

Mathematics. — *Adjustment of N Points (in n-dimensional Space) to the best linear (n—1)-dimensional Space.* II. By Prof. M. J. VAN UVEN. (Communicated by Prof. A. A. NIJLAND).

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We may now calculate $M(\Delta C_{ij} \Delta C_{kl})$.

$$M(\Delta C_{ij} \Delta C_{kl}) = M\left(\sum_{\lambda} \sum_{\mu} C_{ij,\lambda\mu} \Delta c_{\lambda\mu} \cdot \sum_{\rho} \sum_{\sigma} C_{kl,\rho\sigma} \Delta c_{\rho\sigma}\right) = \\ = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} M(\Delta c_{\lambda\mu} \Delta c_{\rho\sigma}),$$

or, by (60'),

$$M(\Delta C_{ij} \Delta C_{kl}) = \frac{\varphi_0}{N-n} \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} (c_{\lambda\rho} a_{\mu\sigma} + c_{\mu\sigma} a_{\lambda\rho} + \\ + c_{\lambda\sigma} a_{\rho\mu} + c_{\rho\mu} a_{\lambda\sigma}) \\ + \frac{2\varphi_0^2}{N-n} \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} (a_{\lambda\rho} a_{\mu\sigma} + a_{\lambda\sigma} a_{\rho\mu}) \\ + \frac{4\varphi_0^2}{N-n} \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} a_{\lambda\mu} a_{\rho\sigma} \\ - \frac{4\varphi_0^2}{(N-n) UC} \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} (a_{\lambda\mu} \sum_{\xi} \sum_{\tau} C_{\xi\tau} a_{\xi\sigma} a_{\rho\tau} + \\ + a_{\rho\sigma} \sum_{\xi} \sum_{\tau} C_{\xi\tau} a_{\xi\mu} a_{\lambda\tau}) \quad | \quad (61)$$

Putting:

$$P_1 = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} c_{\lambda\rho} a_{\mu\sigma}, \quad P_2 = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} c_{\mu\sigma} a_{\lambda\rho}, \\ P_3 = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} c_{\lambda\sigma} a_{\rho\mu}, \quad P_4 = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} c_{\rho\mu} a_{\lambda\sigma}, \\ P = P_1 + P_2 + P_3 + P_4 \\ Q_1 = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} a_{\rho\lambda} a_{\mu\sigma}, \quad Q_2 = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} a_{\lambda\sigma} a_{\rho\mu}, \\ Q = Q_1 + Q_2 \\ R = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} a_{\lambda\mu} a_{\rho\sigma} \\ S_1 = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} \sum_{\xi} \sum_{\tau} C_{ij,\lambda\mu} C_{kl,\rho\sigma} C_{\xi\tau} a_{\lambda\mu} a_{\xi\sigma} a_{\rho\tau}, \\ S_2 = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} \sum_{\xi} \sum_{\tau} C_{ij,\lambda\mu} C_{kl,\rho\sigma} C_{\xi\tau} a_{\rho\sigma} a_{\xi\mu} a_{\lambda\tau}, \quad S = S_1 + S_2 \quad | \quad (62)$$

we may write (61) in the abbreviated form:

$$M(\Delta C_{ij} \Delta C_{kl}) = \frac{\varphi_0}{N-n} \left\{ P + 2\varphi_0 \left(Q + 2R - \frac{2}{UC} S \right) \right\}. \quad (63)$$

Applying the relations (50) and (53), and bearing in mind, that we may not put $C=0$ before the final result, we find:

$$\begin{aligned}
 P_1 &= \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} \frac{C_{ij} C_{\lambda\mu} - C_{i\mu} C_{\lambda j}}{C} C_{kl,\rho\sigma} c_{\lambda\rho} a_{\mu\sigma} = \\
 &= C_{ij} \sum_{\mu} \sum_{\rho} \sum_{\sigma} \delta_{\rho\mu} C_{kl,\rho\sigma} a_{\mu\sigma} - \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{i\mu} \delta_{\rho j} C_{kl,\rho\sigma} a_{\mu\sigma} \\
 &= C_{ij} \sum_{\mu} \sum_{\sigma} C_{kl,\mu\sigma} a_{\mu\sigma} - \sum_{\mu} \sum_{\sigma} C_{i\mu} C_{kl,j\sigma} a_{\mu\sigma} \\
 &= C_{ij} UC_{kl} - \sum_{\mu} \sum_{\sigma} C_{\mu i} \frac{C_{kl} C_{j\sigma} - C_{jl} C_{k\sigma}}{C} a_{\mu\sigma} \\
 &= C_{ij} UC_{kl} - C_{kl} \sum_{\mu} \sum_{\sigma} \left(\frac{C_{ji} C_{\mu\sigma}}{C} - C_{jl,\mu\sigma} \right) a_{\mu\sigma} + \\
 &\quad + C_{jl} \sum_{\mu} \sum_{\sigma} \left(\frac{C_{ki} C_{\mu\sigma}}{C} - C_{ki,\mu\sigma} \right) a_{\mu\sigma} \\
 &= C_{ij} UC_{kl} - C_{kl} UC \cdot \frac{C_{ji}}{C} + C_{kl} UC_{jl} + C_{jl} UC \frac{C_{ki}}{C} - C_{jl} UC_{ki} \\
 &= U(C_{ij} C_{kl}) - \frac{UC}{C} (C_{ji} C_{kl} - C_{jl} C_{ki}) - U(C_{jl} C_{ki}) + C_{ki} UC_{jl} \\
 &= U(C_{ji} C_{kl} - C_{jl} C_{ki}) - UC \cdot C_{ji,kl} + C_{ik} UC_{lj} \\
 &= U(C \cdot C_{ji,kl}) - UC \cdot C_{ji,kl} + C_{ik} UC_{lj} \\
 &= C UC_{ji,kl} + C_{ik} UC_{lj}
 \end{aligned}$$

or, since $C=0$,

$$P_1 = C_{ik} UC_{lj} \dots \dots \dots \dots \quad (64_1)$$

$$P_2 = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} c_{\mu\sigma} a_{\lambda\rho}.$$

By interchanging λ and μ on the one hand, ρ and σ on the other hand, we obtain:

$$P_2 = \sum_{\mu} \sum_{\lambda} \sum_{\sigma} \sum_{\rho} C_{ij,\mu\lambda} C_{kl,\sigma\rho} c_{\lambda\rho} a_{\mu\sigma} = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ji,\lambda\mu} C_{lk,\rho\sigma} c_{\lambda\rho} a_{\mu\sigma}.$$

Comparing this expression with that of P_1 , we observe, that P_2 results from P_1 only by interchanging i and j on the one hand, k and l on the other hand.

So we find for P_2 :

$$P_2 = C_{jl} UC_{ki} = C_{lj} UC_{ik} \dots \dots \dots \quad (64_2)$$

$$P_3 = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} c_{\lambda\sigma} a_{\rho\mu}$$

By interchanging ρ and σ we obtain:

$$P_3 = \sum_{\lambda} \sum_{\mu} \sum_{\sigma} \sum_{\rho} C_{ij,\lambda\mu} C_{kl,\sigma\rho} c_{j\rho} a_{\sigma\mu} = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{lk,\rho\sigma} c_{j\rho} a_{\mu\sigma}.$$

Comparing this expression with that of P_1 , it appears, that only k and l are interchanged, so that we find:

$$P_3 = C_{il} UC_{kj} \dots \dots \dots \dots \quad (64_3)$$

Likewise:

$$\begin{aligned} P_4 &= \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{kl,\rho\sigma} c_{\rho\mu} a_{j\sigma} = \sum_{\mu} \sum_{\lambda} \sum_{\rho} \sum_{\sigma} C_{ij,\mu\lambda} C_{kl,\rho\sigma} c_{\rho\lambda} a_{\mu\sigma} = \\ &= \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ji,\lambda\mu} C_{kl,\rho\sigma} c_{j\rho} a_{\mu\sigma}. \end{aligned}$$

So P_4 derives from P_1 by interchanging i and j , whence:

$$P_4 = C_{jk} UC_{li} = C_{kj} UC_{il} \dots \dots \dots \dots \quad (64_4)$$

Summing up these results, we obtain:

$$P = P_1 + P_2 + P_3 + P_4 = C_{ik} UC_{lj} + C_{lj} UC_{ik} + C_{il} UC_{kj} + C_{kj} UC_{il},$$

or

$$P = U(C_{il} C_{kj} + C_{ik} C_{lj}) \dots \dots \dots \dots \quad (65)$$

In reducing Q_1 we have:

$$\begin{aligned} Q_1 &= \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} \frac{C_{ij} C_{\lambda\mu} - C_{ij\mu} C_{\lambda j}}{C} \cdot \frac{C_{kl} C_{\rho\sigma} - C_{k\sigma} C_{\rho l}}{C} a_{j\rho} a_{\mu\sigma} \\ &= \frac{1}{C^2} (C_{ij} C_{kl} \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{\lambda\mu} C_{\sigma\rho} a_{j\rho} a_{\sigma\mu} - C_{kl} \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{i\mu} C_{\lambda j} C_{\sigma\rho} a_{j\rho} a_{\sigma\mu} - \\ &\quad - C_{ij} \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{\sigma k} C_{l\rho} C_{i\mu} a_{j\rho} a_{\sigma\mu} + \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{i\mu} C_{\lambda j} C_{\sigma k} C_{l\rho} a_{j\rho} a_{\sigma\mu}). \end{aligned}$$

