Mathematics. — *Skew Correlation between Three and More Variables,* II. By Prof. M. J. VAN UVEN. (Communicated by Prof. A. A. NIJLAND).

(Communicated at the meeting of May 25, 1929).

We may still apply the method hitherto followed, when there is *no* linear correlation between the original variables x_1, x_2, x_3 appearing in the given frequency distribution **A**, but when linear correlation can be found between the (unimodular) variables t_1, t_2, t_3, t_a being a function of only x_a $(a = 1, 2, 3)$.

Considering, in particular, $t_a\{x_a\}$ as an ever increasing one-valued function of x_a $(a = 1, 2, 3)$ [1]), $s_a(k_a)$ (**B**a) gives the probability of $x_a < x_a(k_a)$. Thus the equation **B**a associates a value $z_a(k_a)$ with $x_a(k_a)$, and we obtain a series of $\nu_a - 1$ empirical values of the function $z_a\{x_a\}$.

Likewise $\sigma_{b;a}(k_a - \frac{1}{2}, k_b)$ (**B(b; a)**) furnishes the probability of $[x_b < x_b(k_b)$, *being given*: $x_a(k_a - 1) < x_a < x_a(k_a)$]. Hence the equation **B(b; a)** joins a value $\zeta_{b;a}(k_a - \frac{1}{2}, k_b)$ to the set $x_a(k_a - \frac{1}{2}), x_b(k_b)$, and we obtain a series of empirical values of the function $\zeta_{b;a}\{x_a, x_b\}$.

As x_a itself is to be considered as a function of z_a, we may also conceive $\zeta_{b;a}$ as a function of z_a and z_b.

Now z_a is proportional to t_a, so that we (cf **13**) may put **9ter (b; a)** into the form:

$$\left.\begin{aligned}\zeta_{b;a} &= \frac{-A_{ba}\cdot\left(\dfrac{A_{aa}}{A}\right)^{1/2}\cdot z_a + A_{aa}\cdot\left(\dfrac{A_{bb}}{A}\right)^{1/2}\cdot z_b}{\sqrt{A_{aa}}} = \\ &= \frac{-A_{ba}\cdot z_a + \sqrt{A_{aa}A_{bb}}\cdot z_b}{\sqrt{A}} = -\cot\omega_c\cdot z_a + \frac{1}{\sin\omega_c}\cdot z_b\end{aligned}\right\} \quad \textbf{9 quater (b; a)}$$

Hence $\zeta_{b;a}$ appears to be a *linear* function of z_a and z_b.

Finally $S_c(k_a - \frac{1}{2}, k_b - \frac{1}{2}, k_c)$ (**B(c;ba)**) gives the probability of $[x_c < x_c(k_c)$, *being given*: $x_a(k_a - 1) < x_a < x_a(k_a)$, $x_b(k_b - 1) < x_b < x_b(k_b)$]. Thus the equation **B(c; ba)** associates a value $Z_c(k_a - \frac{1}{2}, k_b - \frac{1}{2}, k_c)$ with the set $x_a(k_a - \frac{1}{2}), x_b(k_b - \frac{1}{2}), x_c(k_c)$. This furnishes us a set of empirical values of the function $Z_c\{x_1, x_2, x_3\}$. As x_a depends only on z_a, Z_c is also to be considered as a function of all three variables z_1, z_2, z_3.

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[1]) The limiting agreement that $t\{x\}$ shall be monotone (increasing) is not necessary, but only desirable from a practical point of view and can in certain cases be dropped. (See: M. J. VAN UVEN, *Scheeve Frequentiekrommen*, Versl. K. A. v. W. **25**, p. 709, *Skew Frequency Curves*, Proceed. K. Ak. v. Wet. Amsterdam, Vol. **19**, p. 670).

In virtue of

$$\left.\begin{aligned} Z_c &= \lambda_{ca} \,.\, t_a + \lambda_{bc} \,.\, t_b + t_c = \\ &= \frac{\lambda_{ca} \sqrt{\Lambda_{aa}} \,.\, z_a + \lambda_{bc} \sqrt{\Lambda_{bb}} \,.\, z_b + \sqrt{\Lambda_{cc}} \,.\, z_c}{\sqrt{\Lambda}} \end{aligned}\right\} \quad \textbf{9 quater (c; ba)}$$

Z_c is a linear function of z_1, z_2, z_3.

In constructing empirically the functions $\zeta_{b;a}$ and Z_c it is necessary to determine the required mean values as exactly as possible.

Now the values of the functions $\zeta_{b;a}, Z_c$, which correspond to the sets of class-limits $x_a\,(k_a)$, and to the associated values $z_a\,(k_a)$, are to be calculated by interpolation. If these calculations are made correctly, and if the conditions that the unimodular variables t_1, t_2, t_3 are pure functions of x_1, x_2, x_3 respectively, are really fulfilled, then it must appear, that the functional values $z_a\,(k_a), \zeta_{b;a}\,(k_a, k_b), Z_c\,(k_a, k_b, k_c)$, corresponding to $x_a\,(k_a), x_b\,(k_b), x_c\,(k_c)$, satisfy the two conditions **Ia** and **Ib**. Moreover it must appear that the three magnitudes γ_{ab}, computed from **E^bis^**, are constant, that is to say: that

$$\gamma_{ab} = \frac{A_{ab}}{Q^2_{ab}} = \text{constant} \ (a, b = 1, 2, 3) \quad . \ . \ . \ . \ . \quad \textbf{II}$$

Inversely, when the conditions

$$\left.\begin{aligned} & z^2_a + \zeta^2_{b;a} = z^2_b + \zeta^2_{a;b} \, (= q^2_{ab}), && \qquad . \ . \ . \quad \textbf{Ia} \\ & Z^2_c = H\{z, \zeta\} - q^2_{ab}, && (a, b, c = 1, 2, 3) \quad . \ . \ . \quad \textbf{Ib} \\ & \gamma_{ab} = \frac{A_{ab}}{Q^2_{ab}} = \text{constant}, && \qquad . \ . \ . \quad \textbf{II} \end{aligned}\right.$$

are fulfilled, we may conclude, that *linear correlation* exists between the unimodular variables t_1, t_2, t_3, determined by

$$t_a = \frac{z_a}{\sin \psi_a} = \left(\frac{\Gamma_{aa}}{\Gamma}\right)^{1/2} . \, z_a \qquad (a = 1, 2, 3) \quad . \ . \ . \ . \quad \textbf{J}$$

t_a being (in virtue of $\psi_a = \text{const.}$) a pure function of x_a.

