Mathematics. — *Adjustment of N Points (in n-dimensional Space) to the best linear (n—1)-dimensional Space.* I. By Prof. M. J. VAN UVEN. (Communicated by Prof. A. A. NIJLAND).

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The present paper deals with the problem: to fit a linear space τ of $n-1$ dimensions (hyperplane) through a certain number (N) of points in a linear space of n dimensions, or, expressed analytically, to determine the constants of that equation $p_0 + p_1 x_1 + p_2 x_2 + \ldots + p_n x_n = 0$ which agrees best with the N sets of coordinates $x_1, x_2, \ldots x_n$.

We shall distinguish the given points S from one another by an index in brackets. So the point $S(m)$ has the coordinates $x_i(m)$; $i = 1, \ldots n$; $m = 1, \ldots N$. A summation over the n coordinates will be indicated by $\sum\limits_{\lambda=1}^{n}$, or, if no misunderstanding is to be feared, by $\sum\limits_{\lambda}$, or, more simply, by Σ. On the other hand a summation over the N points will be designated by [ ].

We want then to determine the ratios of the constants $p_0, p_1, p_2, \ldots, p_n$ of the equation

$$p_0 + p_1 x_1 + p_2 x_2 + \ldots + p_n x_n \equiv p_0 + \sum_{\lambda=1}^{n} p_\lambda x_\lambda = 0$$

in such a way, that the given coordinates $x_i(m)$ satisfy this equation as well as possible.

Instead of operating with the ratios of the constants (parameters) $p_0, p_1, p_2, \ldots, p_n$, we may normalize them in some way, either by considering $p_1, p_2, \ldots, p_n$ as the direction-cosines of the normal of τ ($\Sigma p_\lambda^2 = 1$), or by some other method.

§ 1. *Solution of the problem.*

As a rule the best hyperplane τ will not pass exactly through any of the given points. Thus we shall be obliged to shift the points $S(m)$ to other points $T(m)$ (with coordinates $X_i(m)$) which do lie in τ and therefore really satisfy

$$p_0 + p_1 X_1 + p_2 X_2 + \ldots + p_n X_n = p_0 + \sum_{\lambda=1}^{n} p_\lambda X_\lambda = 0. \quad . \quad . \quad (1)$$

The deviations $\overrightarrow{T(m)\,S(m)}$ of the given or "observed" points $S(m)$ from the "adjusted" points $T(m)$ have the projections

$$\xi_i(m) = x_i(m) - X_i(m) \qquad m = 1, \ldots, N \quad ; \quad i = 1, \ldots, n \quad . \quad (2)$$

In the observed point $S(m)$ the expression $p_0 + \Sigma p_\lambda x_\lambda$ assumes the value

$$q_0(m) = p_0 + \sum_{\lambda=1}^{n} p_\lambda x_\lambda(m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This value is, if not equal, at least proportional to the distance of the point $S(m)$ from the hyperplane τ.

Now we have, by (1) and (2),

$$q_0(m) = p_0 + \sum_\lambda p_\lambda X_\lambda(m) + \sum_\lambda p_\lambda \xi_\lambda(m) = \sum_\lambda p_\lambda \xi_\lambda(m) \quad . \quad . \quad (4)$$

We consider the observed point S as that position of T, which is most probable a priori.

The projections $-\xi_i$ of the displacements $\overrightarrow{ST}$ are supposed to be subject to the general n-dimensional probability-law:

$$dW = \left(\frac{F'}{\pi^n}\right)^{1/2} . e^{-f'} . d\xi_1 . d\xi_2 \ldots d\xi_n ,$$

where

$$f' \equiv \sum_{\lambda=1}^{n} \sum_{\mu=1}^{n} f'_{\lambda\mu} \xi_\lambda \xi_\mu$$

is a positive-definite homogeneous quadratic form, and F' the determinant $F' = |f'_{\lambda\mu}|$, the minor (algebraic complement) of f'_{ij} being denoted by F'_{ij}.

We assume that the above n-dimensional probability-formula is the same for all the points of the n-dimensional space. This formula indicates as it were the movability in the different directions. Since we can only make suppositions about the relative movability in the different directions, we cannot prescribe beforehand the coefficients f'_{ij}, but only their ratios.

Thus, putting

$$f'_{ij} = \theta f_{ij}$$

we may give the quantities f_{ij}, leaving the value of the constant θ unsettled for the present.

Putting

$$f = \sum_{\lambda=1}^{n} \sum_{\mu=1}^{n} f_{\lambda\mu} \xi_\lambda \xi_\mu \quad , \quad F = |f_{\lambda\mu}| \quad \text{(with minors } F_{ij}\text{)} \quad . \quad . \quad . \quad (5)$$

we have

$$f' = \theta . f \quad , \quad F' = \theta^n . F \quad , \quad F'_{ij} = \theta^{n-1} . F_{ij} .$$

So the probability-formula for the deviation $(\xi_1, \xi_2, \ldots \xi_n)$ becomes:

$$dW = \left(\frac{\theta^n F}{\pi^n}\right)^{1/2} . e^{-\theta f} d\xi_1 . d\xi_2 \ldots d\xi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This probability-formula shows that the extremities of equally probable displacements lie on a hyper-ellipsoid

$$f = \sum_{\lambda} \sum_{\mu} f_{\lambda\mu}\, \xi_{\lambda}\, \xi_{\mu} = \text{const.}$$

around the centre S.

In order to facilitate the study of the conditions in the given anisotropic space, we shall transform it into an isotropic space. For this purpose we put firstly:

$$f_{ii} = h_i^2 \quad , \quad f_{ij} = g_{ij}\, h_i\, h_j \qquad (\text{whence } g_{ii} = 1) \quad . \quad . \quad . \quad (7)$$

Since f must be positive-definite, the coefficients g_{ij} must lie between -1 and $+1$.

Further, putting

$$h_i\, \xi_i = \eta_i \,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

the form f passes into

$$g \equiv \sum_{\lambda} \sum_{\mu} g_{\lambda\mu}\, \eta_{\lambda}\, \eta_{\mu}$$

Interpreting $\eta_i (i = 1, 2, \ldots n)$ as coordinates in a skew rectilinear system of reference (the axes of η_i and η_j including an angle the cosine of which is g_{ij}), the equation

$$g \equiv \sum_{\lambda} \sum_{\mu} g_{\lambda\mu}\, \eta_{\lambda}\, \eta_{\mu} = r^2 \,(= \text{const.}) \quad . \quad . \quad . \quad . \quad . \quad (9)$$

represents a hypersphere with radius r.