Putting:

$$\sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{\lambda\mu} C_{\sigma\rho} a_{j\rho} a_{\sigma\mu} = Q_{11}, \quad \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{i\mu} C_{\lambda j} C_{\sigma\rho} a_{j\rho} a_{\sigma\mu} = Q_{12},$$

$$\sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{\sigma k} C_{l\rho} C_{i\mu} a_{j\rho} a_{\sigma\mu} = Q_{13}, \quad \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{i\mu} C_{\lambda j} C_{\sigma k} C_{l\rho} a_{j\rho} a_{\sigma\mu} = Q_{14},$$

we may write:

$$Q_1 = \frac{1}{C^2} (C_{ij} C_{kl} Q_{11} - C_{kl} Q_{12} - C_{ij} Q_{13} + Q_{14}) \dots \dots \quad (66_1)$$

Now:

$$Q_{11} = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} (C_{\lambda\rho} C_{\sigma\mu} - C \cdot C_{\lambda\rho, \sigma\mu}) a_{j\rho} a_{\sigma\mu} = (UC)^2 - C \cdot U^2 C; \quad (67_{11})$$

$$\begin{aligned} Q_{12} &= \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{i\mu} (C_{\sigma j} C_{\lambda\rho} - C \cdot C_{\sigma j, \lambda\rho}) a_{j\rho} a_{\sigma\mu} = \\ &= UC \cdot \sum_{\sigma} \sum_{\mu} C_{i\mu} C_{\sigma j} a_{\sigma\mu} - C \cdot \sum_{\mu} \sum_{\sigma} C_{i\mu} UC_{\sigma j} a_{\sigma\mu} \\ &= UC \cdot \sum_{\mu} \sum_{\sigma} (C_{ij} C_{\sigma\mu} - C \cdot C_{ij, \sigma\mu}) a_{\sigma\mu} - C \cdot \sum_{\mu} \sum_{\sigma} C_{i\mu} UC_{\sigma j} a_{\sigma\mu} \\ &= (UC)^2 C_{ij} - C \cdot UC \cdot UC_{ij} - C \cdot \sum_{\mu} \sum_{\sigma} C_{i\mu} UC_{\sigma j} a_{\sigma\mu}. \end{aligned}$$

By interchanging i and j , λ and μ , ϱ and σ , the first member and the first two terms of the second member of this equation remain unaltered, the last term passing into $-C \sum_{\lambda} \sum_{\rho} C_{j\lambda} UC_{\rho i} a_{\rho\lambda}$.

If, in this latter expression, we replace λ by σ , ϱ by μ , it becomes $-C \sum_{\sigma} \sum_{\mu} C_{j\sigma} UC_{\mu i} a_{\mu\sigma} = -C \sum_{\sigma} \sum_{\mu} C_{\sigma j} UC_{i\mu} a_{\sigma\mu}$. We have therefore, as a second expression for Q_{12} :

$$Q_{12} = (UC)^2 C_{ij} - C \cdot UC \cdot UC_{ij} - C \sum_{\mu} \sum_{\sigma} C_{\sigma j} UC_{i\mu} a_{\sigma\mu},$$

whence:

$$\begin{aligned} \sum_{\mu} \sum_{\sigma} C_{i\mu} UC_{\sigma j} a_{\sigma\mu} &= \frac{1}{2} \sum_{\mu} \sum_{\sigma} (C_{i\mu} UC_{\sigma j} + C_{\sigma j} UC_{i\mu}) a_{\sigma\mu} = \frac{1}{2} \sum_{\mu} \sum_{\sigma} U(C_{i\mu} C_{\sigma j}) a_{\sigma\mu} \\ &= \frac{1}{2} U \left\{ \sum_{\sigma} \sum_{\mu} (C_{ij} C_{\sigma\mu} - C \cdot C_{ij, \sigma\mu}) a_{\sigma\mu} \right\} = \\ &= \frac{1}{2} U(C_{ij} \cdot UC - C \cdot UC_{ij}) \end{aligned}$$

or

$$\sum_{\mu} \sum_{\sigma} C_{i\mu} UC_{\sigma j} a_{\sigma\mu} = \frac{1}{2} (C_{ij} \cdot U^2 C - C \cdot U^2 C_{ij}) \quad \dots \quad (68)$$

So we find for Q_{12} :

$$Q_{12} = (UC)^2 C_{ij} - C \cdot UC \cdot UC_{ij} - \frac{1}{2} C \cdot U^2 C \cdot C_{ij} + \frac{1}{2} C^2 \cdot U^2 C_{ij} \quad (67_{12})$$

The expression Q_{13} derives from Q_{12} by replacing i by k , j by l (and also σ by μ , ϱ by λ).

So we find for Q_{13} :

$$Q_{13} = (UC)^2 C_{kl} - C \cdot UC \cdot UC_{kl} - \frac{1}{2} C \cdot U^2 C \cdot C_{kl} + \frac{1}{2} C^2 \cdot U^2 C_{kl} \quad (67_{13})$$

The expression Q_{14} may be reduced as follows:

$$\begin{aligned} Q_{14} &= \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{i\mu} C_{\sigma k} C_{l\rho} C_{\lambda j} a_{\sigma\mu} a_{\lambda\rho} = \\ &= \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} (C_{ik} C_{\sigma\mu} - C \cdot C_{ik, \sigma\mu}) (C_{lj} C_{\lambda\rho} - C \cdot C_{lj, \lambda\rho}) a_{\sigma\mu} a_{\lambda\rho} \\ &= (C_{ik} UC - C \cdot UC_{ik}) (C_{lj} \cdot UC - C \cdot UC_{lj}), \end{aligned}$$

or

$$Q_{14} = (UC)^2 C_{ik} C_{lj} - C \cdot UC \cdot U(C_{ik} C_{lj}) + C^2 \cdot UC_{ik} \cdot UC_{lj} \quad (67_{14})$$

So the equation (66)₁ may, by (67)₁₁, (67)₁₂, (67)₁₃, (67)₁₄, be reduced to

$$\begin{aligned} Q_1 &= \frac{1}{C^2} \{ (UC)^2 \cdot C_{ij} C_{kl} - C \cdot U^2 C \cdot C_{ij} C_{kl} - \\ &- (UC)^2 \cdot C_{ij} C_{kl} + C \cdot UC \cdot C_{kl} UC_{ij} + \frac{1}{2} C \cdot U^2 C \cdot C_{ij} C_{kl} - \frac{1}{2} C^2 \cdot C_{kl} U^2 C_{ij} \\ &- (UC)^2 \cdot C_{kl} C_{ij} + C \cdot UC \cdot C_{ij} UC_{kl} + \frac{1}{2} C \cdot U^2 C \cdot C_{kl} C_{ij} - \frac{1}{2} C^2 \cdot C_{ij} U^2 C_{kl} \\ &\quad + (UC)^2 \cdot C_{ik} C_{lj} - C \cdot UC \cdot U(C_{ik} C_{lj}) + C^2 \cdot UC_{ik} \cdot UC_{lj} \} \end{aligned}$$

$$\begin{aligned} Q_1 &= \frac{1}{C^2} \{ - (UC)^2 \cdot (C_{ij} C_{kl} - C_{ik} C_{lj}) + C \cdot UC \cdot U(C_{ij} C_{kl} - C_{ik} C_{lj}) + \\ &\quad + C^2 UC_{ik} \cdot UC_{lj} - \frac{1}{2} C^2 \cdot C_{kl} U^2 C_{ij} - \frac{1}{2} C^2 \cdot C_{ij} U^2 C_{kl} \} \\ &= \frac{1}{C^2} \{ - (UC)^2 \cdot C \cdot C_{ij, lk} + C \cdot UC \cdot U(C \cdot C_{ij, lk}) + C^2 \cdot UC_{ik} \cdot UC_{lj} - \\ &\quad - \frac{1}{2} C^2 \cdot C_{kl} U^2 C_{ij} - \frac{1}{2} C^2 \cdot C_{ij} U^2 C_{kl} \} \end{aligned}$$

or

$$Q_1 = UC \cdot UC_{ij, lk} + UC_{ik} \cdot UC_{lj} - \frac{1}{2} (C_{kl} U^2 C_{ij} + C_{ij} U^2 C_{kl}) \quad \dots \quad (69_1)$$

In the expression (62) for Q_2 we interchange σ and ρ , and so obtain:

$$Q_2 = \sum_{\lambda} \sum_{\mu} \sum_{\sigma} \sum_{\rho} C_{ij,\lambda\mu} C_{kl,\sigma\rho} a_{i\rho} a_{j\mu} = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} C_{ij,\lambda\mu} C_{lk,\rho\sigma} a_{\lambda\rho} a_{\sigma\mu},$$

which expression derives from Q_1 by interchanging k and l . So we find:

$$Q_2 = UC \cdot UC_{ij,kl} + UC_{il} \cdot UC_{kj} - \frac{1}{2} (C_{kl} U^2 C_{ij} + C_{ij} U^2 C_{kl}). \quad (69_2)$$

For Q we find:

$$Q = UC \cdot (UC_{ij,kl} + UC_{ij,ik}) + UC_{il} \cdot UC_{kj} + UC_{ik} \cdot UC_{lj} - \left. \begin{array}{l} \\ \\ - C_{kl} U^2 C_{ij} - C_{ij} U^2 C_{kl} \end{array} \right\} \dots \quad (70)$$

