Hydrodynamics. — *On the application of* OSEEN's *hydrodynamical equations to the problem of the slipstream from an ideal propeller.* By J. M. BURGERS. (Mededeeling N°. 14 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft). (Communicated by Prof. P. EHRENFEST).

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1. As is well known, OSEEN has given the hydrodynamical equations in a form, especially adapted to the treatment of problems, concerning the disturbances, caused by the presence of a body or by the action of a system of external forces, in a stream of fluid, moving originally with the constant velocity V. OSEEN has shown moreover that these differential equations can be transformed into a system of integral equations (or more exactly integro-differential equations), and much attention has been given to the question whether the latter equations can be solved by means of the method of successive approximations. This question becomes of the greatest interest when we go to the limit of vanishing viscosity (or, what comes to the same thing, of an infinite Reynolds number), as in that case we come to the problems that are of most importance in technical applications. [1])

Until now most attention has been given to the investigation of the limit to which the first approximation tends when the viscosity goes to zero. The results then obtained show some particularities, well corresponding with the results of experimental observations, whereas in other respects they still differ much from reality. On account of the discontinuities which arise in the solution when the viscosity vanishes, it is not easy to start from this limit in order to derive a second approximation. It seems necessary to calculate the second approximation before we go to zero viscosity.

Now the investigation of the flow around a body is a problem of the utmost difficulty on account of the complicated boundary conditions, to which the solution is subjected. So it seems worth while to consider in some detail the much simpler problem of the flow caused by a given system of external forces in an unlimited field. In that case moreover we have a greater freedom, as we may choose the intensity of the forces arbitrarily great or small, a freedom which of course is missing in the case of the flow along a body.

[1]) A clear exposition of OSEEN's researches is to be found in his book „Hydrodynamik" (Mathematik in Monographien und Lehrbüchern, herausgeg. von E. HILB, Bd. I, Leipzig 1927).

From the technical standpoint the most interesting problems of this kind are the case of a system of forces directed parallel to the direction of the general flow, and acting in the points of a surface, perpendicular to the flow — which in fact represents the "*actuating disc*" of the elementary theory of the propeller, as developed by R. E. FROUDE; [1]) and the case of a system of forces, acting in the points of a surface parallel to the flow, and directed normally to the latter — analogous to the *lifting surface*, considered in the theory of aeroplane wings.

In this paper we propose to consider the first case; we shall suppose, that forces of constant intensity f per unit area act in the points of a disc, defined by the equations: $x_1 = 0$, $x^2_2 + x^2_3 \leqslant a^2$ (a being the radius of the disc; the axis Ox_1 is parallel to the direction of the general flow). In this case the configuration of the field depends on two characteristic numbers: the specific loading of the disc $f/\varrho V^2$ (ϱ being the density of the fluid), and the Reynolds number $R = 2aV/\nu$ (ν: the kinematical viscosity).

We shall not come to a full investigation of the results of the successive approximations, and shall limit ourselves to an exposition of the first steps of the process. These steps, however, reveal some interesting particularities in connection with the question, whether the surface of discontinuity, as given by the solution of OSEEN's equations for $\nu \to 0$, will coincide with the boundary of the slipstream (see §§ 7, 8).

2. We shall denote the components of the velocity by $V + u_1$, u_2, u_3; the components of the external forces per unit volume by X_1, X_2, X_3; then for the case of stationary motion OSEEN's equations have the form: [2])

$$\left.\begin{aligned} \varrho\left(\nu \Delta u_i - V \frac{\partial u}{\partial x_1}\right) &= \frac{\partial q}{\partial x_i} - Y_i \\ \sum_j \frac{\partial u_j}{\partial x_j} &= 0 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; (1)$$

where:

$$q = p + \varrho g h + \tfrac{1}{2} \varrho \sum_j u_j^2 \quad . \; . \; . \; . \; . \; . \; . \; (2)$$

(p being the pressure and h the height of a point above the earth's surface), while:

$$Y_i = X_i + y_i \,, \quad y_i = -\varrho \sum_j u_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \quad . \; . \; . \; (3)$$

Introducing the function:

$$\Phi(s) = \frac{1}{k} \int_0^s \frac{1 - e^{-\alpha}}{\alpha} d\alpha \quad . \; . \; . \; . \; . \; . \; . \; (4)$$

[1]) R. E. FROUDE, On the part played in propulsion by differences of fluid pressure, Trans. Instit. Naval Architects, **30**, p. 390, 1889. Comp. further various text books and articles on the theory of the screw propeller.