If t_a is a pure function of x_a $(a = 1, 2, 3)$, each arrangement $z_a, \zeta_{b;a}, Z_c$ associates with the set x_1, x_2, x_3 the same point $P\,(t_1, t_2, t_3)$ in the same (skew) system of coordinates t_1, t_2, t_3. Hence we obtain on the unity-sphere for each set x_1, x_2, x_3 the same spherical triangle (Φ) of reference, thus also the same polar triangle (Ω). In this case the (total) coefficients of correlation of $t_1\{x_1\}, t_2\{x_2\}, t_3\{x_3\}$ are the cosines of the (constant) sides of the polar triangle:

$$\gamma_{23} = \cos \omega_1 \quad , \quad \gamma_{31} = \cos \omega_2 \quad , \quad \gamma_{12} = \cos \omega_3 .$$

If, in particular, t_a is a *linear* function of x_a, $t_a = h_a\,(x_a - \overline{x_a})$, the function z_a, too, will be a linear function of x_a, the function $\zeta_{b;a}$ will be linear in x_a and x_b, Z_c linear in x_1, x_2, x_3. Then the original variables x_1, x_2, x_3 themselves are linearly correlated.

This statement furnishes us a rather reliable test of the linearity of the correlation between x_1, x_2, x_3. [1])

Since no empirical frequency distribution is *rigorously* conform to any ideal theoretical probability scheme, and since we have to reckon with several sources of inaccuracy, as well in calculating mean values (such as $\zeta_{2;1}(k_1 - \frac{1}{2}, k_2)$, $Z_3(k_1 - \frac{1}{2}, k_2 - \frac{1}{2}, k_3)$) as in interpolating (e.g. to compute $\zeta_{2;1}(k_1, k_2)$, $Z_3(k_1, k_2, k_3)$), we cannot reasonably expect, that even in the case $t_a\{x_a\}$ the conditions **Ia, b** and **II** will be satisfied *exactly*. We shall have to content ourselves with an *approximate* validity of these equations. Only when the quotients $q_{a;b} : q_{b;a}$, $Z_c^2 : (H\{z, \zeta\} - Q_{ab}^2)$ and the variables γ_{ab} show some obvious functional dependency on x_1, x_2, x_3, should we be obliged to drop the supposition $t_a\{x_a\}$ $(a = 1, 2, 3)$.

If the condition **II** is not fulfilled, that is to say: if the values of $\gamma_{bc} = \cos\omega_a$ computed for the different sets $x_1(k_1), x_2(k_2), x_3(k_3)$, are *unequal*, so that $\omega_1, \omega_2, \omega_3$ are *variable* magnitudes, then we may *introduce* a *constant* polar triangle (Ω) with sides $\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3$. In order to keep in touch, as much as possible, with the values $\omega_1, \omega_2, \omega_3$ really found: it is preferable to choose for $\overline{\omega}_a$ an *average* value of ω_a. We may obtain such average values by taking the averages of

$$M_1 = \operatorname{tg}\omega_1 = \frac{B_{23}}{A_{23}}\ ,\quad M_2 = \operatorname{tg}\omega_2 = \frac{B_{31}}{A_{31}}\ ,\quad M_3 = \operatorname{tg}\omega_3 = \frac{B_{12}}{A_{12}},$$

according to the precept we have formerly given in S. C. I. *b* (Dutch text p. 976, English text p. 929).

A_{23} and B_{23} are merely functions of x_2 and x_3. So, in averaging M_1, the frequency distribution is to be treated as a two-dimensional one, its frequencies being $Y'(k_2, k_3) = \sum_{i_1=1}^{\nu_1} Y(i_1, k_2, k_3)$. Hence the "weights" introduced in averaging $M_1 = \operatorname{tg}\omega_1$ are built up of these two-dimensional frequencies $Y'(k_2, k_3)$. Likewise for averaging M_2 and M_3.

From the averages $\overline{\omega}_a$ $(a = 1, 2, 3)$ we compute the corresponding arcs $\overline{\varphi}_a$ and $\overline{\psi}_a$ $(a = 1, 2, 3)$.

Then we put, guided by **9bis**,

$$\left.\begin{aligned} z_1 &= \sin\overline{\psi}_1 \,.\, T_1, \\ \zeta_{2;1} &= -\sin\overline{\varphi}_2 \cos\overline{\omega}_3 \,.\, T_1 + \sin\overline{\varphi}_1 \,.\, T_2, \\ Z_3 &= \cos\overline{\varphi}_2 \,.\, T_1 + \cos\overline{\varphi}_1 \,.\, T_2 + T_3. \end{aligned}\right\} \quad . \quad . \quad \mathbf{32} \begin{cases} (1) \\ (2;1) \\ (3;2\,1) \end{cases}$$

[1]) See the footnote further on, preceding equat. **J** in the treatment of n variables.

By confronting these equations with the equations **9bis** we obtain:

$$z_1 \;= \sin \bar{\psi}_1 \,.\, T_1 = \sin \psi_1 \,.\, t_1,$$

$$\zeta_{2:1} = -\sin \bar{\varphi}_2 \cos \bar{\omega}_3 \,.\, T_1 + \sin \bar{\varphi}_1 \,.\, T_2 = -\sin \varphi_2 \cos \omega_3 \,.\, t_1 + \sin \varphi_1 \,.\, t_2,$$

$$Z_3 = \cos \bar{\varphi}_2 \,.\, T_1 + \cos \bar{\varphi}_1 \,.\, T_2 + T_3 = \cos \varphi_2 \,.\, t_1 + \cos \varphi_1 \,.\, t_2 + t_3,$$

whence

$$T_1 = \frac{\sin \psi_1}{\sin \bar{\psi}_1}\, t_1 = \frac{z_1}{\sin \bar{\psi}_1} \quad \ldots\ldots \quad \mathbf{33}$$

$$T_2 = \frac{1}{\sin \bar{\varphi}_1}\left(-\sin \varphi_2 \cos \omega_3 \,.\, t_1 + \sin \varphi_1 \,.\, t_2 + \sin \bar{\varphi}_2 \cos \bar{\omega}_3 \,.\, T_1\right)$$