The expression (62) for R may at once be reduced to

$$R = UC_{ij} \cdot UC_{kl} \dots \dots \dots \quad (71)$$

The expression (62) for S_1 may be transformed in the following way:

$$\begin{aligned} S_1 &= UC_{ij} \cdot \sum_{\rho} \sum_{\sigma} \sum_{\xi} \sum_{\eta} C_{kl,\rho\sigma} C_{\xi\eta} a_{\xi\sigma} a_{\rho\eta} = \\ &= UC_{ij} \cdot \sum_{\rho} \sum_{\sigma} \sum_{\xi} \sum_{\eta} \frac{C_{kl} C_{\rho\sigma} - C_{k\sigma} C_{\rho l}}{C} C_{\xi\eta} a_{\xi\sigma} a_{\rho\eta} \\ &= UC_{ij} \left\{ \frac{C_{kl}}{C} \sum_{\rho} \sum_{\sigma} \sum_{\xi} \sum_{\eta} (C_{\rho\eta} C_{\xi\sigma} - C \cdot C_{\rho\eta, \xi\sigma}) a_{\xi\sigma} a_{\rho\eta} - \right. \\ &\quad \left. - \frac{1}{C} \sum_{\rho} \sum_{\sigma} \sum_{\xi} \sum_{\eta} C_{k\sigma} (C_{\xi l} C_{\rho\eta} - C \cdot C_{\xi l, \rho\eta}) a_{\xi\sigma} a_{\rho\eta} \right\} \\ &= UC_{ij} \left\{ \frac{C_{kl}}{C} [(UC)^2 - C \cdot U^2 C] - \frac{UC}{C} \sum_{\sigma} \sum_{\xi} C_{k\sigma} C_{\xi l} a_{\xi\sigma} + \right. \\ &\quad \left. + \sum_{\sigma} \sum_{\xi} C_{k\sigma} UC_{\xi l} a_{\xi\sigma} \right\} \end{aligned}$$

Making use of the formula (68), which, in this case, takes the form:

$$\sum_{\sigma} \sum_{\xi} C_{k\sigma} UC_{\xi l} a_{\xi\sigma} = \frac{1}{2} (C_{kl} U^2 C - C \cdot U^2 C_{kl}),$$

we find:

$$\begin{aligned} S_1 &= UC_{ij} \cdot \left\{ \frac{(UC)^2}{C} C_{kl} - U^2 C \cdot C_{kl} - \frac{UC}{C} \sum_{\sigma} \sum_{\xi} (C_{kl} C_{\xi\sigma} - C \cdot C_{kl, \xi\sigma}) a_{\xi\sigma} + \right. \\ &\quad \left. + \frac{1}{2} U^2 C \cdot C_{kl} - \frac{1}{2} C \cdot U^2 C_{kl} \right\} \\ &= UC_{ij} \left\{ \frac{(UC)^2}{C} C_{kl} - U^2 C \cdot C_{kl} - \frac{(UC)^2}{C} C_{kl} + UC \cdot UC_{kl} + \right. \\ &\quad \left. + \frac{1}{2} U^2 C \cdot C_{kl} - \frac{1}{2} C \cdot U^2 C_{kl} \right\}, \end{aligned}$$

or

$$S_1 = -\frac{1}{2} UC_{ij} \cdot (U^2 C \cdot C_{kl} - 2 UC \cdot UC_{kl} + C \cdot U^2 C_{kl}) \quad (72_1)$$

The expression (62) for S_2 becomes, by interchanging λ and ρ , μ and σ ,

$$S_2 = \sum_{\rho} \sum_{\sigma} \sum_{\lambda} \sum_{\mu} \sum_{\xi} \sum_{\eta} C_{ij,\rho\sigma} C_{kl,\lambda\mu} a_{\lambda\mu} a_{\xi\sigma} a_{\rho\eta};$$

so it derives from S_1 by interchanging i and k , j and l .

Hence

$$S_2 = -\frac{1}{2} UC_{kl} \cdot (U^2 C \cdot C_{ij} - 2 UC \cdot UC_{ij} + C \cdot U^2 C_{ij}) . \quad (72_2)$$

Summing up these latter results, we find, putting $C=0$,

$$S = S_1 + S_2 = -\frac{1}{2} U^2 C \cdot U(C_{ij} C_{kl}) + 2 UC \cdot UC_{ij} \cdot UC_{kl} . \quad (73)$$

The factor of $2\varphi_0$ in the equation (63) now takes, by (70), (71) and (73), the value:

$$\begin{aligned} Q + 2R - \frac{2}{UC} S &= UC \cdot (UC_{ij,kl} + UC_{ij,lk}) + UC_{il} \cdot UC_{kj} + \\ &\quad + UC_{ik} \cdot UC_{lj} - C_{kl} U^2 C_{ij} - C_{ij} U^2 C_{kl} + 2 UC_{ij} \cdot UC_{kl} + \\ &\quad + \frac{U^2 C}{UC} \cdot U(C_{ij} C_{kl}) - 4 UC_{ij} \cdot UC_{kl} \\ &= UC \cdot (UC_{ij,kl} + UC_{ij,lk}) + UC_{il} \cdot UC_{kj} + UC_{ik} \cdot UC_{lj} - \\ &\quad - 2 UC_{ij} \cdot UC_{kl} - C_{kl} U^2 C_{ij} - C_{ij} U^2 C_{kl} + \\ &\quad + \frac{U^2 C}{UC} \cdot U(C_{ij} C_{kl}). \end{aligned}$$

From

$$\begin{aligned} UC \cdot UC_{ij,kl} &= U(C \cdot UC_{ij,kl}) - C \cdot U^2 C_{ij,kl} = U\{U(C \cdot C_{ij,kl}) - \\ &\quad - UC \cdot C_{ij,kl}\} = 0 \\ &= U^2(C_{ij} C_{kl} - C_{il} C_{kj}) - U^2 C \cdot C_{ij,kl} - UC \cdot UC_{ij,kl} \end{aligned}$$

ensues:

$$2 UC \cdot UC_{ij,kl} = U(C_{kl} UC_{ij} + C_{ij} UC_{kl} - C_{kj} UC_{il} - C_{il} UC_{kj}) - U^2 C \cdot C_{ij,kl},$$

or

$$\begin{aligned} UC \cdot UC_{ij,kl} &= \frac{1}{2} (C_{kl} U^2 C_{ij} + C_{ij} U^2 C_{kl} - C_{kj} U^2 C_{il} - C_{il} U^2 C_{kj} + \\ &\quad + 2 UC_{ij} \cdot UC_{kl} - 2 UC_{il} \cdot UC_{kj} - U^2 C \cdot C_{ij,kl}). \end{aligned}$$

Likewise:

$$\begin{aligned} UC \cdot UC_{ij,lk} &= \frac{1}{2} (C_{lk} U^2 C_{ij} + C_{ij} U^2 C_{lk} - C_{lj} U^2 C_{ik} - C_{ik} U^2 C_{lj} + \\ &\quad + 2 UC_{ij} \cdot UC_{lk} - 2 UC_{ik} \cdot UC_{lj} - U^2 C \cdot C_{ij,lk}), \end{aligned}$$

whence:

$$\begin{aligned} UC \cdot (UC_{ij,kl} + UC_{ij,lk}) &= C_{kl} U^2 C_{ij} + C_{ij} U^2 C_{kl} - \\ &\quad - \frac{1}{2} (C_{kj} U^2 C_{il} + C_{il} U^2 C_{kj} + C_{lj} U^2 C_{ik} + C_{ik} U^2 C_{lj}) + \\ &\quad + 2 UC_{ij} \cdot UC_{kl} - UC_{il} \cdot UC_{kj} - UC_{ik} \cdot UC_{lj} - \frac{1}{2} U^2 C \cdot (C_{ij,kl} + C_{ij,lk}). \end{aligned}$$

So we obtain :

$$\begin{aligned} Q + 2R - \frac{2}{UC} S = & -\frac{1}{2}(C_{kj} U^2 C_{il} + C_{il} U^2 C_{kj} + C_{lj} U^2 C_{ik} + C_{ik} U^2 C_{lj}) + \\ & + \frac{U^2 C}{UC} \{ U(C_{ij} C_{kl}) - \frac{1}{2} UC \cdot (C_{ij,kl} + C_{ij,lk}) \}, \end{aligned}$$

or, by

$$\begin{aligned} UC \cdot (C_{ij,kl} + C_{ij,lk}) &= U(C \cdot C_{ij,kl} + C \cdot C_{ij,lk}) - C \cdot U(C_{ij,kl} + C_{ij,lk}) \\ &= U(C_{ij} C_{kl} - C_{il} C_{kj} + C_{ij} C_{kl} - C_{ik} C_{lj}) = \\ &= 2 U(C_{ij} C_{kl}) - U(C_{il} C_{kj} + C_{ik} C_{lj}), \end{aligned}$$

$$\begin{aligned} Q + 2R - \frac{2}{UC} S = & -\frac{1}{2}(C_{kj} U^2 C_{il} + C_{il} U^2 C_{kj} + C_{lj} U^2 C_{ik} + C_{ik} U^2 C_{lj}) + \\ & + \frac{U^2 C}{2 UC} \cdot U(C_{il} C_{kj} + C_{ik} C_{lj}). \quad \left. \right\} (74) \end{aligned}$$

Thus the equation (63) may, by (65) and (74), be brought into the following form :

$$\begin{aligned} M(\Delta C_{ij} \Delta C_{kl}) &= \frac{\varphi_0}{N-n} [U(C_{il} C_{kj} + C_{ik} C_{lj}) + \\ & + \varphi_0 \{ -(C_{kj} U^2 C_{il} + C_{il} U^2 C_{kj} + C_{lj} U^2 C_{ik} + C_{ik} U^2 C_{lj}) + \\ & + \frac{U^2 C}{UC} \cdot U(C_{il} C_{kj} + C_{ik} C_{lj}) \}], \end{aligned}$$

or

$$\begin{aligned} M(\Delta C_{ij} \Delta C_{kl}) &= \frac{\varphi_0}{N-n} \left\{ U(C_{il} C_{kj} + C_{ik} C_{lj}) \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \right. \\ & \left. - \varphi_0 (C_{kj} U^2 C_{il} + C_{il} U^2 C_{kj} + C_{lj} U^2 C_{ik} + C_{ik} U^2 C_{lj}) \right\} \quad \left. \right\} (75) \end{aligned}$$