[2]) C. W. OSEEN, p. 13, Eq. III^b.

with $k = V/2\nu$ and

$$s = k(r - x_1 + \xi_1), \qquad r = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2},$$

and the tensor:

$$t_{ij} = \delta_{ij} \triangle \Phi - \frac{\partial^2 \Phi}{\partial x_i \, \partial x_j} \quad \left(\delta_{ij} = \begin{cases} 0 \ (i \neq j) \\ 1 \ (i = j) \end{cases} \right) \quad . \quad . \quad . \quad . \quad (5)$$

the equations (1) can be integrated in the form: [1])

$$u_i = \frac{1}{8\pi\nu\varrho} \iiint d\xi_1 \, d\xi_2 \, d\xi_3 \sum_j t_{ij} Y_j \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For the purpose of calculation it is convenient to decompose the solution into two parts, one part representing an irrotational motion, to be derived from a potential φ, while the other part represents a motion, possessing vorticity. This is obtained when we put:

$$u_i = v_i + \partial\varphi/\partial x_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where v_i, φ are defined resp. by:

$$v_i = \frac{1}{4\pi\nu\varrho} \iiint d\xi_1 \, d\xi_2 \, d\xi_3 \frac{e^{-s}}{r} Y_i \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\varphi = \frac{-1}{8\pi\nu\varrho} \iiint d\xi_1 \, d\xi_2 \, d\xi_3 \sum_j \frac{\partial \Phi}{\partial x_j} Y_j \quad . \quad . \quad . \quad . \quad (9)$$

Finally q is given by:

$$q = \frac{1}{4\pi} \iiint d\xi_1 \, d\xi_2 \, d\xi_3 \frac{\sum_j (x_j - \xi_j) Y_j}{r^3} \quad . \quad . \quad . \quad (10)$$

3. In order to come to a first approximation to the solution we neglect the terms y_i, which are of the second degree in u_i, and replace the Y^i in formulae (8) (9) (10) by the corresponding X_i. In our case the X_i represent the system of surface forces, defined in § 1.

It is easily to be seen that this system of surface forces gives a discontinuity in the distribution of the pressure. This is actually observed in the case of a propeller, where the mean pressures just before and behind the propeller disc differ by an amount, depending on the loading. We shall come back to the determination of the pressure afterwards, and begin with that of the velocity.

As we have: $X_2 = X_3 = 0$ we deduce from equation (8) that v_2, v_3 are zero, while the expression for v_1 takes the form:

$$v_1 \text{ or } v = \frac{f}{4\pi\nu\varrho} \iint d\xi_2 \, d\xi_3 \frac{e^{-s}}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

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[1]) C. W. OSEEN, l. c., p. 36, Eq. III^d, with omission of the terms relating to the boundary F.

As further: $\partial \Phi / \partial x_1 = -(1 - e^{-s})/kr$, we find for φ:

$$\varphi = \frac{1}{4\pi\varrho V} \iint d\xi_2 \, d\xi_3 \frac{1 - e^{-s}}{r} = \frac{f}{2\varrho V} \varphi^* - \frac{\nu v}{V} \quad . \quad . \quad . \quad (12)$$

where φ^* denotes the potential due to a circular disc with constant charge 2 per unit area. This potential can be expressed by means of Bessel functions in the following way: [1])

$$\varphi^* = a \int_0^\infty \frac{d\lambda}{\lambda} e^{-\lambda |x_1|} J_0(\lambda \varpi) \, J_1(\lambda a) \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where $\varpi = \sqrt{x_2^2 + x_3^2}$. A corresponding formula can be given for the stream function, which can be transformed in various ways so as to get formulae convenient for numerical calculations. At the surface of the disc we have:

$$\left(\frac{\partial \varphi^*}{\partial x_1}\right)_{x_1 = -0} = +1, \quad \left(\frac{\partial \varphi^*}{\partial x_1}\right)_{x_1 = +0} = -1 \quad . \quad . \quad . \quad . \quad (14)$$

$$\frac{\partial \varphi^*}{\partial \varpi} = -\frac{2a}{\pi \varpi} \left[K\left(\frac{\varpi}{a}\right) - E\left(\frac{\varpi}{a}\right) \right] \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where K, E represent the complete elliptic integrals. [2])