$$= \frac{1}{\sin \bar{\varphi}_1}\left(-\sin \varphi_2 \cos \omega_3 + \frac{\sin \bar{\varphi}_2 \cos \bar{\omega}_3 \sin \psi_1}{\sin \bar{\psi}_1}\right) t_1 + \frac{\sin \varphi_1}{\sin \bar{\varphi}_1} \,.\, t_2 \quad . \quad \mathbf{34}$$

$$= \frac{1}{\sin \bar{\varphi}_1 \sin \bar{\psi}_1}\left(-\sin \varphi_2 \cos \omega_3 \sin \bar{\varphi}_2 \sin \bar{\omega}_3 + \right.$$

$$\left. + \sin \bar{\varphi}_2 \cos \bar{\omega}_3 \sin \varphi_2 \sin \omega_3\right) t_1 + \frac{\sin \varphi_1}{\sin \bar{\varphi}_1} \,.\, t_2$$

$$= \frac{\sin \varphi_2 \sin \bar{\varphi}_2 \sin (\omega_3 - \bar{\omega}_3)}{\sin \bar{\varphi}_1 \sin \bar{\psi}_1}\, t_1 + \frac{\sin \varphi_1}{\sin \bar{\varphi}_1}\, t_2$$

$$= \frac{\sin \varphi_2 \sin (\omega_3 - \bar{\omega}_3)}{\sin \bar{\varphi}_1 \sin \bar{\omega}_3}\, t_1 + \frac{\sin \varphi_1}{\sin \bar{\varphi}_1} \,.\, t_2$$

$$= \frac{\sin \varphi_2 \sin (\omega_3 - \bar{\omega}_3)}{\sin \bar{\psi}_2}\, t_1 + \frac{\sin \varphi_1}{\sin \bar{\varphi}_1}\, t_2$$

$$= \frac{\sin \varphi_2 \sin (\omega_3 - \bar{\omega}_3)}{\sin \bar{\psi}_2} \,.\, \frac{1}{\sin \varphi_2 \sin \omega_3}\, z_1 + \frac{\sin \varphi_1}{\sin \bar{\varphi}_1} \,.\, \frac{1}{\sin \varphi_1 \sin \omega_3}\, z_2 \,.$$

or

$$\left.\begin{aligned} T_2 &= \frac{\sin (\omega_3 - \bar{\omega}_3)}{\sin \bar{\psi}_2 \sin \omega_3}\, z_1 + \frac{1}{\sin \bar{\varphi}_1 \sin \omega_3}\, z_2, \\ &= \frac{\sin (\omega_3 - \bar{\omega}_3)}{\sin \bar{\varphi}_1 \sin \bar{\omega}_3 \sin \omega_3}\, z_1 + \frac{1}{\sin \bar{\varphi}_1 \sin \omega_3}\, z_2, \end{aligned}\right\} \quad . \; . \quad \mathbf{34bis}$$

and

$$T_3 = \cos \varphi_2 \,.\, t_1 + \cos \varphi_1 \,.\, t_2 + t_3 - \cos \bar{\varphi}_2 \,.\, T_1 - \cos \bar{\varphi}_1 \,.\, T_2$$

or, by **34**,

$$T_3 = \cos \varphi_2 \,.\, t_1 + \cos \varphi_1 \,.\, t_2 + t_3 - \cos \bar{\varphi}_2 \frac{\sin \psi_1}{\sin \bar{\psi}_1}\, t_1 -$$

$$- \cos \bar{\varphi}_1 \left(-\frac{\sin \varphi_2 \cos \omega_3}{\sin \bar{\varphi}_1} + \frac{\sin \bar{\varphi}_2 \cos \bar{\omega}_3 \sin \psi_1}{\sin \bar{\varphi}_1 \sin \bar{\psi}_1}\right) t_1 - \cos \bar{\varphi}_1 \frac{\sin \varphi_1}{\sin \bar{\varphi}_1}\, t_2$$

or

$$T_3 = \left\{ \frac{\cos\varphi_2 \sin\bar{\varphi}_1 + \cos\bar{\varphi}_1 \sin\varphi_2 \cos\omega_3}{\sin\bar{\varphi}_1} - \right.$$

$$\left. - \frac{(\cos\bar{\varphi}_2 \sin\bar{\varphi}_1 + \cos\bar{\varphi}_1 \sin\bar{\varphi}_2 \cos\bar{\omega}_3)\sin\psi_1}{\sin\bar{\varphi}_1 \sin\bar{\psi}_1} \right\} t_1 -$$

$$- \frac{\sin\varphi_1 \cos\bar{\varphi}_1 - \cos\varphi_1 \sin\bar{\varphi}_1}{\sin\bar{\varphi}_1} t_2 + t_3.$$

We now have in the "average" spherical triangle $\bar{\Phi}_1 \bar{\Phi}_2 \bar{\Phi}_3$:

$$\cos\bar{\varphi}_2 \sin\bar{\varphi}_1 + \cos\bar{\varphi}_1 \sin\bar{\varphi}_2 \cos\bar{\omega}_3 = \cos\bar{\varphi}_2 \sin\bar{\varphi}_1 - \cos\bar{\varphi}_1 \sin\bar{\varphi}_2 \cos\bar{\Phi}_3 =$$

$$= \sin\bar{\varphi}_3 \cos\bar{\Phi}_2 = - \sin\bar{\varphi}_3 \cos\bar{\omega}_2 ;$$

hence the second term in the coefficient of t_1 becomes:

$$+ \frac{\sin\bar{\varphi}_3 \cos\bar{\omega}_2}{\sin\bar{\varphi}_1 \sin\bar{\psi}_1} . \sin\psi_1 = \frac{\sin\bar{\varphi}_3 \cos\bar{\omega}_2}{\sin\bar{\varphi}_1 \sin\bar{\varphi}_3 \sin\bar{\omega}_2} . \sin\psi_1 = \frac{\cot\bar{\omega}_2}{\sin\bar{\varphi}_1} . \sin\psi_1 .$$

Thus we obtain, by $t_a = \frac{z_a}{\sin\psi_a}$ (a = 1, 2, 3),

$$\left. \begin{aligned} T_3 = \left(\frac{\cos\varphi_2 \sin\bar{\varphi}_1 + \cos\bar{\varphi}_1 \sin\varphi_2 \cos\omega_3}{\sin\bar{\varphi}_1 \sin\psi_1} + \frac{\cot\bar{\omega}_2}{\sin\bar{\varphi}_1} \right) z_1 - \\ - \frac{\sin(\varphi_1 - \bar{\varphi}_1)}{\sin\bar{\varphi}_1 \sin\psi_2} z_2 + \frac{1}{\sin\psi_3} z_3 . \end{aligned} \right\} \quad 35$$