In case two or more of the subscripts i, j, k, l are equal, we obtain :

$$\begin{aligned} l=j \quad M(\Delta C_{ij} \Delta C_{kj}) &= \frac{\varphi_0}{N-n} \left\{ U(C_{ij} C_{kj} + C_{ik} C_{jj}) \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \right. \\ & \left. - \varphi_0 (C_{kj} U^2 C_{ij} + C_{ij} U^2 C_{kj} + C_{jj} U^2 C_{ik} + C_{ik} U^2 C_{jj}) \right\} \quad \left. \right\} (75)[ij,kj] \end{aligned}$$

$$\begin{aligned} i=i, l=j \quad M(\Delta C_{ii} \Delta C_{kl}) &= \frac{2 \varphi_0}{N-n} \left\{ U(C_{ik} C_{il}) \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \right. \\ & \left. - \varphi_0 (C_{il} U^2 C_{ik} + C_{ik} U^2 C_{il}) \right\} \quad \left. \right\} (75)[ii,kl] \end{aligned}$$

$$\begin{aligned} k=i, l=j \quad M(\Delta C_{ij}^2) &= \frac{\varphi_0}{N-n} \left\{ U(C_{ij}^2 + C_{ii} C_{jj}) \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \right. \\ & \left. - \varphi_0 (2 C_{ij} U^2 C_{ij} + C_{jj} U^2 C_{ii} + C_{ii} U^2 C_{jj}) \right\} \quad \left. \right\} (75)[ij,ij] \end{aligned}$$

$$\begin{aligned}
j=i, l=k \quad M(\Delta C_{ii} \Delta C_{kk}) &= \frac{2\varphi_0}{N-n} \left\{ U(C_{ik}^2) \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \right. \\
&\quad \left. - 2\varphi_0 C_{ik} U^2 C_{ik} \right\} = \frac{4\varphi_0 C_{ik}}{N-n} \left\{ UC_{ik} \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \varphi_0 U^2 C_{ik} \right\} \quad (75)[ii, kk] \\
j=i, l=i \quad M(\Delta C_{ii} \Delta C_{ik}) &= \frac{2\varphi_0}{N-n} \left\{ U(C_{ii} C_{ik}) \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \right. \\
&\quad \left. - \varphi_0 (C_{ii} U^2 C_{ik} + C_{ik} U^2 C_{ii}) \right\} \quad (75)[ii, ik] \\
j=k=l=i \quad M(\Delta C_{ii}^2) &= \frac{4\varphi_0 C_{ii}}{N-n} \left\{ UC_{ii} \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \right. \\
&\quad \left. - \varphi_0 U^2 C_{ii} \right\} \quad (75)[ii, ii]
\end{aligned}$$

We will moreover calculate $M(\Delta C_{ij} \Delta UC)$ and $M\{(\Delta UC)^2\}$.
From

$$\Delta UC = \Delta \sum_{\lambda \mu} \Sigma C_{\lambda \mu} a_{\lambda \mu} = \sum_{\lambda \mu} \Sigma a_{\lambda \mu} \Delta C_{\lambda \mu}$$

we obtain

$$\begin{aligned}
M(\Delta C_{ij} \Delta UC) &= \sum_{\lambda \mu} \Sigma M(\Delta C_{ij} \Delta C_{\lambda \mu}) \cdot a_{\lambda \mu} \\
&= \frac{\varphi_0}{N-n} \left\{ \sum_{\lambda \mu} \Sigma U(C_{i\mu} C_{\lambda j} + C_{i\lambda} C_{\mu j}) a_{\lambda \mu} \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \right. \\
&\quad \left. - \varphi_0 \sum_{\lambda \mu} \Sigma (C_{\lambda j} U^2 C_{i\mu} + C_{i\mu} U^2 C_{\lambda j} + C_{\mu j} U^2 C_{i\lambda} + \right. \\
&\quad \left. \left. + C_{i\lambda} U^2 C_{\mu j}) a_{\lambda \mu} \right\} \quad (76)
\right.
\end{aligned}$$

Putting

$$\begin{aligned}
\sum_{\lambda \mu} \Sigma U(C_{i\mu} C_{\lambda j}) a_{\lambda \mu} &= T_1, \quad \sum_{\lambda \mu} \Sigma U(C_{i\lambda} C_{\mu j}) a_{\lambda \mu} = T_2, \quad T = T_1 + T_2, \\
\sum_{\lambda \mu} \Sigma C_{\lambda j} U^2 C_{i\mu} a_{\lambda \mu} &= V_1, \quad \sum_{\lambda \mu} \Sigma C_{i\mu} U^2 C_{\lambda j} a_{\lambda \mu} = V_2, \\
\sum_{\lambda \mu} \Sigma C_{\mu j} U^2 C_{i\lambda} a_{\lambda \mu} &= V_3, \quad \sum_{\lambda \mu} \Sigma C_{i\lambda} U^2 C_{\mu j} a_{\lambda \mu} = V_4, \\
&\quad V = V_1 + V_2 + V_3 + V_4. \quad (77)
\end{aligned}$$

we have

$$M(\Delta C_{ij} \Delta UC) = \frac{\varphi_0}{N-n} \left\{ T \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \varphi_0 V \right\} \quad . . \quad (78)$$

The expression T_1 may be transformed as follows:

$$\begin{aligned}
T_1 &= \sum_{\lambda \mu} \Sigma U(C_{i\mu} C_{\lambda j}) a_{\lambda \mu} = U \{ \sum_{\lambda \mu} \Sigma (C_{ij} C_{\lambda \mu} - C \cdot C_{ij, \lambda \mu}) a_{\lambda \mu} \} = \\
&= U(C_{ij} UC - CUC_{ij}) = C_{ij} U^2 C - C \cdot U^2 C_{ij} = C_{ij} U^2 C.
\end{aligned}$$

Likewise :

$$T_2 = \sum_{\lambda} \sum_{\mu} U(C_{i\lambda} C_{\mu j}) a_{i\mu} = U \left\{ \sum_{\lambda} \sum_{\mu} (C_{ij} C_{\mu\lambda} - C \cdot C_{ij, \mu\lambda}) a_{i\mu} \right\} = C_{ij} U^2 C,$$

whence

$$T = 2 C_{ij} U^2 C \quad \quad (79)$$

In reducing the expression V_1 we observe, that

$$\sum_{\mu} C_{i\mu} a_{\lambda\mu} = \sum_{\mu} C_{i\mu} U c_{\lambda\mu} = U \left(\sum_{\mu} C_{i\mu} c_{\lambda\mu} \right) - \sum_{\mu} U C_{i\mu} \cdot c_{\lambda\mu} = \delta_{i\lambda} U C - \sum_{\mu} U C_{i\mu} \cdot c_{\lambda\mu},$$

thus

$$\sum_{\mu} U C_{i\mu} a_{\lambda\mu} = U \left(\sum_{\mu} C_{i\mu} a_{\lambda\mu} \right) = \delta_{i\lambda} U^2 C - \sum_{\mu} U^2 C_{i\mu} \cdot c_{\lambda\mu} - \sum_{\mu} U C_{i\mu} \cdot a_{\lambda\mu},$$

or

$$2 \sum_{\mu} U C_{i\mu} a_{\lambda\mu} = \delta_{i\lambda} U^2 C - \sum_{\mu} U^2 C_{i\mu} \cdot c_{\lambda\mu}.$$

Hence

$$2 \sum_{\mu} U^2 C_{i\mu} a_{\lambda\mu} = U \left(2 \sum_{\mu} U C_{i\mu} a_{\lambda\mu} \right) = \delta_{i\lambda} U^3 C - \sum_{\mu} U^3 C_{i\mu} \cdot c_{\lambda\mu} - \sum_{\mu} U^2 C_{i\mu} \cdot a_{\lambda\mu},$$

or

$$3 \sum_{\mu} U^2 C_{i\mu} a_{\lambda\mu} = \delta_{i\lambda} U^3 C - \sum_{\mu} U^3 C_{i\mu} \cdot c_{\lambda\mu}.$$

So we find for V_1 :

$$\begin{aligned} V_1 &= \sum_{\lambda} \sum_{\mu} C_{\lambda j} U^2 C_{i\mu} a_{\lambda\mu} = \frac{1}{3} \left(\sum_{\lambda} C_{ij} \delta_{i\lambda} U^3 C - \sum_{\lambda} \sum_{\mu} C_{\lambda j} c_{\lambda\mu} U^3 C_{i\mu} \right) = \\ &= \frac{1}{3} (C_{ij} U^3 C - C \cdot \sum_{\mu} \delta_{j\mu} U^3 C_{i\mu}) \\ &= \frac{1}{3} (C_{ij} U^3 C - C \cdot U^3 C_{ij}). \end{aligned}$$

or by $C = 0$,

$$V_1 = \frac{1}{3} C_{ij} U^3 C.$$

Likewise

$$V_2 = V_3 = V_4 = \frac{1}{3} C_{ij} U^3 C$$

and

$$V = \frac{4}{3} C_{ij} U^3 C \quad \quad (80)$$

Thus the formula (78) may be written in the form:

$$M(\Delta C_{ij} \Delta UC) = \frac{\varphi_0}{N-n} \left\{ 2 C_{ij} U^2 C \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \frac{4}{3} \varphi_0 C_{ij} U^3 C \right\}.$$

or

$$M(\Delta C_{ij} \Delta UC) = \frac{2\varphi_0 C_{ij}}{N-n} \left\{ U^2 C \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \frac{4}{3} \varphi_0 U^3 C \right\}. \quad (81)$$