In the system (η) the hyperplane τ obtains the equation

$$\sum_{\lambda=1}^{n} p_{\lambda}\, \xi_{\lambda} = \sum_{\lambda=1}^{n} \left(\frac{p_{\lambda}}{h_{\lambda}}\right) \eta_{\lambda} = q_0,$$

or, putting

$$\frac{p_i}{h_i} = q_i \,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\sum_{\lambda=1}^{n} q_{\lambda}\, \eta_{\lambda} = q_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

We next consider the distance of the point $S\,(\eta_i = 0)$ from this hyperplane, or, in other words, the radius r of that hypersphere (9) which touches the hyperplane (11).

Denoting by η_i' the coordinates of the point T' of contact, we have for the tangent hyperplane of T'

$$\sum_{\lambda} \left(\sum_{\mu} g_{\lambda\mu}\, \eta_{\mu}'\right) \eta_{\lambda} = r^2.$$

Comparing this equation with (11), we obtain

$$\sum_{\mu} g_{k\mu}\, \eta_{\mu}' = \frac{r^2}{q_0} \cdot q_k \qquad k = 1, 2, \ldots, n \quad . \quad . \quad . \quad . \quad (12)$$

Putting

$$|g_{\lambda\mu}| = G \;\;,\;\; \text{(with minors } G_{ij}\text{)},$$

we derive from (12)

$$\eta'_i = \frac{r^2}{q_0} \sum_\lambda \frac{G_{\lambda i}}{G} \cdot q_\lambda \,.$$

So the condition, that the point T' (η') lies on the hypersphere, furnishes the relation

$$\sum_\rho \sum_\sigma g_{\rho\sigma} \cdot \frac{r^4}{q_0^2\, G^2} \sum_\lambda \sum_\mu G_{\lambda\rho}\, q_\lambda\, G_{\mu\sigma}\, q_\mu = r^2 \quad . \quad . \quad . \quad . \quad (13)$$

Introducing, for the sake of brevity, the symbol δ_{ij}, defined by

$$\delta_{ii} = 1 \quad , \quad \delta_{ij} = 0 \quad \text{for } j \neq i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

we have

$$\sum_\rho g_{\rho\sigma}\, G_{\lambda\rho} = \delta_{\lambda\sigma} \,.\, G, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

whereby (13) is transformed into

$$\frac{r^2}{q_0^2\, G^2} \sum_\lambda \sum_\mu \sum_\sigma \delta_{\lambda\sigma}\, G\, q_\lambda\, G_{\mu\sigma}\, q_\mu = 1,$$

or

$$\frac{r_2}{G} \sum_\lambda \sum_\mu G_{\lambda\mu}\, q_\lambda\, q_\mu = q_0^2,$$

or

$$r^2 = \frac{q_0^2}{\sum_\lambda \sum_\mu \frac{G_{\lambda\mu}}{G}\, q_\lambda\, q_\mu} \,.$$

So we find for the square of the distance $r(m)$ between the point $S(m)$ and the hyperplane τ

$$r^2(m) = \frac{q_0^2(m)}{\sum_\lambda \sum_\mu \frac{G_{\lambda\mu}}{G}\, q_\lambda\, q_\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

It is now easy to formulate the most natural principle of adjustment: In isotropic space we postulate, that the mean square of the distance $r(m)$ shall be a minimum, or

$$\varphi = \frac{[r^2(m)]}{N} \quad \text{minimum} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In order to interpret this condition in the original data, we must return to the coordinates x_i (resp. ξ_i) and the coefficients f_{ij}. From (7) follows

$$F = h_1^2\, h_2^2 \ldots h_n^2 \,.\, G \quad , \quad F_{ij} = \frac{h_1^2\, h_2^2 \ldots h_n^2}{h_i\, h_j} \,.\, G_{ij},$$

whence

$$\frac{G_{ij}}{G} = h_i\, h_j \cdot \frac{F_{ij}}{F} \,.$$

Thus, by (10),

$$\frac{G_{ij}}{G} q_i \, q_j = \frac{F_{ij}}{F} p_i \, p_j \quad . \; . \; . \; . \; . \; . \; . \; . \quad (18)$$

Moreover we have, by (3),

$$q_0 = p_0 + \sum_{\lambda} p_\lambda \, x_\lambda \, .$$

Putting finally

$$\frac{F_{ij}}{F} = a_{ij} , \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

whence

$$| a_{\lambda\mu} | = A = \frac{1}{F} \qquad \left(\text{with minors } A_{ij} = \frac{f_{ij}}{F} \right),$$

we find for φ

$$\varphi = \frac{\frac{1}{N} [(p_0 + p_1 \, x_1 + p_2 \, x_2 + \ldots + p_n \, x_n)^2]}{\sum_{\lambda} \sum_{\mu} a_{\lambda\mu} \, p_\lambda \, p_\mu} \quad . \; . \; . \quad (20)$$

Our problem may therefore be formulated as follows:

To determine the parameters $p_0, p_1, p_2, \ldots, p_n$ of the hyperplane τ in such a way, that the function φ be a minimum.

Putting

$$\frac{[x_i]}{N} = \bar{x}_i \; , \quad x_i = \bar{x}_i + u_i \qquad (\text{whence } [u_i] = 0) \; i = 1, \ldots n, \; . \quad (21)$$

the numerator of (20) passes into

$$\frac{1}{N} [\{ (p_0 + p_1 \, \bar{x}_1 + p_2 \, \bar{x}_2 + \ldots + p_n \, \bar{x}_n) + (p_1 \, u_1 + p_2 \, u_2 + \ldots + p_n \, u_n) \}^2] =$$

$$= (p_0 + p_1 \, \bar{x}_1 + p_2 \, \bar{x}_2 + \ldots + p_n \, \bar{x}_n)^2 + 2 \, (p_0 + p_1 \, \bar{x}_1 + p_2 \, \bar{x}_2 + \ldots + p_n \, \bar{x}_n) \times$$

$$\times \frac{1}{N} [p_1 \, u_1 + p_2 \, u_2 + \ldots + p_n \, u_n] + \frac{1}{N} [(p_1 \, u_1 + p_2 \, u_2 + \ldots + p_n \, u_n)^2],$$

or, by (21) : $[u_i] = 0$, into

$$(p_0 + p_1 \, \bar{x}_1 + p_2 \, \bar{x}_2 + \ldots + p_n \, \bar{x}_n)^2 + \frac{1}{N} \sum_{\lambda=1}^{n} \sum_{\mu=1}^{n} [u_\lambda \, u_\mu] \, p_\lambda \, p_\mu \, .$$