The first term in the coefficient of z_1 can be interpreted geometrically as follows:

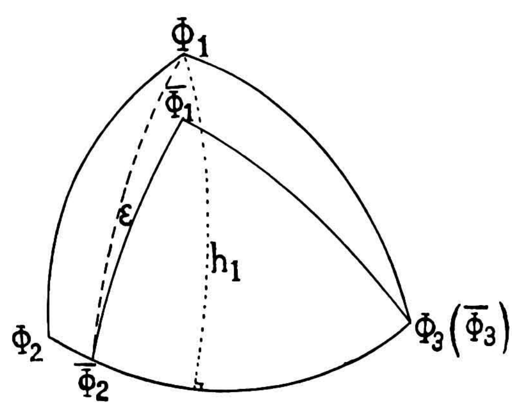

fig.4

We locate the "average" spherical triangle so, that $\bar{\Phi}_3$ coincides with Φ_3, and that $\bar{\Phi}_2 \bar{\Phi}_3$ falls along $\Phi_2 \Phi_3$ (fig. 4).

Putting

$$\angle \Phi_1 \overline{\Phi}_2 \overline{\Phi}_1 = \varepsilon,$$

we have in $\triangle \Phi_1 \overline{\Phi}_2 \Phi_3$:

$$\cos \varphi_2 \sin \overline{\varphi}_1 - \cos \overline{\varphi}_1 \sin \varphi_2 \cos \Phi_3 = \sin \overline{\Phi}_2 \Phi_1 . \cos \Phi_3 \overline{\Phi}_2 \Phi_1,$$

or

$$\cos \varphi_2 \sin \overline{\varphi}_1 + \cos \overline{\varphi}_1 \sin \varphi_2 \cos \omega_3 = \frac{\sin h_1}{\sin (\overline{\Phi}_2 + \varepsilon)} \cos (\overline{\Phi}_2 + \varepsilon) =$$

$$= \sin \psi_1 \cot (\overline{\Phi}_2 + \varepsilon) = - \sin \psi_1 \cot (\overline{\omega}_2 - \varepsilon).$$

Hence the coefficient of z_1 becomes:

$$\frac{1}{\sin \overline{\varphi}_1} \{ -\cot (\overline{\omega}_2 - \varepsilon) + \cot \overline{\omega}_2 \} = \frac{-\sin \varepsilon}{\sin \overline{\varphi}_1 \sin \overline{\omega}_2 \sin (\overline{\omega}_2 - \varepsilon)} = \frac{-\sin \varepsilon}{\sin \overline{\psi}_3 \sin (\overline{\omega}_2 - \varepsilon)}.$$

Thus we find for T_3:

$$T_3 = \frac{-\sin \varepsilon}{\sin \overline{\psi}_3 \sin (\overline{\omega}_2 - \varepsilon)} z_1 - \frac{\sin (\varphi_1 - \overline{\varphi}_1)}{\sin \overline{\varphi}_1 \sin \psi_2} z_2 + \frac{1}{\sin \psi_3} z_3. \quad . \quad \mathbf{35bis}$$

So we have together:

$$\left. \begin{aligned} T_1 &= \frac{1}{\sin \overline{\psi}_1} z_1, \\ T_2 &= \frac{\sin (\omega_3 - \overline{\omega}_3)}{\sin \psi_2 \sin \overline{\omega}_3} z_1 + \frac{1}{\sin \overline{\varphi}_1 \sin \omega_3} z_2 \\ T_3 &= \frac{-\sin \varepsilon}{\sin \overline{\psi}_3 \sin (\overline{\omega}_2 - \varepsilon)} z_1 - \frac{\sin (\varphi_1 - \overline{\varphi}_1)}{\sin \overline{\varphi}_1 \sin \psi_2} z_2 + \frac{1}{\sin \psi_3} z_3 \\ &\text{where } \varepsilon \text{ is determined by} \\ \cot (\overline{\omega}_2 - \varepsilon) &= - \frac{\sin \overline{\varphi}_1 \cos \varphi_2 + \cos \overline{\varphi}_1 \sin \varphi_2 \cos \omega_3}{\sin \psi_1}. \end{aligned} \right\} \quad . \quad . \quad \mathbf{K}$$

In this way we have constructed three variables T_1, T_2, T_3, which are *linearly* correlated, having the (total) coefficients of correlation:

$$\overline{\gamma}_{23} = \cos \overline{\omega}_1 \quad , \quad \overline{\gamma}_{31} = \cos \overline{\omega}_2 \quad , \quad \overline{\gamma}_{12} = \cos \overline{\omega}_3.$$

T_1 is a function of only z_1, thus a pure function of x_1.

Since $\omega_3 = \text{arc tg} \frac{B_{12}}{A_{12}}$ is built up out of $z_1, z_2, \zeta_{2;1}, \zeta_{1;2}$, z_1 depending only on x_1, z_2 only on x_2, $\zeta_{2;1}$ and $\zeta_{1;2}$ only on x_1 and x_2, we recognize ω_3 to be a function only of x_1 and x_2. Thus:

T_2 is a function only of x_1 and x_2 $\left(\frac{\partial T_2}{\partial x_3} = 0 \right)$.

T_3 however is a function of all three variables x_1, x_2, x_3.

Hence:

If the conditions **Ia** and **Ib** are fulfilled, but **II** is not, we may, by introducing the constant (average) magnitudes

$\overline{\omega}_1$, $\overline{\omega}_2$, $\overline{\omega}_3$, establish a *linear* correlation between $T_1\{x_1\}$, $T_2\{x_1, x_2\}$, $T_3\{x_1, x_2, x_3\}$. Then the (total) coefficients of correlation are $\gamma_{23} = \cos \overline{\omega}_1$, $\gamma_{31} = \cos \overline{\omega}_2$, $\gamma_{12} = \cos \overline{\omega}_3$.

If, the magnitudes ω_1 and ω_2 being variable, the magnitude $\omega_3 = \text{arc tg}\,\dfrac{B_{12}}{A_{12}}$ is found to be constant, then we naturally choose this constant value for $\overline{\omega}_3$ (fig. 5), whence $\omega_3 - \overline{\omega}_3 = 0$.
In this case T_2 becomes a pure function of x_2.