4. We now come to the determination of v. From the point $P(x_1, x_2, x_3)$ we draw a perpendicular to the plane Ox_2x_3; let $N(0, x_2, x_3)$ be the footpoint of this perpendicular (comp. fig. 1). If $Q(0, \xi_2, \xi_3)$ denotes an arbitrary point of the area of the disc, we write $\xi_2 - x_2 = \alpha \cos \beta$, $\xi_3 - x_3 = \alpha \sin \beta$. Then in the integral (11) $d\xi_2 \, d\xi_3$ can be replaced by $\alpha \, d\alpha \, d\beta$. As r, the distance from P to Q, is equal to $\sqrt{x_1^2 + \alpha^2}$, we have $\alpha \, d\alpha = r \, dr$; hence:

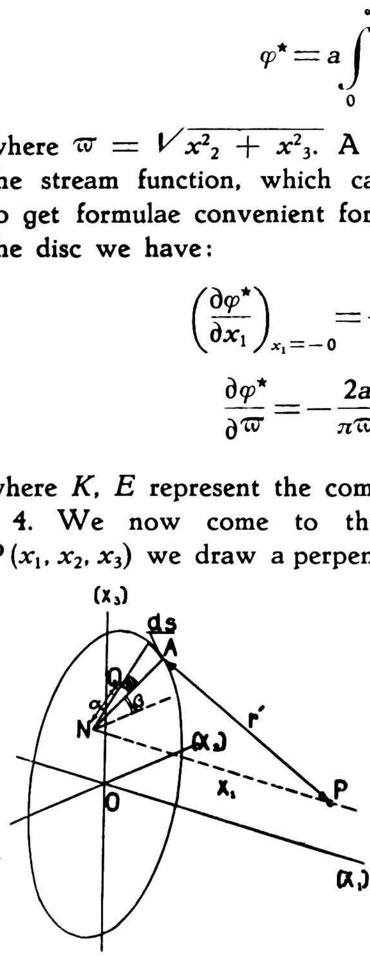

Fig. 1.

$$v = \frac{f}{4\pi\nu\varrho} \int d\beta \int dr \, e^{-k(r - x_1)}.$$

The integration according to dr can be effected, and it will easily be seen that both for the case when N lies in the interior of the disc area, and for the case when it lies outside of it, we may assume that

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[1]) Comp. H. LAMB, Hydrodynamics (Cambridge, 1917), p. 131, Art. 102, 2⁰.

[2]) The latter expression becomes (logarithmically) infinite at the boundary of the disc; this might be obviated by supposing the forces f to diminish to zero when we approach to this boundary, which would imply some changes in our formulae. For our further considerations this point, however, is of small importance.

the point Q (for constant β) moves from N to a point A on the boundary of the disc; then we may write for v:

$$v = \frac{f}{2\pi\varrho V} \int ds \frac{d\beta}{ds} [e^{-k(|x_1| - x_1)} - e^{-k(r' - x_1)}] \quad . \quad . \quad . \quad (16)$$

where ds denotes an element of the contour of the disc and r' the distance of this element from P. In the first case mentioned β increases by 2π as ds moves around the whole contour; in the second case it oscillates between two limits.

We shall suppose that the Reynolds number R is large. Then we introduce the surface:

$$(a - \varpi)^2 = \frac{2M}{k} x_1 + \frac{M^2}{k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where M represents a fixed number. This surface is obtained by the rotation of a parabola around the axis Ox_1 (comp. fig. 2); the focus of

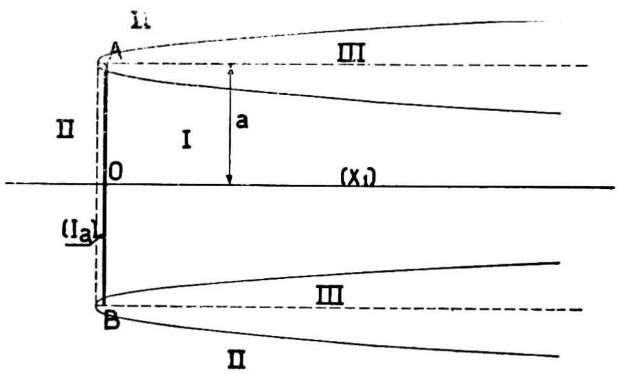

Fig. 2.

the parabola lies in the boundary of the disc (points A, B) and its axis is parallel to Ox_1. It can be shown that for all points outside of this surface the quantity $k(r' - x_1)$, which occurs in the exponent of e in the second term of formula (16), is greater than M, so that, when we choose for M say 5 or 10, the term $e^{-k(r'-x_1)}$ may be put equal to zero with sufficient accuracy. In that case we get:

$$v = \frac{f}{2\pi\varrho V} e^{-k(|x_1| - x_1)} \int ds \frac{d\beta}{ds} .$$

When we divide the whole of space into the four parts I, Ia, II, III, as indicated in fig. 2, we get:

for points within II: $v = 0$,

within I: $v = \dfrac{f}{\varrho V}$,

within Ia: $v = \dfrac{f}{\varrho V} e^{2kx_1}$.