We put

$$\frac{1}{N} [u_i \, u_j] = b_{ij} \quad , \quad B = | b_{\lambda\mu} | \qquad (\text{with minors } B_{ij}) \; . \; . \quad (22)$$

moreover

$$\sum_{\lambda} \sum_{\mu} a_{\lambda\mu} \, p_\lambda \, p_\mu = \alpha , \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (23)$$

$$\sum_{\lambda} \sum_{\mu} b_{\lambda\mu} \, p_\lambda \, p_\mu = \beta , \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (24)$$

whence we may write for φ (20):

$$\varphi = \frac{(p_0 + p_1 \bar{x}_1 + p_2 \bar{x}_2 + \ldots + p_n \bar{x}_n)^2 + \beta}{\alpha} \quad . \quad . \quad . \quad (25)$$

The condition φ minimum requires:

$$\frac{\partial \varphi}{\partial p_0} = 0 \ , \ \frac{\partial \varphi}{\partial p_1} = 0 \ , \ \frac{\partial \varphi}{\partial p_2} = 0, \ldots, \frac{\partial \varphi}{\partial p_n} = 0.$$

Since neither α nor β contains the parameter p_0, we have

$$\frac{\partial \varphi}{\partial p_0} = \frac{2\,(p_0 + p_1 \bar{x}_1 + p_2 \bar{x}_2 + \ldots + p_n \bar{x}_n)}{\alpha},$$

so that $\frac{\partial \varphi}{\partial p_0} = 0$ is equivalent to

$$p_0 + p_1 \bar{x}_1 + p_2 \bar{x}_2 + \ldots + p_n \bar{x}_n = 0 \quad . \quad . \quad . \quad . \quad . \quad (26)$$

This equation expresses that the "best" hyperplane τ must pass through the "mean" point $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$.

Thus the form φ is reduced to

$$\varphi = \frac{\beta}{\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where α and β are positive-definite quadratic functions of $p_1, p_2, \ldots p_n$.

From

$$\log \varphi = \log \beta - \log \alpha$$

ensues

$$\frac{1}{\varphi} \cdot \frac{\partial \varphi}{\partial p_i} = \frac{1}{\beta} \cdot \frac{\partial \beta}{\partial p_i} - \frac{1}{\alpha} \cdot \frac{\partial \alpha}{\partial p_i},$$

so that for the condition $\frac{\partial \varphi}{\partial p_i} = 0$ can be written:

$$\frac{\dfrac{\partial \beta}{\partial p_i}}{\dfrac{\partial \alpha}{\partial p_i}} = \frac{\beta}{\alpha} = \varphi,$$

or, by

$$\frac{\partial \alpha}{\partial p_i} = 2 \sum_{\lambda=1}^{n} a_{\lambda i} p_\lambda \ , \ \frac{\partial \beta}{\partial p_i} = 2 \sum_{\lambda=1}^{n} b_{\lambda i} p_\lambda \ ,$$

$$\frac{\sum_\lambda b_{\lambda i} p_\lambda}{\sum_\lambda a_{\lambda i} p_\lambda} = \varphi \qquad i = 1, 2, \ldots, n,$$

or

$$\sum_\lambda (b_{\lambda i} - \varphi\, a_{\lambda i})\, p_\lambda = 0 \qquad i = 1, 2, \ldots, n \quad . \quad . \quad . \quad . \quad (28)$$

Putting

$$b_{ij} - \varphi\, a_{ij} = c_{ij}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

the conditions (28) run

$$\sum_{\lambda=1}^{n} c_{\lambda i}\, p_\lambda = 0 \qquad i = 1, 2, \ldots, n \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

So we arrive at n homogeneous linear equations in the n coefficients $p_1, p_2, \ldots p_n$, considered as variables.

In order that they shall be soluble, it is necessary that

$$C = |\, c_{\lambda\mu}\,| = |\, b_{\lambda\mu} - \varphi\, a_{\lambda\mu}\,| = 0 \; . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

As a_{ij} and b_{ij} are the coefficients of positive-definite quadratic forms, the equation (31), of the n^{th} degree in φ, has n positive real roots. The smallest of these (φ_0) furnishes the minimum-value of φ.

The equations (28) which now take the form

$$\sum_{\lambda=1}^{n} (b_{\lambda i} - \varphi_0\, a_{\lambda i})\, p_\lambda = 0 \qquad i = 1, 2, \ldots, n \quad . \quad . \quad . \quad . \qquad (32)$$

determine the ratios of the parameters $p_1, p_2, \ldots, p_n$, while the condition (26) furnishes the corresponding value of p_0.

In what follows we shall still denote $b_{ij} - \varphi_0\, a_{ij}$ by c_{ij}, and the determinant $|\, b_{\lambda\mu} - \varphi_0\, a_{\lambda\mu}\,|$ by C.

Leaving aside one of the n equations $\sum_{\lambda=1}^{n} c_{\lambda i} p_\lambda = 0$ (32), for instance $\sum_{\lambda=1}^{n} c_{\lambda j} p_\lambda = 0$, we find

$$\frac{p_1}{C_{1j}} = \frac{p_2}{C_{2j}} = \ldots = \frac{p_n}{C_{nj}} = \omega_j \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (33)$$

for each index j.

Hence

$$\frac{p_i}{p_k} = \frac{C_{ij}}{C_{kj}} = \frac{C_{ii}}{C_{ki}} = \frac{C_{ik}}{C_{kk}},$$

or, taking account of the symmetry of C ($C_{ik} = C_{ki}$),

$$\left(\frac{p_i}{p_k}\right)^2 = \frac{C_{ii}}{C_{ki}} \times \frac{C_{ik}}{C_{kk}} = \frac{C_{ii}}{C_{kk}},$$

whence

$$\frac{p_1}{\sqrt{C_{11}}} = \frac{p_2}{\sqrt{C_{22}}} = \ldots = \frac{p_n}{\sqrt{C_{nn}}} = \omega_0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

and

$$a = \sum_\lambda \sum_\mu a_{\lambda\mu}\, p_\lambda\, p_\mu = \omega_0^2 \sum_\lambda \sum_\mu a_{\lambda\mu} \sqrt{C_{\lambda\lambda}\, C_{\mu\mu}} = \omega_0^2 \sum_\lambda \sum_\mu C_{\lambda\mu}\, a_{\lambda\mu} \, . \quad (35)$$

The form $\sum_\lambda \sum_\mu C_{\lambda\mu} a_{\lambda\mu} = \sum_\lambda \sum_\mu a_{\lambda\mu} \dfrac{\partial C}{\partial c_{\lambda\mu}}$ is the first so-called "emanant" of the determinant C, with respect to the determinant $A = |\, a_{\lambda\mu}\,|$ (Aronholdprocess). Denoting the 1st, 2nd, ... k^{th} emanant by $UC, U^2C, \dots U^kC$, we have:

$$\left.\begin{aligned} UC &= \sum_\lambda \sum_\mu C_{\lambda\mu}\, a_{\lambda\mu} \\ U^2C &= \sum_\lambda \sum_\mu \sum_\rho \sum_\sigma C_{\lambda\mu,\rho\sigma}\, a_{\lambda\mu}\, a_{\rho\sigma} \\ &\vdots \\ U^nC &= n!\, A \end{aligned}\right\} \quad \dots\dots \quad (36)$$

§ 2. *Degree of Uncertainty of the parameters* $p_0, p_1, p_2, \dots p_n$.