Finally we calculate $M\{(\Delta UC)^2\}$:

$$M\{(\Delta UC)^2\} = \sum_{\lambda} \sum_{\mu} M(\Delta C_{i\mu} \cdot \Delta UC) a_{i\mu},$$

or, by (81),

$$M\{(\Delta UC)^2\} = \frac{2\varphi_0}{N-n} (\sum_{\lambda} \sum_{\mu} C_{\lambda\mu} a_{\lambda\mu}) \cdot \left\{ U^2 C \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \frac{2}{3} \varphi_0 U^3 C \right\},$$

thus:

$$M\{(\Delta UC)^2\} = \frac{2\varphi_0 UC}{N-n} \left\{ U^2 C \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \frac{2}{3} \varphi_0 U^3 C \right\}. \quad . \quad (82)$$

or

$$M\{(\Delta UC)^2\} = \frac{2\varphi_0}{N-n} \{ UC \cdot U^2 C + \varphi_0 (U^2 C)^2 - \frac{2}{3} \varphi_0 UC \cdot U^3 C \} \quad (82')$$

By means of these results we are now able to compute the mean squares and products of the errors of p_1, p_2, \dots, p_n .

We must first normalize these direction-parameters. This may be done in several ways, but in any case we must divide the parameter p_i by a homogeneous function of the p_1, p_2, \dots, p_n of degree unity. As this divisor may not be zero, it must take the form of the $(2\nu)^{\text{th}}$ root of a positive-definite form of degree 2ν .

We shall confine ourselves to the simplest case, that the divisor is the square root of a positive-definite quadratic form:

$$\varepsilon = \sum_{\lambda=1}^n \sum_{\mu=1}^n e_{\lambda\mu} p_{\lambda} p_{\mu}. \quad . \quad (83)$$

Then, denoting the normalized parameter by r_i , we have

$$r_i^2 = \frac{p_i^2}{\varepsilon} = \frac{1}{\sum_{\lambda} \sum_{\mu} e_{\lambda\mu} \frac{p_{\lambda} p_{\mu}}{p_i^2}} = \frac{1}{\sum_{\lambda} \sum_{\mu} e_{\lambda\mu} \frac{\sqrt{C_{\lambda\lambda} C_{\mu\mu}}}{C_{ii}}} = \frac{C_{ii}}{\sum_{\lambda} \sum_{\mu} e_{\lambda\mu} C_{\lambda\mu}},$$

or, putting

$$\sum_{\lambda} \sum_{\mu} e_{\lambda\mu} C_{\lambda\mu} = VC, \quad . \quad (84)$$

$$r_i^2 = \frac{C_{ii}}{VC} \quad . \quad (85)$$

Hence

$$2 r_i \Delta r_i = \frac{VC \cdot \Delta C_{ii} - C_{ii} \sum_{\lambda} \sum_{\mu} e_{\lambda\mu} \Delta C_{\lambda\mu}}{(VC)^2},$$

likewise

$$2 r_k \Delta r_k = \frac{VC \cdot \Delta C_{kk} - C_{kk} \sum_{\sigma} e_{\rho\sigma} \Delta C_{\rho\sigma}}{(VC)^2},$$

so that

$$\begin{aligned} 4 r_i r_k M(\Delta r_i \Delta r_k) &= \frac{1}{(VC)^4} \{ (VC)^2 \cdot M(\Delta C_{ii} \Delta C_{kk}) - \\ &- VC \cdot C_{kk} \sum_{\rho} \sum_{\sigma} e_{\rho\sigma} M(\Delta C_{ii} \Delta C_{\rho\sigma}) - VC \cdot C_{ii} \sum_{\lambda} \sum_{\mu} e_{\lambda\mu} M(\Delta C_{kk} \Delta C_{\lambda\mu}) + \\ &+ C_{ii} C_{kk} \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} e_{\lambda\mu} e_{\rho\sigma} M(\Delta C_{\lambda\mu} \Delta C_{\rho\sigma}) \}, \end{aligned}$$

or

$$\left. \begin{aligned} M(\Delta r_i \Delta r_k) = & \frac{1}{4 \sqrt{C_{ii} C_{kk}} \cdot (VC)^3} \{ (VC)^2 \cdot M(\Delta C_{ii} \Delta C_{kk}) - \\ & - VC \cdot C_{kk} \sum_{\rho} \sum_{\sigma} e_{\rho\sigma} M(\Delta C_{ii} \Delta C_{\rho\sigma}) - \\ & - VC \cdot C_{ii} \sum_{\lambda} \sum_{\mu} e_{\lambda\mu} M(\Delta C_{kk} \Delta C_{\lambda\mu}) + \\ & + C_{ii} C_{kk} \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} e_{\lambda\mu} e_{\rho\sigma} M(\Delta C_{\lambda\mu} \Delta C_{\rho\sigma}) \}; \end{aligned} \right\} \quad (86)[i,k]$$

in particular,

$$\left. \begin{aligned} M(\Delta r_i^2) = & \frac{1}{4 C_{ii} (VC)^3} \{ (VC)^2 \cdot M(\Delta C_{ii}^2) - \\ & - 2 VC \cdot C_{ii} \sum_{\lambda} \sum_{\mu} e_{\lambda\mu} M(\Delta C_{ii} \Delta C_{\lambda\mu}) + \\ & + C_{ii}^2 \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} e_{\lambda\mu} e_{\rho\sigma} M(\Delta C_{\lambda\mu} \Delta C_{\rho\sigma}) \} \end{aligned} \right\} \quad (86)[i,i]$$

In order to compute the uncertainty of the corresponding parameter $r_0 = \frac{p_0}{\sqrt{\epsilon}}$, we observe, that from

$$r_0 = - \sum_{\lambda=1}^n r_{\lambda} \bar{x}_{\lambda}$$

follows:

$$\Delta r_0 = - \sum_{\lambda=1}^n r_{\lambda} \Delta \bar{x}_{\lambda} = \sum_{\lambda=1}^n \bar{x}_{\lambda} \Delta r_{\lambda}, \dots \dots \dots \quad (87)$$

whence

$$M(\Delta r_0 \Delta r_i) = - \sum_{\lambda=1}^n r_{\lambda} M(\Delta \bar{x}_{\lambda} \Delta r_i) = \sum_{\lambda=1}^n \bar{x}_{\lambda} M(\Delta r_{\lambda} \Delta r_i). \quad (88)$$

The uncertainty of $r_i = \frac{p_i}{\sqrt{\epsilon}}$ is exclusively due to that of the quantities b_{kl} ($k, l = 1, \dots, n$); thus

$$\Delta r_i = \sum_{\rho} \sum_{\sigma} r_{i;\rho\sigma} \Delta b_{\rho\sigma}$$

and

$$M(\Delta r_i \Delta \bar{x}_j) = \sum_{\rho} \sum_{\sigma} r_{i;\rho\sigma} M(\Delta b_{\rho\sigma} \Delta \bar{x}_j) \dots \dots \dots \quad (89)$$

From

$$\bar{x}_j = \frac{[x_j]}{N}, b_{\rho\sigma} = \frac{[u_{\rho} u_{\sigma}]}{N}$$

ensues

$$\Delta \bar{x}_j = \frac{[\Delta x_j]}{N}, \Delta b_{\rho\sigma} = \frac{[u_{\sigma} \Delta u_{\rho}] + [u_{\rho} \Delta u_{\sigma}]}{N},$$

whence

$$M(\Delta b_{\rho\sigma} \Delta \bar{x}_j) = \frac{1}{N^2} \{ M([u_{\sigma} \Delta u_{\rho}] [\Delta x_j]) + M([u_{\rho} \Delta u_{\sigma}] [\Delta x_j]) \}.$$

Since Δx_j , and consequently $\Delta u_\rho, \Delta u_\sigma$ are independent of u_ρ, u_σ , we have

$$M([u_\sigma \Delta_\rho] [\Delta x_j]) = M[u_\sigma] \cdot M(\Delta u_\rho [\Delta x_j]),$$

or, by $[u_\sigma] = 0$ (21),

$M([u_\sigma \Delta u_\rho] [\Delta x_j]) = 0$; likewise: $M([u_\rho \Delta u_\sigma] [\Delta x_j]) = 0$, whence

$$M(\Delta b_{\rho\sigma} \Delta \bar{x}_j) = 0 \dots \dots \dots \dots \quad (90)$$

and, by (89),

$$M(\Delta r_i \Delta \bar{x}_j) = 0 \dots \dots \dots \dots \quad (91)$$

So (88) reduces to

$$M(\Delta r_0 \Delta r_i) = - \sum_{\lambda=1}^n \bar{x}_\lambda M(\Delta r_\lambda \Delta r_i) \dots \dots \quad (92)$$

Finally we derive from (87)

$$M(\Delta r_0^2) = - \sum_{\mu=1}^n r_\mu M(\Delta r_0 \Delta \bar{x}_\mu) - \sum_{\mu=1}^n \bar{x}_\mu M(\Delta r_0 \Delta r_\mu),$$

or

$$\left. \begin{aligned} M(\Delta r_0^2) = & + \sum_{\lambda} \sum_{\mu} r_\lambda r_\mu M(\Delta \bar{x}_\lambda \Delta \bar{x}_\mu) + 2 \sum_{\lambda} \sum_{\mu} r_\lambda \bar{x}_\mu M(\Delta r_\mu \Delta \bar{x}_\lambda) + \\ & + \sum_{\lambda} \sum_{\mu} \bar{x}_\lambda \bar{x}_\mu M(\Delta r_\lambda \Delta r_\mu). \end{aligned} \right\} \quad (93)$$

