U' \right) + \frac{12}{R} \left(\int dy\, \gamma\, y + 1 \right) \end{aligned} \right\} \quad (74)$$

It appears that the part depending on Y is definite negative. The part depending on ψ has a form which reminds immediately the formulae of § 4, Part II; when we put:

$$\varphi^{\star}(y) = R^{-1} \left(-12\, y \int dy\, \gamma\, y + \gamma - 12\, y \right) \quad . \quad . \quad . \quad (75)$$

we may write for it:

$$-\frac{R}{L} \iint dx\, dy \left\{ R^{-2} (\triangle\, \psi)^2 - \varphi^{\star} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \right\}.$$

Hence we see that this part will be definite negative only when the function $\varphi^{\star}$ fulfills the same condition as was asked for the function φ investigated in § 4. Thus for $y < 0$ we have (comp. eq. (36) above, where $y' = \frac{1}{2} + y$):

$$-12\, y \int dy\, \gamma\, y + \gamma - 12\, y < \frac{A}{R\, (\frac{1}{2} + y)^2} \quad . \quad . \quad . \quad . \quad (76)$$

When β and $\gamma(y)$ are known, the mean value of U' is derived from:

$$\overline{U'} = \frac{\Sigma\, U'\, e^{-\beta R^{-1} \int dy\, (U'^2 - \gamma\, U')}}{\Sigma\, e^{-\beta R^{-1} \int dy\, (U'^2 - \gamma\, U')}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

It is readily seen that the function Y_m which makes the exponent of e a maximum, is determined by:

$$2\, Y_m^{IV} = 2\, U_m''' = \gamma''.$$

Hence:

$$\gamma = 2\,U_m' + c_1\,y + c_2.$$

We may suppose that $\gamma = 0$ in the axis of the channel, and so we can discard c_2. Substitution in (76) then gives:

$$2\,U_m' + 12\,y < \frac{A}{R\,(\frac{1}{2} + y)^2}.$$

This condition recalls the result of the end of § 4, but it has to be observed that now U_m is not equal to the mean value of U.

Further investigations will be necessary to throw more light on this point.

I should like to mention two more questions which arise in connection with our deductions.

A thing which is not cleared up is the part played by the number of coordinates in the ξ-space, particularly in so far as it depends on the magnitude of ε (the spacing of the lattice introduced into the x,y-plane). Allusion has already been made to this circumstance in § 2, Part I, in connection with equation (16). Another point is this: the results of § 9 seem to make it probable that when L is taken large enough, the values of E for the most frequent fields will differ from zero only by amounts of the order $L/\beta = 2\,LK^2/NH = 2\,\varepsilon^2\,K^2/H$. It is somewhat surprising that the value of ε here suddenly turns up.

The other point I would consider is about the weights given to the various fields present in the sequence. We have given equal weights to equal volumes of the ξ-space, and leaving aside the considerations of § 2 connected with LIOUVILLE's theorem, the question again might be put forward whether we are right in doing so? Might not it be that some fields a priori had a greater probability than others? From the point of view accepted in statistical mechanics this could only be the case when our fields could be built up in various ways from other units. But our fields are already fully specified. And the introduction of "units" for ψ by means of which its value in every separate point could be built up in various ways, would seem rather absurd.

The introduction of principal solutions (Eigenlösungen) of any fourth order differential equation for ψ does not promise any help. Every field we considered can be built up from such functions in one way only. The use of such functions comes down to the introduction of a new system of coordinates in the ξ-space, to be derived from the ξ's by means of an orthogonal transformation; so it does not change the constant proportion between weight and volume. Principal solutions are useful only, when they make it easier to express the energy or some suchlike quantity.

Hence it would appear that there is no indication which forces us to look for other weights.