Our next step is to estimate the accuracy of the solution.

An alteration of the position of the observed points will cause a displacement of the best hyperplane τ. So we have to investigate the oscillations to which the parameters $p_0, p_1, p_2, \dots p_n$ are subject. The coefficients a_{ij}, being given a priori, are not affected by observational errors. So the uncertainty of the p_i is merely due to that of the quantities C_{ij}, these latter being functions of the quantities c_{ij}. From (28) ensues that the uncertainty of these c_{ij} depends only on that of the quantities b_{ij} and φ_0. Hence we must first determine the degree of uncertainty (error) of the quantities b_{ij} and of φ_0, which in its turn depends on the b_{ij}.

Operating with one variable only, the observed values must be adjusted to a "most probable" value (as a rule: the arithmetical mean). Calling this most probable value: the "solution-value", the difference between the observed value and the solution-value is called the apparent error, in distinction from the essentially unknown true error.

Likewise we shall, also in the present case, denote the coordinates $X_i^*(m)$ of the adjusted point $T(m)$ by: the "solution-values" of the coordinates $x_i(m)$, and the differences, viz. the quantities $\xi_i(m)$ will be called: the "apparent errors" of the coordinates of $S(m)$.

Next to these apparent errors of the coordinates $x_i(m)$ we consider the — essentially unknown — "true errors" $\triangle x_i(m)$. These true errors of the coordinates $x_i(m)$ are transmitted to the quantities $u_i(m)$ (defined by (21)) and likewise to the quantities $b_{ij} = \frac{1}{N}[u_i u_j]$; afterwards to φ_0 and to the quantities c_{ij} and C_{ij}.

So we first proceed to the investigation of the true errors of the b_{ij}, and will, more particularly, try to determine the mean value $M(\triangle b_{ij} \triangle b_{kl})$ of the product of the true errors $\triangle b_{ij}$ and $\triangle b_{kl}$.

As $\triangle x_i(m)$ is the true error of $x_i(m)$, we derive from

$$u_i(m) = x_i(m) - \overline{x}_i = \frac{-x_i(1) - x_i(2) - \dots + (N-1)\, x_i(m) - \dots - x_i(N)}{N}$$

the formula

$$\triangle u_i(m) = \frac{-\triangle x_i(1) - \triangle x_i(2) - \dots + (N-1) \triangle x_i(m) - \dots - \triangle x_i(N)}{N}. \quad (37)$$

From $b_{ij} = \frac{1}{N}[u_i\ u_j]$ ensues

$$N \triangle b_{ij} = [u_j \triangle u_i] + [u_i \triangle u_j] \quad , \quad N \triangle b_{kl} = [u_l \triangle u_k] + [u_k \triangle u_l],$$

whence

$$\left. \begin{aligned} N^2 \triangle b_{ij} \triangle b_{kl} = [u_j \triangle u_i]\,[u_l \triangle u_k] + [u_j \triangle u_l]\,[u_k \triangle u_l] + \\ + [u_i \triangle u_j]\,[u_l \triangle u_k] + [u_i \triangle u_j]\,[u_k \triangle u_l]. \end{aligned} \right\} \quad (38)$$

Considering the first term of the second member apart, we have:

$$\begin{aligned} [u_j \triangle u_i]\,[u_l \triangle u_k] = [u_j\,(\mu)\, u_l\,(\mu)\,.\,\triangle u_i\,(\mu) \triangle u_k\,(\mu)] + \\ + [[u_j\,(\mu)\, u_l\,(\nu)\,.\,\triangle u_i\,(\mu) \triangle u_k\,(\nu)\,]], \end{aligned}$$

where the sum [[ ]] extends to the $N(N-1)$ terms in which $\nu \neq \mu$.

We must now occupy ourselves with the mean values of these expressions. Denoting the mean value of the quantity R by $M(R)$, and taking into account that the variations $\triangle u_i(m)$ are independent of the quantities $u_i(m)$ themselves (these latter were merely introduced for the purpose of calculation), we may write:

$$\begin{aligned} M([u_j \triangle u_i]\,[u_l \triangle u_k]) = M([u_j\,(\mu)\,.\,u_l\,(\mu)]) \times M(\triangle u_i\,(\mu)\,.\,\triangle u_k\,(\mu)) + \\ + M([[u_j\,(\mu)\,.\,u_l\,(\nu)]]) \times M(\triangle u_i\,(\mu)\,.\,\triangle u_k\,(\nu)). \end{aligned}$$

From (21) and (22) ensues:

$$\begin{aligned} 0 = [u_j\,(\mu)]\,[u_l\,(\mu)] = [u_j\,(\mu)\,.\,u_l\,(\mu)] + [[u_j\,(\mu)\,.\,u_l\,(\nu)]] = \\ = N\,b_{jl} + [[u_j\,(\mu)\,.\,u_l\,(\nu)\,]], \end{aligned}$$

whence

$$M([[u_j\,(\mu)\,.\,u_l\,(\nu)\,]]) = -\,N\,.\,M\,(b_{jl}).$$

Thus:

$$\left. \begin{aligned} M([u_j \triangle u_i]\,[u_l \triangle u_k]) = \\ = N\,.\,M\,(b_{jl})\,\{M(\triangle u_i\,(\mu)\,.\,\triangle u_k\,(\mu)) - M(\triangle u_i\,(\mu)\,.\,\triangle u_k\,(\nu))\} \end{aligned} \right\}. \quad (39)$$