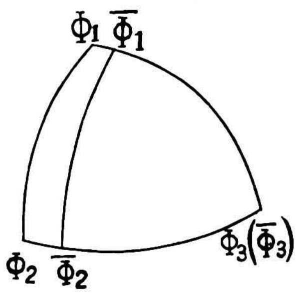

fig. 5

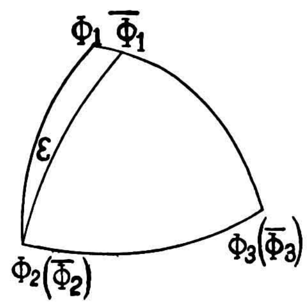

fig. 6

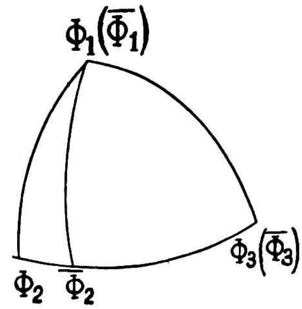

fig. 7

If it is found that not only ω_3, but also φ_1 is constant (fig. 6), the expression

$$\cos \varphi_1 = \frac{\cos \omega_2 \cos \omega_3 - \cos \omega_1}{\sin \omega_2 \sin \omega_3} = \frac{A_{31}\, A_{12}\, Q_{23}^2 - A_{23}\, Q_{31}^2\, Q_{12}^2}{B_{31}\, B_{12}\, Q_{23}^2}$$

turning out to take the same value for each set $x_1(k_1)$, $x_2(k_2)$, $x_3(k_3)$, then we have also

$$\sin \psi_2 = \sin \omega_3 \sin \varphi_1 = \text{constant.}$$

We now obtain (see fig. 6):

$$\varepsilon = \Phi_2 - \overline{\Phi}_2 = -(\omega_2 - \overline{\omega}_2) \quad , \quad \overline{\omega}_2 - \varepsilon = \omega_2.$$

In this case the expression 35bis for T_3 gets rid of its term with z_2, and is reduced to

$$T_3 = \frac{\sin(\omega_2 - \overline{\omega}_2)}{\sin \psi_3 \sin \overline{\omega}_2} z_1 + \frac{1}{\sin \psi_3} z_3.$$

Here ω_2 is a function only of z_1 and z_3; so $\sin \psi_3$ ($= \sin \varphi_1 \sin \omega_2$) is.
Therefore: $\omega_3 = \text{const.}$ and $\varphi_1 = \text{const.}$ furnishes $T_1\{x_1\}$, $T_2\{x_2\}$, $T_3\{x_1, x_3\}$.

If, besides ω_3, also φ_2 turns out to be constant, the expression

$$\cos \varphi_2 = \frac{A_{12}\, A_{23}\, Q_{31}^2 - A_{31}\, Q_{12}^2\, Q_{23}^2}{B_{12}\, B_{23}\, Q_{31}^2}$$

taking the same value for each set $x_1(k_1), x_2(k_2), x_3(k_3)$, then $\varepsilon = 0$ (see fig. 7), so that T_3 drops its term with z_1, and is reduced to

$$T_3 = \frac{-\sin(\varphi_1 - \overline{\varphi_1})}{\sin \overline{\varphi_1} \sin \psi_2} z_2 + \frac{1}{\sin \psi_3} z_3.$$

Here $\sin \psi_3 = \sin \varphi_2 \sin \omega_1 \{x_2, x_3\}$ is a function only of x_2 and x_3; however the coefficient of z_2 still depends on all the variables x_1, x_2, x_3. Hence in this case T_3 is still a function of all three variables x_a.

By interchanging the subscripts we may — in the same manner — arrive at analogous relations of dependency.

We may observe, that we have computed the elements of the spherical triangle of reference exclusively by means of **E**, hence out of the magnitudes z_a, $\zeta_{b;a}$. So we have only used these magnitudes, which are much more accurate than the magnitudes Z_c, 1⁰ because, for calculating Z_c, we have much more to do with mean values (e.g. $z_a(k_a - \frac{1}{2})$) and with interpolated values (e.g. $Z_c(k_1, k_2, k_3)$), 2⁰ because the magnitudes Z_c (see **B**(3;21)) are computed from — generally small — one-dimensional frequency sums.

If **Ia** is satisfied, but **Ib** is not, then the 6 triplets $z_a, \zeta_{b;a}, Z_c$ still join to each set $x_1(k_1), x_2(k_2), x_3(k_3)$ one and the same image point $\Pi(k_1, k_2, k_3)$ on the unity-sphere (see fig. 3), the 3 expressions $r^2_{ab} = q^2_{ab} + Z^2_c$ giving however values for r^2 different from the value of $H\{z, \zeta\}$ as it is deduced solely from the magnitudes $z_a, \zeta_{b;a}$. We may now introduce new magnitudes Z'_1, Z'_2, Z'_3, which do satisfy **Ib**, in other words: we put:

$$Z'^2_c = H\{z, \zeta\} - q^2_{ab} \quad , \quad (c = 1, 2, 3) \quad . \quad . \quad . \quad . \quad \mathbf{36}$$

and consider these Z'_c as "adjusted" values of Z_c. So we replace the empirical triplets $z_a, \zeta_{b;a}, Z_c$, computed from the *given* frequency distribution, by the triplets $z_a, \zeta_{b;a}, Z'_c$. By computing back the frequency distribution corresponding to the *adjusted* triplets, this frequency distribution will appear to be somewhat different from the given one. Then we must judge whether the discrepancy thus found between the given and the computed frequency distribution may be considered as a small accidental deviation.

We are inclined to adjust in the first place the magnitudes Z_c, these being subject to the greatest uncertainty. It is, on the other hand, the great uncertainty of Z_c which makes the frequency distribution rather insensible to an alteration of Z_c.

When we, nevertheless, in computing back the frequency distribution, desire to get back *exactly* the *given* frequencies, we may succeed by keeping one of the magnitudes Z_c, for instance Z_3, unaltered. [1]) Then

[1]) We shall preferably keep that magnitude Z_c, which satisfies **Ib** best

the triplets $z_1, \zeta_{2:1}, Z_3$ and $z_2, \zeta_{1:2}, Z_3$ will give back exactly the original frequency distribution.