When we restrict ourselves to that part of the region III which lies not too far from the disc, so that the thickness of the parabolical mantle is still small compared to the radius a, then in the calculation of the integral $\int ds \frac{d\beta}{ds} e^{-k(r'-x_1)}$ we may neglect the curvature of the boundary of the disc, and use the approximation (comp. fig. 3 for the definition of the angle β):

$$r' = \sqrt{x_1^2 + \frac{(a-\varpi)^2}{\cos^2\beta}}\,.$$

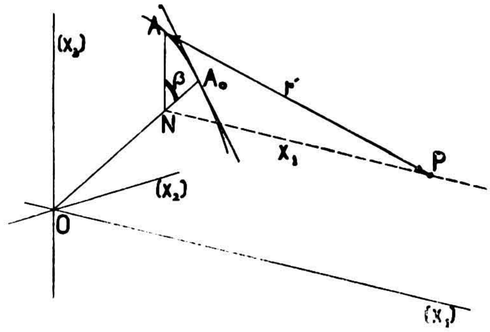

Fig. 3. $OA_0 = a$, $ON = \varpi$, $NA_0 = a - \varpi$, $NA \cong (a-\varpi)/\cos\beta$.

We then have for $\varpi < a$:

$$v = \frac{f}{\varrho V}\left[1 - \frac{1}{\pi}\int_0^{\pi/2} d\beta\, e^{-k(r'-x_1)}\right] \quad . \quad . \quad . \quad . \quad . \quad (18^a)$$

and for $\varpi > a$:

$$v = \frac{f}{\varrho V}\cdot\frac{1}{\pi}\int_0^{\pi/2} d\beta\, e^{-k(r'-x_1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18^b)$$

5. According to formula (7) we now have:

$$u_1 = v + \frac{\partial\varphi}{\partial x_1} = v - \frac{\nu}{V}\frac{\partial v}{\partial x_1} + \frac{f}{2\varrho V}\frac{\partial\varphi^*}{\partial x_1} \quad . \quad . \quad . \quad (19^a)$$

and further, writing u_ω for $\sqrt{u_2{}^2 + u_3{}^2}$:

$$u_\omega = \frac{\partial\varphi}{\partial\varpi} = -\frac{\nu}{V}\frac{\partial v}{\partial\varpi} + \frac{f}{2\varrho V}\frac{\partial\varphi^*}{\partial x_1} \quad . \quad . \quad . \quad . \quad . \quad (19^b)$$

Making use of the expressions found for v, we obtain:

within region I: $\quad v - \frac{\nu}{V}\frac{\partial v}{\partial x_1} = \frac{f}{\varrho V}$; $\quad \frac{\partial v}{\partial\varpi} = 0$

within regions Ia and II: $\quad$ „ $= 0$; $\quad$ „ $= 0$.

Hence we see that — apart from the general velocity V of the original flow — in the regions Ia and II we only have the motion defined by the potential $\frac{f}{2\varrho V}\varphi^{\star}$; in the region I we besides have a constant velocity $\frac{f}{\varrho V}$ superposed on it. As at the surface of the disc $\varphi^{\star}$ shows the discontinuity given in (14), we see that the resultant motion is continuous here.

The distribution of the velocity within the region III is given by more complicated expressions, which for the present we shall not consider. It is of interest, however, to calculate the vorticity within this region. The vortex lines are circles having the axis Ox_1 as their common axis; the vorticity (directed along the tangent to these circles) is given by:

$$w_t \text{ or } w = -\frac{\partial v}{\partial \varpi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Taking r' as the variable according to which is integrated in formulae (18^a) and (18^b), we can transform the expression for w into an integral, known from the theory of Bessel functions. In this way we obtain:

$$w = \frac{f}{2\pi\nu\varrho} e^{kx_1} K_0\left(k\sqrt{x_1^2 + (\varpi - a)^2}\right) \quad . \quad . \quad . \quad . \quad (22)$$

where K_0 denotes a Bessel function of the second kind; [1]) this expression s valid both for $\varpi < a$ and for $\varpi > a$.