From (37) follows:

$$\triangle u_i\,(1) \triangle u_k\,(1) = \frac{(N-1) \triangle x_i\,(1) - \triangle x_i\,(2) - \ldots - \triangle x_i\,(N)}{N} \times$$

$$\times \frac{(N-1) \triangle x_k\,(1) - \triangle x_k\,(2) - \ldots - \triangle x_k\,(N)}{N}$$

$$= \frac{\begin{aligned} (N-1)^2 \triangle x_i(1) \triangle x_k(1) + \triangle x_i(2) \triangle x_k(2) + \ldots \\ + \triangle x_i\,(N) \triangle x_k\,(N) + [[R_{\mu\nu} \triangle x_i\,(\mu) \triangle x_k(\nu)]] \end{aligned}}{N^2}$$

and

$$\triangle u_i\,(1) \triangle u_k\,(2) = \frac{(N-1) \triangle x_i\,(1) - \triangle x_i\,(2) - \triangle x_i\,(3) - \ldots - \triangle x_i\,(N)}{N} \times$$

$$\times \frac{-\triangle x_k\,(1) + (N-1) \triangle x_k\,(2) - \triangle x_k\,(3) - \ldots - \triangle x_k\,(N)}{N}$$

$$= \frac{\begin{aligned} -(N-1) \triangle x_i\,(1) \triangle x_k\,(1) - (N-1) \triangle x_i\,(2) \triangle x_k\,(2) + \\ + \triangle x_i\,(3) \triangle x_k(3) + \ldots + \triangle x_i\,(N) \triangle x_k\,(N) + [[S_{\mu\nu} \triangle x_i\,(\mu) \triangle x_k\,(\nu)]] \end{aligned}}{N^2}$$

As the law of movability is assumed to be the same for each point of the n-dimensional space, the expressions $M(\triangle x_i(1) \triangle x_k(1))$, $M(\triangle x_i(2) \triangle x_k(2)), \ldots$ will be equal, and their common value will be denoted by $M(\triangle x_i \triangle x_k)$.

Since the points $S(m)$ are supposed to be observed independently, we have

$$M(\triangle x_i(\mu) \triangle x_k(\nu)) = 0 \qquad \text{(also for } k = i\text{)}.$$

So we obtain

$$M(\triangle u_i(1) \triangle u_k(1)) = \frac{(N-1)^2 + N-1}{N^2} M(\triangle x_i \triangle x_k) = \frac{N-1}{N} M(\triangle x_i \triangle x_k),$$

$$M(\triangle u_i(1) \triangle u_k(2)) = \frac{-2(N-1)+N-2}{N^2} M(\triangle x_i \triangle x_k) = -\frac{1}{N} M(\triangle x_i \triangle x_k),$$

or, in general,

$$M(\triangle u_i(\mu) \triangle u_k(\mu)) = \frac{N-1}{N} M(\triangle x_i \triangle x^k),$$

$$M(\triangle u_i(\mu) \triangle u_k(\nu)) = -\frac{1}{N} M(\triangle x_i \triangle x_k).$$

Hence the equation (39) passes into

$$M([u_j \triangle u_i][u_l \triangle u_k]) = N\, M(b_{jl}) \times \left(\frac{N-1}{N} + \frac{1}{N}\right) M(\triangle x_i \triangle x_k),$$

or

$$M([u_j \triangle u_i][u_l \triangle u_k]) = N \,.\, M(b_{jl}) \,.\, M(\triangle x_i \triangle x_k) \quad . \quad . \quad (40)$$

Therefore the equation (38) furnishes:

$$\left.\begin{aligned} M(\triangle b_{ij} \triangle b_{kl}) = \frac{1}{N} \{ & M(b_{jl}) M(\triangle x_i \triangle x_k) + M(b_{jk}) M(\triangle x_i \triangle x_l) + \\ & + M(b_{il}) M(\triangle x_j \triangle x_k) + M(b_{ik}) M(\triangle x_j \triangle x_l)\} \end{aligned}\right\} . \quad (41)$$

Since the adjustment of the N points to the hyperplane τ begins only after the n^{th} point, so that only $N-n$ points require adjustment, we have:

$$M(\triangle x_i \triangle x_k) = \frac{[\xi_i\, \xi_k]}{N-n} = \frac{N\, M(\xi_i\, \xi_k)}{N-n}.$$

In this formula ξ_i appears as the apparent error, in distinction from the true error $\triangle x_i$.

Now we find for $M(\xi_i \xi_k)$ by the probability-formula (5):

$$M(\xi_i\, \xi_k) = \int \xi_i\, \xi_k\, dW = \left(\frac{\theta^n F}{\pi^n}\right)^{1/2} \int\limits_{\xi_1=-\infty}^{+\infty} \cdots \int\limits_{\xi_n=-\infty}^{+\infty} \xi_i\, \xi_k\, e^{-\theta f}\, d\xi_1 \ldots d\xi_n \,.$$

From

$$\int dW = \left(\frac{\theta^n F}{\pi^n}\right)^{1/2} \int\limits_{\xi_1=-\infty}^{+\infty} \cdots \int\limits_{\xi_n=-\infty}^{+\infty} e^{-\theta f}\, d\xi_1 \ldots d\xi_n = 1$$

follows

$$I = \int\limits_{\xi_1=-\infty}^{+\infty} \cdots \int\limits_{\xi_n=-\infty}^{+\infty} e^{-\theta f}\, d\xi_1 \ldots d\xi_n = \left(\frac{\pi}{\theta}\right)^{n/2} . F^{-1/2}.$$

By differentiating with respect to f_{ik} we get

$$\frac{\partial I}{\partial f_{ik}} = -\theta \int\limits_{\xi_1=-\infty}^{+\infty} \cdots \int\limits_{\xi_n=-\infty}^{+\infty} \frac{\partial f}{\partial f_{ik}}\, e^{-\theta f}\, d\xi_1 \ldots d\xi_n = \left(\frac{\pi}{\theta}\right)^{n/2} \times -\frac{1}{2} F^{-3/2} \times \frac{\partial F}{\partial f_{ik}}.$$

If, $k = i$, we have

$$\frac{\partial f}{\partial f_{ii}} = \xi_i^2 \; , \; \frac{\partial F}{\partial f_{ii}} = F_{ii} \, .$$

If $k \neq i$, we have, on account of the symmetry of f and F $(f_{ik} = f_{ki})$,

$$\frac{\partial f}{\partial f_{ik}} = 2\; \xi_i\; \xi_k \, , \; \frac{\partial F}{\partial f_{ik}} = 2 F_{ik}.$$