We have now

$$M(\Delta \bar{x}_\lambda \Delta \bar{x}_\mu) = \frac{M(\Delta x_\lambda \Delta x_\mu)}{N},$$

or, by (45),

$$M(\Delta \bar{x}_\lambda \Delta \bar{x}_\mu) = \frac{\varphi_0}{N-n} a_{\lambda\mu} \dots \dots \dots \quad (94)$$

Thus the equation (93) may, by (91) and (94), be reduced to

$$M(\Delta r_0^2) = \frac{\varphi_0}{N-n} \sum_{\lambda} \sum_{\mu} r_\lambda r_\mu a_{\lambda\mu} + \sum_{\lambda} \sum_{\mu} \bar{x}_\lambda \bar{x}_\mu M(\Delta r_\lambda \Delta r_\mu),$$

or, since

$$\sum_{\lambda} \sum_{\mu} r_\lambda r_\mu a_{\lambda\mu} = \frac{\sum_{\lambda} \sum_{\mu} \sqrt{C_{\lambda\lambda} C_{\mu\mu}} \cdot a_{\lambda\mu}}{VC} = \frac{\sum_{\lambda} \sum_{\mu} a_{\lambda\mu} C_{\lambda\mu}}{VC} = \frac{UC}{VC},$$

$$M(\Delta r_0^2) = \frac{\varphi_0}{N-n} \cdot \frac{UC}{VC} + \sum_{\lambda} \sum_{\mu} \bar{x}_\lambda \bar{x}_\mu M(\Delta r_\lambda \Delta r_\mu) \dots \quad (95)$$

In case p_i is normalized to the direction-cosine, denoted by s_i , we have $e_{\lambda\mu} = \delta_{\lambda\mu}$ ($= 1$ for $\mu = \lambda$, and $= 0$ for $\mu \neq \lambda$), whence VC passes into

$$\sum_{\lambda=1}^n C_{\lambda\lambda} = V_1 C \dots \dots \dots \quad (96)$$

So the formula (86) takes the form:

$$\begin{aligned} M(\Delta s_i \Delta s_k) = & \frac{1}{4 \sqrt{C_{ii} C_{kk}} \cdot (V_1 C)^3} \{ (V_1 C)^2 M(\Delta C_{ii} \Delta C_{kk}) - \\ & - V_1 C \cdot C_{kk} \sum_{\rho} M(\Delta C_{ii} \Delta C_{\rho\rho}) - V_1 C \cdot C_{ii} \sum_{\lambda} M(\Delta C_{kk} \Delta C_{\lambda\lambda}) + \\ & + C_{ii} C_{kk} \sum_{\lambda} \sum_{\rho} M(\Delta C_{\lambda\lambda} \Delta C_{\rho\rho}) \}, \end{aligned}$$

or, by (75) [ii, kk],

$$M(\Delta s_i \Delta s_k) =$$

$$\begin{aligned} & = \frac{1}{4 \sqrt{C_{ii} C_{kk}} \cdot (V_1 C)^3} \left((V_1 C)^2 \cdot \frac{4\varphi_0 \sqrt{C_{ii} C_{kk}}}{N-n} \left\{ \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) UC_{ik} - \right. \right. \\ & \left. \left. - \varphi_0 U^2 C_{ik} \right\} - V_1 C \cdot C_{kk} \frac{4\varphi_0}{N-n} \sum_{\rho} \sqrt{C_{ii} C_{\rho\rho}} \cdot \left\{ \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) UC_{i\rho} - \varphi_0 U^2 C_{i\rho} \right\} - \right. \\ & \left. - V_1 C \cdot C_{ii} \frac{4\varphi_0}{N-n} \sum_{\lambda} \sqrt{C_{kk} C_{\lambda\lambda}} \cdot \left\{ \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) UC_{k\lambda} - \varphi_0 UC_{k\lambda} \right\} + \right. \\ & \left. + C_{ii} C_{kk} \frac{4\varphi_0}{N-n} \sum_{\lambda} \sum_{\rho} C_{\lambda\rho} \left\{ \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) UC_{\lambda\rho} - \varphi_0 U^2 C_{\lambda\rho} \right\} \right) \end{aligned}$$

$$\begin{aligned} M(\Delta s_i \Delta s_k) = & \frac{\varphi_0}{(N-n)(V_1 C)^3} \left(\left(1 + \varphi_0 \frac{U^2 C}{UC} \right) \{ (V_1 C)^2 \cdot UC_{ik} - \right. \\ & - V_1 C \cdot \sum_{\rho} C_{k\rho} UC_{i\rho} - V_1 C \cdot \sum_{\lambda} C_{i\lambda} UC_{k\lambda} + C_{ik} \sum_{\lambda} \sum_{\rho} C_{\lambda\rho} UC_{\lambda\rho} \} - \\ & - \varphi_0 \{ (V_1 C)^2 U^2 C_{ik} - V_1 C \cdot \sum_{\rho} C_{k\rho} U^2 C_{i\rho} - \right. \\ & \left. - V_1 C \cdot \sum_{\lambda} C_{i\lambda} U^2 C_{k\lambda} + C_{ik} \sum_{\lambda} \sum_{\rho} C_{\lambda\rho} U^2 C_{\lambda\rho} \} \right) \end{aligned}$$

or

$$\begin{aligned} M(\Delta s_i \Delta s_k) = & \frac{\varphi_0}{(N-n)(V_1 C)^3} \left(\left(1 + \varphi_0 \frac{U^2 C}{UC} \right) \{ (V_1 C)^2 \cdot UC_{ik} - \right. \\ & - V_1 C \cdot \sum_{\lambda} U(C_{i\lambda} C_{\lambda k}) + \frac{C_{ik}}{2} \sum_{\lambda} \sum_{\mu} U(C_{\lambda\mu}^2) \} - \right. \\ & \left. - \varphi_0 \{ (V_1 C)^2 \cdot U^2 C_{ik} - V_1 C \cdot \sum_{\lambda} C_{\lambda k} U^2 C_{i\lambda} - \right. \\ & \left. - V_1 C \cdot \sum_{\lambda} C_{i\lambda} U^2 C_{\lambda k} + C_{ik} \sum_{\lambda} \sum_{\mu} C_{\lambda\mu} U^2 C_{\lambda\mu} \} \right), \end{aligned} \quad \left. \right\} \quad (97) [i, k]$$

and, in particular,

$$\begin{aligned} M(\Delta s_i^2) = & \frac{\varphi_0}{(N-n)(V_1 C)^3} \left(\left(1 + \varphi_0 \frac{U^2 C}{UC} \right) \{ (V_1 C)^2 \cdot UC_{ii} - \right. \\ & - V_1 C \cdot \sum_{\lambda} U(C_{i\lambda})^2 + \frac{C_{ii}}{2} \sum_{\lambda} \sum_{\mu} U(C_{\lambda\mu}^2) \} - \right. \\ & \left. - \varphi_0 \{ (V_1 C)^2 \cdot U^2 C_{ii} - 2V_1 C \cdot \sum_{\lambda} C_{i\lambda} U^2 C_{i\lambda} + C_{ii} \sum_{\lambda} \sum_{\mu} C_{\lambda\mu} U^2 C_{\lambda\mu} \} \right), \end{aligned} \quad \left. \right\} \quad (97) [i, i]$$

The movability of the points S in the different directions is expressed by the coefficients a_{ij} . So it may sometimes be preferable to normalize the p_i by dividing them by

$$\sqrt{a} = \sqrt{\sum \sum a_{\lambda\mu} p_\lambda p_\mu}.$$

Doing so, we obtain the parameter

$$t_i = \frac{p_i}{\sqrt{a}} = \sqrt{\frac{C_{ii}}{UC}} \dots \dots \dots \quad (98)$$

Instead of substituting $a_{\lambda\mu}$ for $\epsilon_{\lambda\mu}$ in the equations (86), we may directly calculate the uncertainty of the t_i ($i = 1, \dots, n$) making use of (81) and (82).