In this case it is necessary to take for r^2:

$$r'^2 = r_{12}^2 = q_{12}^2 + Z_3^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad 37$$

Hence $H\{z, \zeta\}$ must be adjusted to this value r_{12}^2. Thus some of the magnitudes $z_3, \zeta_{1;3}, \zeta_{3;1}, \zeta_{2;3}, \zeta_{3;2}$ must certainly undergo alteration. We naturally prefer to leave the comparatively accurate magnitude z_3 unaltered. As the two conditions

$$\zeta_{2;3}^2 - \zeta_{3;2}^2 = z_2^2 - z_3^2 \quad , \quad \zeta_{3;1}^2 - \zeta_{1;3}^2 = z_3^2 - z_1^2,$$

deriving from **Ia**, must in any case be satisfied, we must vary simultaneously either $\zeta_{2;3}$, and $\zeta_{3;2}$, or $\zeta_{3;1}$ and $\zeta_{1;3}$.

Leaving, for instance, $\zeta_{3;1}$ and $\zeta_{1;3}$ unaltered, the alteration required in adjusting $H\{z, \zeta\}$ will be effectuated solely on $\zeta_{2;3}$ and $\zeta_{3;2}$. The magnitudes $\zeta_{3;1}$ and $\zeta_{1;3}$ keeping their values and q_{31} also doing so, we must first adjust Z_2 into Z'_2, determined by

$$Z_2'^2 = r'^2 - q_{31}^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad 38$$

When we, in applying the equations **D** and **E**, interchange the spherical triangle (Φ) and its polar triangle (Ω), and take into account, that, in interchanging (Φ) and (Ω), the magnitude z_a (projection of OP on $O\Omega_a$) passes into the magnitude Z_a (projection of OP on $O\Phi_a$), the magnitude $\zeta_{b;a}$ (projection of OP on the line $O\Psi_{b;a}$ of intersection of $O\varphi_a$ and $O\omega_c$) passing into $\zeta_{b;c}$ (projection of OP on the line $O\Psi_{b;c}$ of intersection of $O\omega_a$ and $O\varphi_c$) (see fig. 1 and 3), then we obtain from $\operatorname{tg} \omega_1 = \dfrac{z_2 \zeta_{2;3} + z_3 \zeta_{3;2}}{z_2 z_3 - \zeta_{3;2} \zeta_{2;3}}$:

$$\operatorname{tg} \varphi_1 = \frac{Z_2 \zeta_{2;1} + Z_3 \zeta_{3;1}}{Z_2 Z_3 - \zeta_{3;1} \zeta_{2;1}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad 39$$

Replacing in this expression Z_2 by Z'_2, we obtain an adjusted value φ'_1 of φ_1, determined by

$$\operatorname{tg} \varphi_1' = \frac{Z_2' \zeta_{2;1} + Z_3 \zeta_{3;1}}{Z_2' Z_3 - \zeta_{3;1} \zeta_{2;1}} \quad . \; . \; . \; . \; . \; . \; . \quad 39bis$$

The arcs ω_2 and ω_3 remaining unaltered, the arc ω_1, which depends on $\zeta_{2;3}$ and $\zeta_{3;2}$, undergoes alteration. Here the formula:

$\cos \omega_1 = \cos \omega_2 \cos \omega_3 - \sin \omega_2 \sin \omega_3 \cos \varphi_1$ furnishes, if applied to the adjusted values:

$$\cos \omega_1' = \cos \omega_2 \cos \omega_3 - \sin \omega_2 \sin \omega_3 \cos \varphi_1' \quad . \; . \; . \; . \quad 40$$

The relation

$$q_{23}^2 = \sin^2 \varphi_3 \, . \, t_2^2 - 2 \cos \omega_1 \sin \varphi_2 \sin \varphi_3 \, . \, t_2 \, t_3 + \sin^2 \varphi_2 \, . \, t_3^2 =$$
$$= \frac{z_2^2 - 2 \cos \omega_1 \, . \, z_2 \, z_3 + z_3^2}{\sin^2 \omega_1}$$

enables us to compute the new value q'_{23} of q_{23}, viz.:

$$q'^2_{23} = \frac{z_2^2 - 2\cos\omega'_1 \,.\, z_2 z_3 + z_3^2}{\sin^2 \omega'_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathbf{41}$$

Putting:

$$q'^2_{23} = q^2_{23} + \delta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathbf{42}$$

the equations

$$\zeta'^2_{2;3} = \zeta^2_{2;3} + \delta \quad , \quad \zeta'^2_{3;2} = \zeta^2_{3;2} + \delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathbf{43}$$

furnish the adjusted values $\zeta'_{2;3}$ and $\zeta'_{3;2}$ of $\zeta_{2;3}$ and $\zeta_{3;2}$.

We naturally choose the average value $\overline{\omega'}_1$ of the new values ω'_1 for the constant $\overline{\omega_1}$ in the formulae **K** for T_1, T_2, T_3. Here we use the *adjusted* spherical triangles (Φ') and (Ω'), determined by the unaltered sides ω_2 and ω_3, and by the adjusted side ω'_1. This latter must be computed from **37**, **38**, **39bis**, **40**, thus according to the scheme:

$$\left.\begin{aligned}
& r'^2 = r^2_{12} = q^2_{12} + Z^2_3, \\
& Z'^2_2 = r'^2 - q^2_{31}, \\
& \text{tg}\,\varphi'_1 = \frac{Z'_2 \zeta_{2;1} + Z_3 \zeta_{3;1}}{Z'_2 Z_3 - \zeta_{3;1} \zeta_{2;1}} \\
& \cos\omega'_1 = \cos\omega_2 \cos\omega_3 - \sin\omega_2 \sin\omega_3 \cos\varphi'_1
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \mathbf{L}$$

If none of the conditions **Ia, b, II** is satisfied, then some of the magnitudes z_a, $\zeta_{b;a}$ must also be adjusted. Unless we desire, intentionally, to keep at least one of the magnitudes Z_c, we prefer to determine ω_1, ω_2, ω_3 again by the equations **E**bis, in which A_{ab} and B_{ab} are constructed out of the *unadjusted* values z_a, z_b, $\zeta_{b;a}$, $\zeta_{a;b}$.

Now we must make $q_{b;a}$ agree with $q_{a;b}$.