So we obtain in either case:

$$\theta \int\limits_{\xi_1=-\infty}^{+\infty} \cdots \int\limits_{\xi_n=-\infty}^{+\infty} \xi_i\; \xi_k\; e^{-\theta f}\, d\xi_1 \ldots d\xi_n = \frac{1}{2} \left(\frac{\pi}{\theta}\right)^{n/2} . F^{-3/2} . F_{ik}$$

whence,

$$M(\xi_i\, \xi_k) = \left(\frac{\theta^n F}{\pi^n}\right)^{1/2} \int\limits_{\xi_1=-\infty}^{+\infty} \cdots \int\limits_{\xi_n=-\infty}^{+\infty} \xi_i\, \xi_k\, e^{-\theta f}\, d\xi_1 \ldots d\xi_n = \frac{1}{2\theta} \cdot \frac{F_{ik}}{F}$$

or, by (15),

$$M(\xi_i\, \xi_k) = \frac{a_{ik}}{2\theta} \cdot \quad \ldots \ldots \ldots \quad (42)$$

We can now find the value of θ, corresponding to the data.
From

$$q_0 = \sum_\lambda p_\lambda\, \xi_\lambda$$

we have

$$[q_0^2] = [\sum_\lambda \sum_\mu p_\lambda\, p_\mu\, \xi_\lambda\, \xi_\mu] = \sum_\lambda \sum_\mu p_\lambda\, p_\mu\, [\xi_\lambda\, \xi_\mu] =$$

$$= N \sum_\lambda \sum_\mu p_\lambda\, p_\mu\, M(\xi_\lambda\, \xi_\mu) = N \sum_\lambda \sum_\mu p_\lambda\, p_\mu \frac{a_{\lambda\mu}}{2\theta},$$

or, by (23),

$$[q_0^2] = \frac{N}{2\theta} \sum_\lambda \sum_\mu a_{\lambda\mu} p_\lambda p_\mu = \frac{Na}{2\theta} \quad . \; . \; . \; . \; . \quad (43)$$

For $[q_0^2]$ and α we must take here their solution-values. Then the equation (20) or $\varphi = \frac{[q_0^2]}{N\alpha}$, considered in its solutionary state, gives

$$\theta = \frac{1}{2\varphi_0} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (44)$$

So (42) gives for the solution-value of $M(\xi_i \, \xi_k)$

$$M(\xi_i \, \xi_k) = \varphi_0 \, a_{ik} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (45)$$

and

$$M(\triangle x_i \, \triangle x_k) = \frac{N}{N-n} \varphi_0 \, a_{ik} \quad . \; . \; . \; . \; . \; . \; . \quad (46)$$

Replacing in (41) the unknown values $M(b_{jl})$ etc. by the actually found values b_{jl} etc. we finally arrive at:

$$M(\triangle b_{ij} \, \triangle b_{kl}) = \frac{\varphi_0}{N-n} (a_{ik} b_{jl} + a_{jl} b_{ik} + a_{il} b_{kj} + a_{kj} b_{il}) \; . \quad . \quad (47)$$

So we have obtained the general formula for the uncertainty of the quantities b_{ij}.

We now proceed to investigate the uncertainty of φ_0.

From $C = | c_{\lambda\mu} | = | b_{\lambda\mu} - \varphi_0 \, a_{\lambda\mu} | = 0$ follows

$$\triangle C = \sum_\lambda \sum_\mu C_{\lambda\mu} \triangle c_{\lambda\mu} = \sum_\lambda \sum_\mu C_{\lambda\mu} \triangle b_{\lambda\mu} - \sum_\lambda \sum_\mu C_{\lambda\mu} a_{\lambda\mu} \, . \, \triangle \varphi_0 = 0,$$

whence

$$UC \, . \, \triangle \varphi_0 = \sum_\lambda \sum_\mu C_{\lambda\mu} \triangle b_{\lambda\mu} \quad . \; . \; . \; . \; . \; . \; . \quad (48)$$

Thus:

$$UC \, . \, M(\triangle \varphi_0 \, \triangle b_{ij}) = \sum_\lambda \sum_\mu C_{\lambda\mu} M(\triangle b_{ij} \, \triangle b_{\lambda\mu}),$$

or, by (47)

$$UC \, . \, M(\triangle \varphi_0 \, \triangle b_{ij}) = \frac{\varphi_0}{N-n} \sum_\lambda \sum_\mu C_{\lambda\mu} (a_{i\lambda} b_{j\mu} + a_{j\mu} b_{i\lambda} + a_{i\mu} b_{\lambda j} + a_{\lambda j} b_{i\mu})$$

$$= \frac{\varphi_0}{N-n} \sum_\lambda \sum_\mu C_{\lambda\mu} \{ a_{i\lambda} (c_{j\mu} + \varphi_0 \, a_{j\mu}) + a_{j\mu} (c_{i\lambda} + \varphi_0 \, a_{i\lambda}) +$$

$$+ a_{i\mu} (c_{\lambda j} + \varphi_0 \, a_{\lambda j}) + a_{\lambda j} (c_{i\mu} + \varphi_0 \, a_{i\mu}) \}$$

or

$$\left. \begin{aligned} UC \, . \, M(\triangle \varphi_0 \, \triangle b_{ij}) = \frac{\varphi_0}{N-n} \sum_\lambda \sum_\mu C_{\lambda\mu} (c_{j\mu} a_{i\lambda} + c_{i\lambda} a_{j\mu} + c_{\lambda j} a_{i\mu} + c_{i\mu} a_{\lambda j}) + \\ + \frac{2\varphi_0^2}{N-n} \sum_\lambda \sum_\mu C_{\lambda\mu} (a_{i\lambda} \, a_{j\mu} + a_{i\mu} \, a_{j\lambda}). \end{aligned} \right\} . \quad (49)$$