From (98) follows

$$2t_i \Delta t_i = \frac{UC \cdot \Delta C_{ii} - C_{ii} \Delta UC}{(UC)^2}, \quad 2t_k \Delta t_k = \frac{UC \cdot \Delta C_{kk} - C_{kk} \Delta UC}{(UC)^2},$$

whence

$$4t_i t_k M(\Delta t_i \Delta t_k) = \frac{1}{(UC)^4} \left((UC)^2 M(\Delta C_{ii} \Delta C_{kk}) - \right. \\ \left. - UC \cdot C_{kk} M(\Delta C_{ii} \Delta UC) - UC \cdot C_{ii} M(\Delta C_{kk} \Delta UC) + C_{ii} C_{kk} M\{(\Delta UC)^2\} \right).$$

or

$$M(\Delta t_i \Delta t_k) = \frac{1}{4\sqrt{C_{ii} C_{kk}} \cdot (UC)^3} \left((UC)^2 M(\Delta C_{ii} \Delta C_{kk}) - \right. \\ \left. - UC \cdot C_{kk} M(\Delta C_{ii} \Delta UC) - UC \cdot C_{ii} M(\Delta C_{kk} \Delta UC) + \right. \\ \left. + C_{ii} C_{kk} M\{(\Delta UC)^2\} \right).$$

Applying (75) [i, k, k], (81) and (82), we obtain

$$M(\Delta t_i \Delta t_k) = \frac{1}{4C_{ik}(UC)^3} \left((UC)^2 \frac{4\varphi_0 C_{ik}}{N-n} \left\{ \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) UC_{ik} - \varphi_0 U^2 C_{ik} \right\} - \right. \\ \left. - UC \cdot C_{kk} \frac{2\varphi_0 C_{ii}}{N-n} \left\{ \left(1 - \varphi_0 \frac{U^2 C}{UC} \right) U^2 C - \frac{2}{3} \varphi_0 U^3 C \right\} - \right. \\ \left. - UC \cdot C_{ii} \frac{2\varphi_0 C_{kk}}{N+n} \left\{ \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) U^2 C - \frac{2}{3} \varphi_0 U^3 C \right\} + \right. \\ \left. + C_{ii} C_{kk} \cdot \frac{2\varphi_0 UC}{N-n} \left\{ \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) U^2 C - \frac{2}{3} \varphi_0 U^3 C \right\} \right).$$

or

$$M(\Delta t_i \Delta t_k) = \frac{\varphi_0}{(N-n)(UC)^2} \left\{ \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) (UC \cdot UC_{ik} - \right. \\ \left. - \frac{1}{2} C_{ik} U^2 C) - \varphi_0 (UC \cdot U^2 C_{ik} - \frac{1}{3} C_{ik} U^2 C) \right\} \quad (99) [i, k]$$

and, in particular,

$$M(\Delta t_i^2) = \frac{\varphi_0}{(N-2)(UC)^2} \left\{ \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) (UC \cdot UC_{ii} - \frac{1}{2} C_{ii} U^2 C) - \varphi_0 (UC \cdot U^2 C_{ii} - \frac{1}{3} C_{ii} U^3 C) \right\}^{(99)[i,i]}$$

Moreover (95) passes into:

$$M(\Delta t_0^2) = \frac{\varphi_0}{N-n} + \sum_{\lambda} \sum_{\mu} \bar{x}_{\lambda} \bar{x}_{\mu} M(\Delta t_{\lambda} \Delta t_{\mu}) \dots \quad (100)$$

The parameters t_i are distinguished by a particular property.

From

$$2t_i \Delta t_i = \frac{UC \cdot \Delta C_{ii} - C_{ii} \Delta UC}{(UC)^2} \quad i = 1, 2, \dots, n$$

follows:

$$2t_i M(\Delta t_i \Delta \varphi_0) = \frac{UC \cdot M(\Delta C_{ii} \Delta \varphi_0) - C_{ii} M(\Delta UC \cdot \Delta \varphi_0)}{(UC)^2} \quad i = 1, 2, \dots, n.$$

By (58) and (59) we obtain

$$2t_i M(\Delta t_i \Delta \varphi_0) = \frac{UC \cdot \frac{-2\varphi_0^2}{N-n} \cdot \frac{U^2 C}{UC} \cdot C_{ii} - C_{ii} \cdot \frac{-2\varphi_0^2}{N-n} U^2 C}{(UC)^2} = 0 \quad i = 1, 2, \dots, n$$

or

$$M(\Delta t_i \Delta \varphi_0) = 0. \quad i = 1, 2, \dots, n. \dots \quad (101)$$

Since

$$t_0 = - \sum_{i=1}^n t_i \bar{x}_i$$

we have

$$\Delta t_0 = - \sum_{\lambda} t_{\lambda} \Delta \bar{x}_{\lambda} - \sum_{\lambda} \bar{x}_{\lambda} \Delta t_{\lambda}$$

and

$$M(\Delta t_0 \Delta \varphi_0) = - \sum_{\lambda} t_{\lambda} M(\Delta \bar{x}_{\lambda} \Delta \varphi_0) - \sum_{\lambda} \bar{x}_{\lambda} M(\Delta t_{\lambda} \Delta \varphi_0). \quad (102)$$

The uncertainty of φ_0 being merely due to that of the b_{kl} , so that $\Delta \varphi_0 = \sum_{\rho} \sum_{\sigma} \varphi_{0;\rho\sigma} \Delta b_{\rho\sigma}$, we have, on account of (90),

$$M(\Delta \bar{x}_j \cdot \Delta \varphi_0) = \sum_{\rho} \sum_{\sigma} \varphi_{0;\rho\sigma} M(\Delta \bar{x}_j \Delta b_{\rho\sigma}) = 0; \quad \dots \quad (103)$$

so we obtain for (102), by (101) and (103),

$$M(\Delta t_0 \Delta \varphi_0) = 0. \quad \dots \quad (104)$$

Thus the variability of all the t_i ($i = 0, 1, 2, \dots, n$) is independent of that of φ_0 .

The normalisation of the parameters p_i to the parameters t_i may be interpreted as follows:

From

$$t_i = \frac{p_i}{\sqrt{a}} = \frac{p_i}{\sqrt{\sum_{\lambda} \sum_{\mu} a_{\lambda\mu} p_{\lambda} p_{\mu}}}$$

and

$$s_i = \frac{p_i}{\sqrt{\sum_{\lambda} p_{\lambda}^2}} \quad (s_i = \text{direction-cosine})$$

ensues

$$t_i = \frac{s_i}{\sqrt{\sum_{\lambda} \sum_{\mu} a_{\lambda\mu} s_{\lambda} s_{\mu}}} \quad \dots \quad (105)$$

The hyperellipsoids $f \equiv \sum_{\lambda} \sum_{\mu} f_{\lambda\mu} \xi_{\lambda} \xi_{\mu} = \text{const.}$ are homothetic with the standard-hyperellipsoid

$$\sum_{\lambda} \sum_{\mu} f_{\lambda\mu} x_{\lambda} x_{\mu} = 1 \quad \dots \quad (106)$$

The tangent hyperplane of the point $R'(x'_i)$ is represented by

$$\sum_{\lambda} \sum_{\mu} (f_{\lambda\mu} x'_{\mu}) x_{\lambda} = 1 \quad \dots \quad (107)$$

In order that this tangent hyperplane be parallel to τ , the conditions

$$\sum_{\mu} f_{k\mu} x'_{\mu} = H s_k \quad (k = 1, \dots, n)$$

must be satisfied. Hence

$$x'_{\lambda} = H \frac{\sum_{\mu} F_{\lambda\mu} s_{\lambda}}{F}.$$

Since $R'(x'_i)$ lies on the standard-hyperellipsoid (106), we have

$$\sum_{\rho} \sum_{\sigma} f_{\rho\sigma} x'_{\rho} x'_{\sigma} = 1,$$

or

$$\begin{aligned} \sum_{\rho} \sum_{\sigma} f_{\rho\sigma} \cdot \frac{H}{F} \sum_{\lambda} F_{\lambda\rho} s_{\lambda} \cdot \frac{H}{F} \sum_{\mu} F_{\mu\sigma} s_{\mu} &= \frac{H^2}{F} \sum_{\lambda} \sum_{\mu} \sum_{\sigma} \delta_{\lambda\sigma} F_{\mu\sigma} s_{\lambda} s_{\mu} = \\ &= H^2 \sum_{\lambda} \sum_{\mu} \frac{F_{\lambda\mu}}{F} s_{\lambda} s_{\mu} = H^2 \sum_{\lambda} \sum_{\mu} a_{\lambda\mu} s_{\lambda} s_{\mu} = 1, \end{aligned}$$

whence

$$\sum_{\lambda} \sum_{\mu} a_{\lambda\mu} s_{\lambda} s_{\mu} = \frac{1}{H^2} \quad \dots \quad (108)$$

The distance of the origin O from this tangent hyperplane being denoted by d , we have, by (107),

$$d = \sqrt{\sum_{\lambda} \sum_{\mu} (\sum_{\rho} f_{\rho\mu} x'_{\rho})^2} = \frac{1}{H \sqrt{\sum_{\lambda} s_{\lambda}^2}} = \frac{1}{H}$$

or, by (108),

$$d = \sqrt{\sum_{\lambda} \sum_{\mu} a_{\lambda\mu} s_{\lambda} s_{\mu}}, \quad \dots \quad (109)$$

Hence

$$t_i = \frac{s_i}{d} \quad \quad (110)$$

Thus we obtain the parameter t_i by dividing the direction-cosine s_i by the distance d of the centre O of the standard-hyperellipsoid (106) from its tangent hyperplane parallel to τ .

RECAPITULATION.

Being given N points $S(m)$ ($m = 1, \dots, N$), with coordinates $x_i(m)$ ($i = 1, \dots, n$) in a linear n -dimensional space, to determine the hyperplane $\tau: p_0 + p_1 x_1 + \dots + p_n x_n = 0$, which is best fitted to the given N points.

The displacements (ξ_i) are supposed to follow the probability-law

$$dW = \left(\frac{\theta^n F}{\pi^n} \right)^{1/2} e^{-\theta f} d\xi_1 \dots d\xi_n, \quad \quad (6)$$

where $f \equiv \sum_{\lambda} \sum_{\mu} f_{\lambda\mu} \xi_{\lambda} \xi_{\mu}$, $F = |f_{\lambda\mu}|$ (5), θ an as yet unknown constant factor.

§ 1. Determination of the parameters p_0, p_1, \dots, p_n .

Putting $\frac{F_{ij}}{F} = a_{ij}$ (19), the principle of adjustment may be enunciated by the postulate:

To minimize the function

$$\varphi = \frac{1}{N} \left[(p_0 + \sum_{\lambda=1}^n p_{\lambda} x_{\lambda})^2 \right] - \frac{\sum_{\lambda} \sum_{\mu} a_{\lambda\mu} p_{\lambda} p_{\mu}}{\sum_{\lambda} \sum_{\mu} a_{\lambda\mu} p_{\lambda} p_{\mu}} \quad ([] = \text{sum over the } N \text{ points}) \quad . \quad (20)$$

by disposing conveniently of the parameters p_0, p_1, \dots, p_n .