Referring to S. C. II, p. 408, we observe that the most preferable methods of adjustment are the following two:

a.: we keep z_a and z_b, derive ω_c from **E**bis, and adjust the values $\zeta_{b;a}$ and $\zeta_{a;b}$ to the values $\zeta'_{b;a}$ and $\zeta'_{a;b}$, which are, in virtue of **9ter (b; a)** determined by

$$\zeta'_{b;a} = \frac{z_b - \cos\omega_c \,.\, z_a}{\sin\omega_c} \quad , \quad \zeta'_{a;b} = \frac{z_a - \cos\omega_c \,.\, z_b}{\sin\omega_c} \quad . \quad . \quad . \quad \mathbf{44}$$

(see fig. 8).

So we adjust $q_{b;a}$ and $q_{a;b}$ to $q'_{b;a} = q'_{a;b} = q'_{ab}$, determined by

$$q'^2_{ab} = z^2_a + \zeta'^2_{b;a} = z^2_b + \zeta'^2_{a;b} = \frac{z^2_a - 2\cos\omega_c \,.\, z_a z_b + z^2_b}{\sin^2\omega_c} \quad . \quad . \quad \mathbf{45}$$

Then the function $r^2 = H\{z, \zeta\}$, being (by **31**) built up of the magnitudes z_a and ω_a ($a = 1, 2, 3$) may be computed immediately by means of the values $\omega_1 . \omega_2, \omega_3$ calculated from $\mathbf{E}^{\mathbf{bis}}$.

So we have evidently:

$$H\{z, \zeta'\} = H\{z, \zeta\}.$$

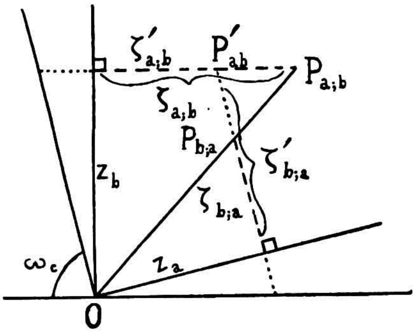

fig.8

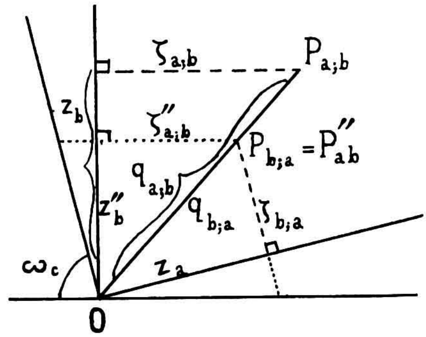

fig.9

Then the magnitudes Z_c are adjusted into Z'_c according to **36**.

The spherical triangles (Φ) and (Ω) remaining unaltered, we derive the unimodular, linearly correlated variables T_1, T_2, T_3 still from the same equations **K**, we formerly constructed.

In computing back the frequency distribution out of one of the triplets $z_a, \zeta'_{b;a}, Z'_c$, we must be prepared to find deviations from the original frequency distribution. The advantage of this method is, however, that we make as much use as possible of the magnitudes z_1, z_2, z_3, which are the most accurate ones.

b. We keep z_a and $\zeta_{b;a}$, thus also $q^2_{b;a} = z^2_a + \zeta^2_{b;a}$, and derive ω_c still from $\mathbf{E}^{\mathbf{bis}}$ (fig. 9). The magnitude z_b however must now be replaced by z''_b, which is, in virtue of $\mathbf{9}^{\mathbf{ter}}$ **(b; a)**, determined by

$$z''_b = \cos \omega_c \, . \, z_a + \sin \omega_c \, . \, \zeta_{b;a} \quad \ldots\ldots \quad \mathbf{M}$$

By means of **E** we may transform this expression into:

$$z''_b = \frac{z_a A_{ab} + \zeta_{b;a} B_{ab}}{q_{b;a} \cdot q_{a;b}} = \frac{z_a (z_a z_b - \zeta_{b;a} \zeta_{a;b}) + \zeta_{b;a} (z_a \zeta_{a;b} + z_b \zeta_{b;a})}{q_{b;a} \cdot q_{a;b}} = \frac{z_b \cdot q^2_{b;a}}{q_{b;a} \cdot q_{a;b}},$$

or

$$z''_b = \frac{q_{b;a}}{q_{a;b}} \, . \, z_b \quad \ldots\ldots\ldots \quad \mathbf{M}^{\mathbf{bis}}$$

Therefore, if we wish to keep, for instance, z_1 and $\zeta_{2;1}$, we must replace z_2 by $z''_2 = \frac{q_{2;1}}{q_{1;2}} . z_2$, and substitute this value z''_2 for z_2 in the equations **K**. The method (*b*) is to be followed, if we desire, intentionally, to get back the original frequency distribution out of one of the triplets. In this case we have, of course, to leave also Z_c (i.c. Z_3)

unaltered, and to apply the corresponding method of adjustment, however on the understanding, that we must take for r^2 the value:

$$r''^2 = r^2_{2;1} = q^2_{2;1} + Z^2_3 = z^2_1 + \zeta^2_{2;1} + Z^2_3, \quad \ldots\ldots \quad \mathbf{46}$$

so that also the values of Z'_2, φ'_1, ω'_1 turn out to be different from those derived from **L**.

If we desire to keep $\zeta_{3;1}$ and $\zeta_{1;3}$, we must, while retaining ω_2, replace these magnitudes, according to **44**, by

$$\zeta''_{3;1} = \frac{z_3 - \cos\omega_2 \cdot z_1}{\sin\omega_2}, \quad \zeta''_{1;3} = \frac{z_1 - \cos\omega_2 \cdot z_3}{\sin\omega_2} \quad \ldots \quad \mathbf{47}$$

Then we must take for q^2_{31} (see **41**):

$$q''^2_{31} = \frac{z^2_3 - 2\cos\omega_2 \cdot z_3 z_1 + z^2_1}{\sin^2\omega_2} \quad \ldots\ldots \quad \mathbf{48}$$

So we obtain for Z_2 the adjusted value Z''_2, determined by

$$Z''^2_2 = r''^2 - q''^2_{31} \quad \ldots\ldots \quad \mathbf{49}$$

By means of the value $\zeta''_{3;1}$, found from **47**, and the value Z''_2, derived from **49**, we now compute the adjusted value φ''_1 from

$$\operatorname{tg}\varphi''_1 = \frac{Z''_2\,\zeta_{2;1} + Z_3\,\zeta''_{3;1}}{Z''_2\,Z_3 - \zeta''_{3;1}\,\zeta_{2;1}} \quad \ldots\ldots \quad \mathbf{50}$$