The upper part of fig. 4 gives a picture of the field determined by the

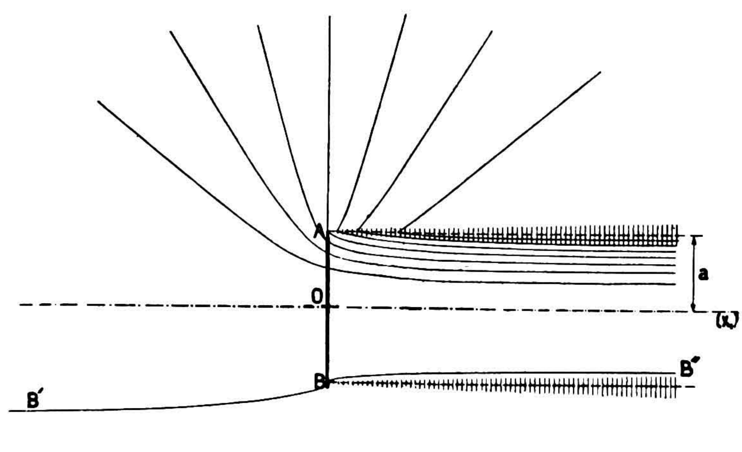

Fig. 4.

[1]) Comp. f.i. A. GRAY, G. B. MATHEWS and T. M. MAC ROBERT, Bessel functions (London, 1922); the formula to be applied is given on p. 50, eq. (29). The Bessel function $K_0(x)$ is denoted by $\frac{i\pi}{2} H_0^{(1)}(ix)$ in JAHNKE-EMDE's Funktionentafeln (Teubner 1909).

components u_1, u_{ϖ} for the case $R = \infty$. When R has the value 16.000, the streamlines have to be altered within the shaded area only (M being taken 10); the discontinuities at the cylindrical surface $\varpi = a$ then make place for a continuous change of direction. In the lower part of fig. 4 the boundary of the slipstream is indicated, which is obtained when a constant velocity V is superposed on the field of the upper part; the value of $f/\varrho V^2$ has been taken equal to 2. The form of the line $B'BB''$ is to be derived from the equation:

$$\frac{d\varpi}{dx_1} = \frac{u_{\varpi}}{V + u_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

For great values of Reynolds number the part $-\frac{\nu}{V}\frac{\partial v}{\partial \varpi}$ in u_{ϖ} is of small importance compared to the part depending on $\varphi^{\star}$. Then u_{ϖ} is negative. When $\frac{f}{\varrho V^2}$ is small, we may write the integral of (23) in the approximative form:

$$\varpi = a + \int_0^{x_1} \frac{u_{\varpi}}{V} dx_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where for u_{ϖ} we may use the approximate value:

$$u_{\varpi} \simeq \frac{f}{2\varrho V}\left(\frac{\partial \varphi^{\star}}{\partial \varpi}\right)_{\varpi = a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24^a)$$

This expression is independent of Reynolds number. On the other hand, as this number increases, the „vortex mantle" (i.e. the region III) is concentrated more and more into the cylindrical surface $\varpi = a$ ($x_1 > 0$). Hence when in the first approximation we go to infinitely great values of R, the vortex mantle does *not* coincide with the boundary of the slipstream.

Formula (10) gives us for q:

$$q = \frac{f}{4\pi}\iint d\xi_2\, d\xi_3 \frac{x_1}{r^3} = -\frac{f}{2}\frac{\partial \varphi^{\star}}{\partial x_1} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Making use of (14) we see that q — and in consequence also the ordinary pressure p — has a discontinuity of the amount f at the surface of the actuating disc. In the rest of the field q is continuous. As u_1, however, is discontinuous along the cylinder $\varpi = a$ ($x_1 > 0$), when we make R infinite, this would imply a discontinuity for p along that cylinder.

6. In order to obtain a second approximation, we calculate the quantities y_i defined in (3) from the values of u_i given by the first approximation. These quantities y_i represent the vectorial product of the velocity into the vorticity, multiplied by the density of the fluid.

They are zero everywhere in the regions I, Ia, II, where the vorticity is zero. In the region III we find, writing y_{ϖ} for $\sqrt{y^2_2 + y^2_3}$ (so that y_{ϖ} denotes the component directed radially outward from the axis Ox_1):

$$y_1 = + \varrho\, u_{\varpi}\, w\ , \qquad y_{\varpi} = - \varrho\, u_1\, w \quad . \ . \ . \ . \quad (26)$$

Some insight into the influence of the component y_{ϖ} can be get already, when we replace the actual distribution of y_{ϖ} through the region III by a surface distribution of the intensity $f'_{\varpi} = \int d\varpi\, y_{\varpi}$ on the cylinder $\varpi = a$ $(x_1 > 0)$.