Making use of the symbol δ_{ij}, introduced by (14):

$$\delta_{ii} = 1 \ , \ \delta_{ij} = 0 \quad \text{for} \quad j \neq i,$$

we have

$$\sum_{\mu} C_{\lambda\mu} \, c_{j\mu} = \delta_{\lambda j} \, C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

thus:

$$UC \, . \, M(\triangle \varphi_0 \triangle b_{ij}) = \frac{\varphi_0}{N-n} C (\sum_{\lambda} \delta_{\lambda j} a_{i\lambda} + \sum_{\mu} \delta_{i\mu} a_{j\mu} + \sum_{\mu} \delta_{j\mu} a_{i\mu} + \sum_{\lambda} \delta_{i\lambda} a_{\lambda j}) +$$

$$+ \frac{2\varphi_0^2}{N-n} \sum_{\lambda} \sum_{\mu} C_{\lambda\mu} (a_{i\lambda} a_{j\mu} + a_{i\mu} a_{j\lambda})$$

$$= \frac{4\varphi_0}{N-n} C a_{ij} + \frac{2\varphi_0^2}{N-n} \sum_{\lambda} \sum_{\mu} C_{\lambda\mu} (a_{i\lambda} a_{j\mu} + a_{i\mu} a_{j\lambda}),$$

or, on account of $C = 0$, and $\sum\limits_{\lambda} \sum\limits_{\mu} C_{\lambda\mu} a_{i\lambda} a_{j\mu} = \sum\limits_{\mu} \sum\limits_{\lambda} C_{\mu\lambda} a_{i\mu} a_{j\lambda} = \sum\limits_{\lambda} \sum\limits_{\mu} C_{\lambda\mu} a_{i\mu} a_{j\lambda}$,

$$UC \, . \, M(\triangle \varphi_0 \triangle b_{ij}) = \frac{4\varphi_0^2}{N-n} \sum_{\lambda} \sum_{\mu} C_{\lambda\mu} a_{i\mu} a_{\lambda j} \quad . \quad . \quad . \quad (51)$$

Moreover, from (48) ensues:

$$UC \, . \, M(\triangle \varphi_0^2) = \sum_{\rho} \sum_{\sigma} C_{\rho\sigma} M(\triangle \varphi_0 \triangle b_{\rho\sigma}),$$

or, by (51),

$$UC \, . \, M(\triangle\varphi_0^2) = \frac{4\varphi_0^2}{N-n} \cdot \frac{1}{UC} \cdot \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{\rho\sigma} C_{\lambda\mu} a_{\rho\mu} a^{\lambda\sigma}. \quad . \quad (52)$$

In reducing the second member of (52) and analogous sums, we may not make use of the particular circumstance, that $C = 0$. On the contrary we must start from the general formula:

$$C_{ij} \, C_{kl} - C_{il} \, C_{kj} = C \, . \, C_{ij,\,kl} = C \, . \, C_{jl,\,lk} = C \, . \, C_{il,\,kj}, \quad . \quad . \quad (53)$$

where $C_{ij,\,kl}$ means the minor of the element c_{kl} in the determinant C_{ij}, this latter not being symmetrical.

Applying (53), we may write:

$$\sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{\rho\sigma} C_{\lambda\mu} a_{\rho\mu} a_{\lambda\sigma} = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} (C_{\rho\mu} C_{\lambda\sigma} - C \, . \, C_{\rho\mu,\lambda\sigma}) \, a_{\rho\mu} \, a_{\lambda\sigma} =$$

$$= \sum_{\rho} \sum_{\mu} C_{\rho\mu} a_{\rho\mu} \sum_{\lambda} \sum_{\sigma} C_{\lambda\sigma} a_{\lambda\sigma} - C \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{\rho\mu,\lambda\sigma} a_{\rho\mu} a_{\lambda\sigma},$$

or, by (36),

$$\sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{\rho\sigma} C_{\lambda\mu} a_{\rho\mu} a_{\lambda\sigma} = (UC)^2 - C \, . \, U^2 C \quad . \quad . \quad . \quad (54)$$

In this final form, we may put $C = 0$. Then from (52) we derive

$$M(\triangle \varphi_0^2) = \frac{4\varphi_0^2}{N-n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Hence the mean error of φ_0 amounts to

$$E(\varphi_0) = \sqrt{M(\triangle \varphi_0{}^2)} = \frac{2\varphi_0}{\sqrt{N-n}}. \quad . \quad . \quad . \quad . \quad . \quad (56)$$

We shall now compute $M(\triangle C_{ij} \triangle \varphi_0)$.
We have first:

$$M(\triangle c_{kl} \triangle \varphi_0) = M(\triangle b_{kl} \triangle \varphi_0) - a_{kl}\, M(\triangle \varphi_0{}^2) =$$

$$= \frac{4\varphi_0{}^2}{N-n}\left(\frac{\sum\limits_{\lambda}\sum\limits_{\mu} C_{\lambda\mu}\, a_{k\mu}\, a_{\lambda l}}{UC} - a_{kl}\right). \quad . \quad . \quad (57)$$

further:

$$M(\triangle C_{ij} \triangle \varphi_0) = \sum_{\rho}\sum_{\sigma} C_{ij,\rho\sigma}\, M(\triangle c_{\rho\sigma} \triangle \varphi_0) =$$

$$= \frac{4\varphi_0{}^2}{N-n}\left(\frac{\sum\limits_{\lambda}\sum\limits_{\mu}\sum\limits_{\rho}\sum\limits_{\sigma} C_{ij,\rho\sigma}\, C_{\lambda\mu}\, a_{\rho\mu}\, a_{\lambda\sigma}}{UC} - \sum_{\rho}\sum_{\sigma} C_{ij,\rho\sigma} a_{\rho\sigma}\right)$$

$$= \frac{4\varphi_0{}^2}{(N-n)\,UC}\left\{\sum_{\lambda}\sum_{\mu}\sum_{\rho}\sum_{\sigma} \frac{C_{ij}\, C_{\rho\sigma} - C_{i\sigma} C_{\rho j}}{C} C_{\lambda\mu}\, a_{\rho\mu}\, a_{\lambda\sigma} - UC\,.\,UC_{ij}\right\}$$

$$= \frac{4\varphi_0{}^2}{(N-n)\,UC}\left\{\frac{C_{ij}}{C}\sum_{\lambda}\sum_{\mu}\sum_{\rho}\sum_{\sigma}(C_{\rho\mu}\, C_{\lambda\sigma} - C\,.\,C_{\rho\mu,\lambda\sigma})\, a_{\rho\mu}\, a_{\lambda\sigma} - \right.$$

$$\left. - \frac{1}{C}\sum_{\lambda}\sum_{\mu}\sum_{\rho}\sum_{\sigma} C_{i\sigma}(C_{\lambda j}\, C_{\rho\mu} - C\,.\,C_{\lambda j,\rho\mu})\, a_{\rho\mu}\, a_{\lambda\sigma} - UC\,.\,UC_{ij}\right\}$$

$$= \frac{4\varphi_0{}^2}{(N-n)\,UC}\left\{\frac{C_{ij}(UC)^2}{C} - C_{ij}\, U^2 C - \frac{UC}{C}\sum_{\lambda}\sum_{\sigma} C_{i\sigma} C_{\lambda j}\, a_{\lambda\sigma} + \right.$$