At last we find the adjusted value ω''_1 from

$$\cos\omega''_1 = \cos\omega_2\cos\omega_3 - \sin\omega_2\sin\omega_3\cos\varphi''_1 \quad \ldots \quad \mathbf{51}$$

Then we determine the average value $\overline{\omega''_1}$ of this adjusted value ω''_1, after which we substitute this value $\overline{\omega''_1}$ for the constant $\overline{\omega_1}$ in the formulae **K**. Hence, in the actual case (z_1, $\zeta_{2;1}$, Z_3 being kept) the formulae **K** are altered in such a way that we replace $\underline{z_2}$ by $\underline{z''_2}$ (determined by **M** (**Mbis**)) and $\overline{\omega_1}$ by $\overline{\omega''_1}$ (determined by **46—51**). So the adjustment is to be effectuated according to the scheme:

$$\left.\begin{aligned}
z''_2 &= \cos\omega_3 \cdot z_1 + \sin\omega_3 \cdot \zeta_{2;1} = \frac{q_{2;1}}{q_{1;2}} \cdot z_2 \\
r''^2 &= r^2_{2;1} = z^2_1 + \zeta^2_{2;1} + Z^2_3 \\
\zeta''_{3;1} &= \frac{z_3 - \cos\omega_2 \cdot z_1}{\sin\omega_2} \\
q''^2_{31} &= \frac{z^2_3 - 2\cos\omega_2 \cdot z_3 z_1 + z^2_1}{\sin^2\omega_2} \\
Z''^2_2 &= r''^2 - q''^2_{31} \\
\operatorname{tg}\varphi''_1 &= \frac{Z''_2\,\zeta_{2;1} + Z_3\,\zeta''_{3;1}}{Z''_2\,Z_3 - \zeta''_{3;1}\,\zeta_{2;1}} \\
\cos\omega''_1 &= \cos\omega_2\cos\omega_3 - \sin\omega_2\sin\omega_3\cos\varphi''_1
\end{aligned}\right\} \quad \ldots \quad \mathbf{N}$$

Of course the other elements φ_a, ψ_a of the spherical triangles occurring in **K**, are to be recomputed from $\omega''_1, \omega_2, \omega_3$. Thereby they undergo alteration, and so do the constants $\overline{\varphi}_a, \overline{\psi}_a$ derived from $\overline{\omega''}_1, \overline{\omega}_2, \overline{\omega}_3$.

Summary of the Treatment of Skew Correlation between Three Variables:

From the given frequency distribution **A** the magnitudes $z_a, \zeta_{b;a}, Z_c$ are calculated by means of **B**.

The equations **C** (**C**bis), **D**, **E** (**E**bis), **F**, **G**, **H** define the magnitudes $q_{b;a}, Q_{ab}, A_{ab}, B_{ab}, M_{ab}, \gamma_{ab}, r^2 = H\{z, \zeta\}$, by means of which we can formulate the conditions **Ia, b, II**:

$$q_{b;a} = q_{a;b} \; (= q_{ab}) \qquad (a, b = 1, 2, 3) \quad \ldots\ldots \quad \mathbf{Ia}$$

$$Z_c^2 = H\{z, \zeta\} - q_{ab}^2 \qquad (c = 1, 2, 3) \quad \ldots\ldots \quad \mathbf{Ib}$$

$$\frac{\sqrt{1-\gamma_{ab}^2}}{\gamma_{ab}} = M_{ab} = \frac{B_{ab}}{A_{ab}} = \text{constant} \quad \ldots\ldots \quad \mathbf{II}$$

If both **Ia, b** and **II** are satisfied for each set $x_1 (k_1), x_2 (k_2), x_3 (k_3)$, there exists *linear* correlation between the unimodular variables $t_1\{x_1\}, t_2\{x_2\}, t_3\{x_3\}$, determined by the equations **J**, the total coefficients of correlation being $\gamma_{23}, \gamma_{31}, \gamma_{12} \left(\frac{\gamma_{ab}}{A_{ab}} > 0\right)$.

If **Ia, b** are satisfied, but **II** is not, then we can establish linear correlation between the (unimodular) variables $T_1\{x_1\}$, $T_2\{x_1, x_2\}$, $T_3\{x_1, x_2, x_3\}$ given by the equations **K**, the total coefficients of correlation being $\gamma_{23}, \gamma_{31}, \gamma_{12}$.

If **Ib** is not satisfied, then we may, by abandoning — if necessary — all the magnitudes Z_c, keep the variables $T_1\{x_1\}, T_2\{x_1, x_2\}, T_3\{x_1, x_2, x_3\}$ determined by **K**. If, however, we insist upon keeping Z_3 intact, then either ω_1 or ω_2 must be altered. When we keep ω_2, the magnitude ω_1 must be adjusted to the magnitude ω'_1, to be computed by means of the equations **L**. Then this ω'_1 and its average $\overline{\omega'}_1$ take the place of ω_1 and $\overline{\omega}_1$ in the formulae **K**.

If even **Ia** is not fulfilled, we may, by abandoning — if necessary — all the magnitudes $\zeta_{b;a}$, but keeping the magnitudes $\omega_1, \omega_2, \omega_3$, computed from **E**bis, retain, even in this case, the variables $T_1\{x_1\}, T_2\{x_1, x_2\}$, $T_3\{x_1, x_2, x_3\}$ defined by **K**.

If, on the contrary, we wish to keep $\zeta_{2;1}$ intact, we must replace z_2 by the magnitude z''_2, determined by **M** (**M**bis). If we desire to keep not only $\zeta_{2;1}$ but also Z_3 unaltered, we must replace the magnitudes z_2 and ω_1 by the magnitudes z''_2 and ω''_1 determined by **N**, and reconstruct the formulae **K**, using those adjusted values z''_2, ω''_1 and the average $\overline{\omega''}_1$. Then the arcs φ_a, ψ_a (a = 1, 2, 3) must first be computed from $\omega''_1, \omega_2, \omega_3$, and their averages $\overline{\varphi}_a, \overline{\psi}_a$ from $\overline{\omega''}_1, \overline{\omega}_2, \overline{\omega}_3$.

(To be continued).