When in (26) we insert the value of u_1, given by (19[a]) and observe that $\partial v/dx$ changes sign at the same time with $a - \varpi$, whereas w is an even function of $a - \varpi$, we obtain :

$$f'_{\varpi} = \int d\varpi\, y_{\varpi} = - \frac{f^2}{2\varrho V^2} \left\{ 1 + \left(\frac{\partial \varphi^*}{\partial x_1} \right)_{\varpi = a} \right\} \quad . \ . \ . \quad (27)$$

A system of surface forces of this kind gives rise to a field of motion that can be decomposed in the way as indicated in formula (7). For our purpose the most interesting part is v', which gives rise to rotational motion. For not too great values of x, we get this motion only within the region III, and in calculating the component v'_{ϖ} we may neglect the curvature of the surface of the cylinder. In this way we get:

$$v'_{\varpi} = \frac{1}{4\pi\nu\varrho} \int d\xi_1\, f'_{\varpi} \int_{-\infty}^{+\infty} d\sigma\, \frac{e^{-s}}{r},$$

where σ denotes a tangential coordinate along a circle on the cylinder considered above, and $r = \sqrt{(x_1 - \xi_1)^2 + (\varpi - a)^2 + \sigma^2}$, while $s = k(r - x_1 + \xi_1)$. This integral can be expressed by means of the Bessel function K_0 as follows : [1])

$$v'_{\varpi} = \frac{1}{2\pi\nu\varrho} \int d\xi_1\, f'_{\varpi}\, e^{k(x_1 - \xi_1)}\, K_0\left(k \sqrt{(x_1 - \xi_1)^2 + (\varpi - a)^2}\right) \quad . \quad (28)$$

The vorticity calculated from this component has the value :

$$w' = \frac{\partial v'_{\varpi}}{\partial x_1} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (29)$$

Now the vortex mantle of the first approximation had the intensity (per unit length) :

$$\gamma = \int d\varpi\, w = - \int d\varpi\, \frac{\partial v}{\partial \varpi} = \frac{f}{\varrho V} \quad . \ . \ . \ . \ . \quad (30)$$

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[1]) Comp. A. GRAY etc., l.c., p. 50, eq. (30) and the formula following it.

For the amount by which this intensity is changed in the second approximation we get, making use of some approximations for the K_0 function:

$$\gamma' = \frac{1}{\varrho V} \frac{\partial}{\partial x_1} \int d\xi_1 f'_{\varpi} e^{-k \{|x_1-\xi_1| - (x_1-\xi_1)\}} \simeq \frac{f'_{\varpi}}{\varrho V} \quad . \quad . \quad . \quad (31)$$

Hence the intensity becomes, when we make use of (27):

$$\gamma + \gamma' = \frac{f}{\varrho V}\left[1 - \frac{f}{2\varrho V^2}\left\{1 + \left(\frac{\partial \varphi^*}{\partial x_1}\right)_{\varpi = a}\right\}\right] \quad . \quad . \quad . \quad (32)$$

As the mean velocity of the vortex mantle (taken over its thickness) differs only by an amount of the order f^2 from:

$$U_m = V + \frac{f}{2\varrho V}\left\{1 + \left(\frac{\partial \varphi^*}{\partial x_1}\right)_{\varpi = a}\right\} \quad . \quad . \quad . \quad . \quad . \quad (33)$$

we see that to the order of f^2 we have

$$U_m (\gamma + \gamma') = \frac{f}{\varrho} = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

This result coincides with the well known condition for a vortex layer in a stationary field [1]).

The system of surface forces f'_{ω} along the cylindrical mantle at the same time gives a discontinuity in the quantity q. It is easily to be seen, however, that this discontinuity in q just annihilates the discontinuity that would arise in p, when we should go to an infinite value of the Reynolds number already in the first approximation.

7. In order to study the influence of the quantity y_1 it is not allowed to concentrate this force on the surface of the cylinder, as in that case we may get integrals that are no longer convergent.