$$\left. + \sum_{\lambda}\sum_{\sigma} C_{i\sigma}\, UC_{\lambda j}\, a_{\lambda\sigma} - UC\,.\,UC_{ij}\right\}$$

$$= \frac{4\varphi_0{}^2}{(N-n)\,UC}\left\{\frac{C_{ij}(UC)^2}{C} - C_{ij} U^2 C - \right.$$

$$\left. - \frac{UC}{C}\sum_{\lambda}\sum_{\sigma}(C_{ij} C_{\lambda\sigma} - C.C_{ij,\lambda\sigma})\, a_{\lambda\sigma} + \sum_{\lambda}\sum_{\sigma} C_{i\sigma} UC_{\lambda j}\, a_{\lambda\sigma} - UC.UC_{ij}\right\}$$

$$= \frac{4\varphi_0{}^2}{(N-n)\,UC}\left\{\frac{C_{ij}(UC^2)}{C} - C_{ij}U^2C - \frac{C_{ij}(UC)^2}{C} + \right.$$

$$\left. + UC\,.\,UC_{ij} + \sum_{\lambda}\sum_{\sigma} C_{i\sigma}\, UC_{\lambda j}\, a_{\lambda\sigma} - UC\,.\,UC_{ij}\right\}$$

$$= \frac{4\varphi_0{}^2}{(N-n)\,UC}\left\{- C_{ij}\,.\,U^2C + \sum_{\lambda}\sum_{\sigma} C_{i\sigma}\, U C_{\lambda j}\, a_{\lambda\sigma}\right\}.$$

Since $M(\triangle C_{ij} \triangle \varphi_0)$ must be symmetrical with respect to i and j, we have

$$\sum_\lambda \sum_\sigma C_{i\sigma} UC_{\lambda j} a_{\lambda\sigma} = \sum_\lambda \sum_\sigma C_{j\sigma} UC_{\lambda i} a_{\lambda\sigma} = \sum_\sigma \sum_\lambda C_{j\lambda} UC_{\sigma i} a_{\sigma\lambda} = \sum_\lambda \sum_\sigma C_{\lambda j} UC_{i\sigma} a_{\lambda\sigma}$$

$$= \tfrac{1}{2} \sum_\lambda \sum_\sigma (C_{i\sigma} UC_{\lambda j} + C_{\lambda j} UC_{i\sigma}) a_{\lambda\sigma} = \tfrac{1}{2} \sum_\lambda \sum_\sigma U(C_{i\sigma} C_{\lambda j}) \,.\, a_{\lambda\sigma}$$

$$= \tfrac{1}{2} U\{\sum_\lambda \sum_\sigma (C_{ij} C_{\lambda\sigma} - C \,.\, C_{ij,\lambda\sigma}) a_{\lambda\sigma}\} = \tfrac{1}{2} U\{C_{ij} UC - C.UC_{ij}\}$$

$$= \tfrac{1}{2} C_{ij} \,.\, U^2C - \tfrac{1}{2} C \,.\, U^2 C_{ij}$$

or, by $C = 0$,

$$\sum_\lambda \sum_\sigma C_{i\sigma} UC_{\lambda j} a_{\lambda\sigma} = \tfrac{1}{2} C_{ij} U^2C.$$

Hence we find

$$M(\triangle C_{ij} \triangle \varphi_0) = \frac{-2\varphi_0^2}{N-n} \cdot \frac{U^2C}{UC} \cdot C_{ij}. \quad . \quad . \quad . \quad . \quad (58)$$

From

$$\triangle\, UC = \triangle \sum_\lambda \sum_\mu C_{\lambda\mu} a_{\lambda\mu} = \sum_\lambda \sum_\mu a_{\lambda\mu} \triangle C_{\lambda\mu}$$

follows

$$M(\triangle\, UC \,.\, \triangle \varphi_0) = \sum_\lambda \sum_\mu a_{\lambda\mu} M(\triangle C_{\lambda\mu} \triangle \varphi_0) =$$

$$= \frac{-2\varphi_0^2}{N-n} \cdot \frac{U^2C}{UC} \sum_\lambda \sum_\mu a_{\lambda\mu} C_{\lambda\mu} = \frac{-2\varphi_0^2}{N-n} \cdot \frac{U^2C}{UC} \,.\, UC,$$

thus

$$M(\triangle\, UC \,.\, \triangle \varphi_0) = \frac{-2\varphi_0^2}{N-n} U^2 C \quad . \quad . \quad . \quad . \quad . \quad (59)$$

We next consider $M(\triangle c_{ij} \triangle c_{kl})$:

$$M(\triangle c_{ij} \triangle c_{kl}) = M\{(\triangle b_{ij} - a_{ij} \triangle \varphi_0)(\triangle b_{kl} - a_{kl} \triangle \varphi_0)\} =$$
$$= M(\triangle b_{ij} \triangle b_{kl}) - a_{ij} M(\triangle b_{kl} \triangle \varphi_0) - a_{kl} M(\triangle b_{ij} \triangle \varphi_0) + a_{ij} a_{kl} M(\triangle \varphi_0^2),$$

or, by (47), (51) and (55),

$$\left. \begin{aligned} M(\triangle c_{ij} \triangle c_{kl}) = \frac{\varphi_0}{N-n} \Big\{ & (b_{ik} a_{jl} + b_{jl} a_{ik} + b_{il} a_{kj} + b_{kj} a_{il}) - \\ & - \frac{4\varphi_0 a_{ij}}{UC} \sum_\xi \sum_\eta C_{\xi\eta} a_{\xi l} a_{k\eta} - \frac{4\varphi_0 a_{kl}}{UC} \sum_\xi \sum_\eta C_{\xi\eta} a_{\xi j} a_{i\eta} + 4\varphi_0 a_{ij} a_{kl} \Big\} \end{aligned} \right\} . \quad (60)$$

or

$$\left. \begin{aligned} M(\triangle c_{ij} \triangle c_{kl}) = \frac{\varphi_0}{N-n} \Big\{ & (c_{ik} a_{jl} + c_{jl} a_{ik} + c_{il} a_{kj} + c_{kj} a_{il}) + \\ & + 2\varphi_0 (a_{ik} a_{jl} + a_{il} a_{kj}) + 4\varphi_0 a_{ij} a_{kl} - \\ & - \frac{4\varphi_0}{UC} (a_{ij} \sum_\xi \sum_\eta C_{\xi\eta} a_{\xi l} a_{k\eta} + a_{kl} \sum_\xi \sum_\eta C_{\xi\eta} a_{\xi j} a_{i\eta}) \Big\} \end{aligned} \right\} . \quad (60')$$

(To be continued).