Now it follows from the formulae of § 5 that a distribution of surface forces of intensity 1 per unit area over a circular area defined by $x_1 = \xi_1$, $x_2^2 + x_3^2 \leqq \varpi'^2$ gives rise to a distribution of vortex motion, determined by the expression (compare eq. 22):

$$\frac{1}{2\pi\nu\varrho} e^{k(x_1-\xi_1)} K_0 \left(k\sqrt{(x_1-\xi_1)^2 + (\varpi - \varpi')^2}\right).$$

Hence a system of forces, acting with the intensities $y_1 \, d\xi_1 \, d\varpi'$ in

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[1]) When in a stationary field of flow two regions of irrotational motion are separated by a vortex layer, we have in the first place to satisfy the condition that the normal component of the velocity at this layer has to be zero, whereas the tangential components u'_t, u''_t on both sides of the layer have to fulfill the relation: $(u'_t)^2 - (u''_t)^2 = \text{constant}$, in order that the application of BERNOULLI's theorem may give a continuous distribution of the pressure. Putting $U_m = 1/2(u'_t + u''_t)$, $\gamma = u'_t - u''_t$, we get $U_m \gamma = \text{constant}$.

the direction of the axis Ox_1 in the points of the circles $x_1 = \xi_1$, $x_2^2 + x_3^2 = \varpi'^2$, will give rise to a distribution of vortices, determined by

$$w' = \frac{1}{2\pi\nu\varrho}\int d\xi_1 \int d\varpi' \, y_1 \, e^{k(x_1-\xi_1)} \frac{\partial K_0 \left(k\sqrt{(x_1-\xi_1)^2+(\varpi-\varpi')^2}\right)}{\partial \varpi'}. \quad (35)$$

The general aspect of the distribution of vorticity thus obtained shows a direction of rotation opposite to that of the vortices obtained in the first approximation in the points *outside* of the cylinder $\varpi = a$ $(x_1 > 0)$, whereas *within* this cylinder it shows the same direction of rotation as the original vortices. Hence the effect of the correction is to diminish the intensity of the vorticity outside of the cylinder, and to increase the intensity inside of it; this can be interpreted as a tendency to a *contraction* of the vortex mantle. — The total intensity per unit length of the additional vorticity, as defined by the integral $\int d\varpi \, w'$, has the value zero.

We may get an estimate of the contraction mentioned by calculating the impulse per unit length of the system of vortices w'. The impulse of a vortex ring of radius ϖ and intensity $w' d\varpi$ is given by the expression $\pi \varrho \varpi^2 w' d\varpi$; inserting the value (35) and integrating, first with respect to $d\varpi$, then with respect to $d\varpi'$ and $d\xi_1$, and besides making use of some approximations for the K_0 function, we get with sufficient accuracy for our present purpose (for not too great values of x_1):

$$J' = \frac{2\pi f a}{V^2}\int_0^{x_1} (u_\varpi)_{\varpi=a} \, d\xi_1 \quad . \; . \; . \; . \; . \; . \quad (36)$$

where for u_ϖ we may use the approximate value (24^a). As the impulse of the vortex layer in the first approximation had the value (per unit length) $\pi \varrho a^2 \gamma = \pi a^2 f / V$ and as the intensity per unit length is not changed by the addition of w', we may calculate the change δa of the "equivalent radius" by means of the formula:

$$\delta (\pi a^2 f / V) = 2\pi a \, \delta a \, f / V = J'.$$

This gives us:

$$\delta a = \frac{J' V}{2\pi a f} = \int_0^{x_1} \frac{(u_\varpi)_{\varpi=a}}{V} d\xi_1 \quad . \; . \; . \; . \; . \; . \; . \quad (37)$$

It is easily to be seen that this „equivalent radius" may be considered as the distance of the „centre of gravity" of an element of the vortex layer from the axis Ox_1. Comparing the result given by (37) with that of equation (24) we see that the „equivalent radius" of the vortex layer in the second approximation has just the value it would have obtained when the vortex layer was concentrated wholly along the boundary of the slipstream.

8. Our survey of the corrections obtained from the second approximation is still very incomplete, especially as we have not considered the change produced in the potential φ. However, judging by the results of §§ 6, 7, it seems allowed to state that for great values of Reynolds number there are some tendencies to approach to the picture of the motion, which is to be expected on general grounds. The intensity per unit length of the vortex layer becomes inversely proportional to its mean velocity; the pressure becomes continuous; and the distribution of the vorticity over the thickness of the layer is changed in such a way, that the „centre of gravity" of an element of the layer falls in the boundary of the slipstream.

But when now at once we should draw the conclusion that the vortex layer for $R \rightarrow \infty$ coincides with the boundary of the slipstream, we might make a mistake. When in the second approximation we go to $R \rightarrow \infty$, keeping $f/\varrho V^2$ constant, we get again the vortex layer on the cylindrical surface $\varpi = a$ $(x_1 > 0)$, combined with a vortex *double layer* of such an intensity, that the system as regards to its impulse is equivalent to a vortex layer, lying on the boundary of the slipstream.

Until now we have limited ourselves to the consideration of the first and second approximations. In order to obtain the successive approximations in regular order, it is advisable to write the solution in the form of series proceeding according to powers of $f/\varrho V^2$; then the successive terms of these series can be obtained from equation (6) one after the other. It would seem that the terms of these series will show singularities for $R = \infty$; it is well possible, however, that the series rest summable in a certain way and that their sums will represent the real motion with a vortex layer coinciding with the slipstream boundary.

That such a behaviour of the series is not wholly improbable may be seen from the following, much simplified example. We shall suppose that the various terms of the series for the vorticity $\Sigma\, w^{(n)}$ are obtained by means of the formula (compare formulae 35 and 26):

$$w^{(n)} = \frac{1}{2\pi\nu} \int d\xi_1 \int d\varpi'\, u_{\varpi}\, w^{(n-1)}\, e^{k(x_1-\xi_1)} \frac{\partial K_0 \left(k \sqrt{(x_1-\xi_1)^2 + (\varpi-\varpi')^2}\right)}{\partial \varpi'} .$$

Here for u_{ϖ} we shall take the *same* quantity in all integrals; moreover we shall suppose that it is a function of ξ_1 only (not depending on ϖ'). Besides we replace the Bessel function by the first term of its asymptotic expansion, so that for $x_1 > \xi_1$:

$$e^{k(x_1-\xi_1)} K_0 \left(k \sqrt{(x_1-\xi_1)^2 + (\varpi-\varpi')^2}\right) \simeq \sqrt{\frac{\pi}{2k(x_1-\xi_1)}}\, e^{-\frac{(\varpi-\varpi')^2}{2k(x_1-\xi_1)}} ,$$

while for $\xi_1 > x_1$ this expression is replaced by zero. In this way we obtain, when we replace $\partial/\partial\varpi'$ by $-\partial/\partial\varpi$:

$$w^{(n)} = \frac{-1}{2\pi\nu} \frac{\partial}{\partial\varpi} \int_0^{x_1} d\xi_1 \int d\varpi'\, u_{\omega}\, w^{(n-1)} \sqrt{\frac{\pi}{2k(x_1-\xi_1)}}\, e^{-\frac{(\varpi-\varpi')^2}{2k(x_1-\xi_1)}} . \quad (38)$$

For $w^{(0)}$ we take (compare formula 22):

$$w^{(0)} = \frac{f}{2\pi\nu\varrho}\sqrt{\frac{\pi}{2kx_1}}\, e^{-\frac{k(\varpi - a)^2}{2x_1}} \quad . \; . \; . \; . \; . \; . \; . \quad (a)$$

Formula (38) then gives:

$$w^{(1)} = \frac{f}{2\pi\nu\varrho}\,\psi\sqrt{\frac{\pi}{2kx_1}}\,\frac{\partial}{\partial a}\, e^{-\frac{k(\varpi - a)^2}{2x_1}} \quad . \; . \; . \; . \; . \; . \quad (b)$$

where $\partial/\partial\varpi$ now has been replaced by $-\partial/\partial a$, while $\psi = \int\limits_0^{x_1} \frac{u_{\varpi}}{V}\, d\xi_1$.

The next integration gives us:

$$w^{(2)} = \frac{f}{2\pi\nu\varrho}\,\frac{\psi^2}{2}\sqrt{\frac{\pi}{2kx_1}}\,\frac{\partial^2}{\partial a^2}\, e^{-\frac{k(\varpi - a)^2}{2x_1}} \quad . \; . \; . \; . \; . \; . \quad (c)$$

In this way we obtain the series:

$$\Sigma\, w^{(n)} = \frac{f}{2\pi\nu\varrho}\sqrt{\frac{\pi}{2kx_1}}\,\sum_0^{\infty}\frac{\psi^n}{n!}\,\frac{\partial^n}{\partial a^n}\, e^{-\frac{k(\varpi - a)^2}{2x_1}} \quad . \; . \; . \; . \quad (39)$$

As this series has the form of a Taylor series, we are induced to write:

$$\Sigma\, w^{(n)} = \frac{f}{2\pi\nu\varrho}\sqrt{\frac{\pi}{2kx_1}}\, e^{-\frac{k(\varpi - a - \psi)^2}{2x_1}} \quad . \; . \; . \; . \quad (40)$$

which expression in fact represents a vortex layer, distributed symmetrically to both sides of the surface:

$$\varpi = a + \psi = a + \int\limits_0^{x_1} \frac{u_{\varpi}}{V}\, d\xi_1.$$

Comparison with formula (24) shows that the latter surface represents the boundary of the slipstream.

Of course more rigourous calculations are necessary, before the question of the summation of the series can be settled.