Introducing $\bar{x}_i = \frac{[x_i]}{N}$, $u_i = x_i - \bar{x}_i$ (21), and putting $\frac{1}{N} [u_i u_j] = b_{ij}$ (22), $b_{ij} - \varphi a_{ij} = c_{ij}$ (29), the problem is solved by

$$C \equiv |c_{\lambda\mu}| = 0, \quad \quad (31)$$

$$\sum_{\lambda=1}^n c_{\lambda i} p_{\lambda} = 0, \quad (i = 1, \dots, n) \quad \quad (30)$$

$$p_0 + \sum_{\lambda=1}^n p_{\lambda} \bar{x}_{\lambda} = 0 \quad \quad (26)$$

The smallest of the n real positive roots of the equation (31), of the n^{th} degree in φ , is the required minimum-value φ_0 of φ . The solution-value φ_0 of φ being known, also the quantities c_{ij} are determined.

Then the ratios of the p_i ($i = 1, \dots, n$) are given by

$$\frac{p_1}{\sqrt{C_{11}}} = \frac{p_2}{\sqrt{C_{22}}} = \dots = \frac{p_n}{\sqrt{C_{nn}}} \quad \quad (34)$$

Finally (26) furnishes the corresponding value of p_0 .

The factor θ turns out to be $\theta = \frac{1}{2\varphi_0}$ (44).

§ 2. Uncertainty of the normalized parameters.

The uncertainty of the different quantities x_i , \bar{x}_i , b_{ij} , c_{ij} , C_{ij} , φ_0 is characterized by the mean value of the squares and products of their errors.

Introducing $U\Phi \equiv \sum_{\lambda} \sum_{\mu} a_{\lambda\mu} \frac{\partial \Phi}{\partial C_{\lambda\mu}}$ (thus: $UC = \sum_{\lambda} \sum_{\mu} a_{\lambda\mu} C_{\lambda\mu}$, $U^2C = \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} a_{\lambda\mu} a_{\rho\sigma} C_{\lambda\mu\rho\sigma}$, etc., (36), $UC_{ij} = \sum_{\lambda} \sum_{\mu} a_{\lambda\mu} C_{ij,\lambda\mu}$ etc.), we obtain

$$M(\Delta x_i \Delta x_k) = \frac{N}{N-n} \varphi_0 a_{ik}, \quad M(\Delta \bar{x}_i \Delta \bar{x}_k) = \frac{1}{N-n} \varphi_0 a_{ik}. \quad (46)$$

$$M(\Delta b_{ij} \Delta b_{kl}) = \frac{\varphi_0}{N-n} (a_{ik} b_{jl} + a_{jl} b_{ik} + a_{il} b_{kj} + a_{kj} b_{il}), \quad . . . \quad (47)$$

$$M(\Delta \varphi_0^2) = \frac{4\varphi_0^2}{N-n} \quad . . . \quad (55) \quad \text{or} \quad E(\varphi_0) = \frac{2\varphi_0}{\sqrt{N-n}}, \quad . . . \quad (56)$$

$$M(\Delta c_{ij} \Delta c_{kl}) = \frac{\varphi_0}{N-n} \left\{ (b_{ik} a_{jl} + b_{jl} a_{ik} + b_{il} a_{kj} + b_{kj} a_{il}) - \right. \\ \left. - \frac{4\varphi_0 a_{ij}}{UC} \sum_{\xi} \sum_{\eta} C_{\xi\eta} a_{\xi l} a_{k\eta} - \frac{4\varphi_0 a_{kl}}{UC} \sum_{\xi} \sum_{\eta} C_{\xi\eta} a_{\xi j} a_{i\eta} + 4\varphi_0 a_{ij} a_{kl} \right\}, \quad (60)$$

$$M(\Delta C_{ij} \Delta C_{kl}) = \frac{\varphi_0}{N-n} \left\{ U(C_{il} C_{kj} + C_{ik} C_{lj}) \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \right. \\ \left. - \varphi_0 (C_{kj} U^2 C_{il} + C_{il} U^2 C_{kj} + C_{lj} U^2 C_{ik} + C_{ik} U^2 C_{lj}) \right\}, \quad (75)$$

with the main particular cases

$$M(\Delta C_{ii} \Delta C_{kk}) = \frac{4\varphi_0 C_{ik}}{N-n} \left\{ UC_{ik} \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \varphi_0 U^2 C_{ik} \right\}, \quad (75)[ii,kk]$$

$$M(\Delta C_{ii}^2) = \frac{4\varphi_0 C_{ii}}{N-n} \left\{ UC_{ii} \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \varphi_0 U^2 C_{ii} \right\}; \quad (75)[ii,ii]$$

$$M(\Delta C_{ij} \Delta UC) = \frac{2\varphi_0 C_{ij}}{N-n} \left\{ U^2 C \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \frac{2}{3} \varphi_0 U^3 C \right\}. \quad . . . \quad (81)$$

$$M \{ (\Delta UC)^2 \} = \frac{2\varphi_0 UC}{N-n} \left\{ U^2 C \cdot \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) - \frac{2}{3} \varphi_0 U^3 C \right\}. \quad . . . \quad (82)$$

As a rule the p_i are normalized by dividing them by the square root of a positive-definite quadratic function of p_1, \dots, p_n :

$$\varepsilon = \sum_{\lambda=1}^n \sum_{\mu=1}^n e_{\lambda\mu} p_{\lambda} p_{\mu} \quad . . . \quad (83) \quad \text{Putting } \sum_{\lambda} \sum_{\mu} e_{\lambda\mu} C_{\lambda\mu} = V(C), \quad . . . \quad (84)$$

the parameter p_i is normalized to $r_i = \sqrt{\frac{C_{ii}}{VC}}$ (85), whence

$$\left. \begin{aligned} M(\Delta r_i \Delta r_k) &= \frac{1}{4\sqrt{C_{ii} C_{kk}} (VC)^3} \{ (VC)^2 M(\Delta C_{ii} \Delta C_{kk}) - \\ &\quad - VC \cdot C_{kk} \sum_{\rho} \sum_{\sigma} e_{\rho\sigma} M(\Delta C_{ii} \Delta C_{\rho\sigma}) - \\ &\quad - VC \cdot C_{ii} \sum_{\lambda} \sum_{\mu} e_{\lambda\mu} M(\Delta C_{kk} \Delta C_{\lambda\mu}) + \\ &\quad + C_{ii} C_{kk} \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} e_{\lambda\mu} e_{\rho\sigma} M(\Delta C_{\lambda\mu} \Delta C_{\rho\sigma}) \}, \end{aligned} \right\} \quad (86) [i, k]$$

in particular:

$$\left. \begin{aligned} M(\Delta r_i^2) &= \frac{1}{4C_{ii}(VC)^3} \{ (VC)^2 M(\Delta C_{ii}^2) - \\ &\quad - 2 VC \cdot C_{ii} \sum_{\lambda} \sum_{\mu} e_{\lambda\mu} M(\Delta C_{ii} \Delta C_{\lambda\mu}) + \\ &\quad + C_{ii}^2 \sum_{\lambda} \sum_{\mu} \sum_{\rho} \sum_{\sigma} e_{\lambda\mu} e_{\rho\sigma} M(\Delta C_{\lambda\mu} \Delta C_{\rho\sigma}) \}. \end{aligned} \right\} \quad (86) [i, i]$$

For $r_0 = \frac{p_0}{\sqrt{\varepsilon}}$ we have

$$M(\Delta r_0 \Delta r_i) = - \sum_{j=1}^n \bar{x}_j M(\Delta r_j \Delta r_i). \quad \quad (92)$$

$$M(\Delta r_0^2) = \frac{\varphi_0}{N-n} \cdot \frac{UC}{VC} + \sum_{i} \sum_{\mu} \bar{x}_i \bar{x}_{\mu} M(\Delta r_i \Delta r_{\mu}). \quad . . . \quad (95)$$

Specialisations of e_{ij} :

I. $e_{ij} = \delta_{ij}$ ($\delta_{ii} = 1$, $\delta_{ij} = 0$ for $j \neq i$), $\varepsilon = \sum_{\lambda=1}^n p_{\lambda}^2$, $VC = \sum_{\lambda} C_{\lambda\lambda} = V_1 C$,

$$s_i = \sqrt{\frac{p_i}{\sum_{\lambda} p_{\lambda}^2}} = \sqrt{\frac{C_{ii}}{\sum_{\lambda} C_{\lambda\lambda}}},$$

$$\left. \begin{aligned} M(\Delta s_i \Delta s_k) &= \frac{\varphi_0}{(N-n)(V_1 C)^3} \left(\left(1 + \varphi_0 \frac{UC}{VC} \right) \{ (V_1 C)^2 \cdot UC_{ik} - \right. \\ &\quad \left. - V_1 C \cdot \sum_{\lambda} U(C_{i\lambda} C_{\lambda k}) + \frac{C_{ik}}{2} \sum_{\lambda} \sum_{\mu} U(C_{\lambda\mu}^2) \} \right) - \varphi_0 \{ (V_1 C)^2 \cdot U^2 C_{ik} - \\ &\quad - V_1 C \cdot \sum_{\lambda} C_{i\lambda} U^2 C_{\lambda k} - V_1 C \cdot \sum_{\lambda} C_{i\lambda} U^2 C_{\lambda k} + C_{ik} \sum_{\lambda} \sum_{\mu} C_{\lambda\mu} U^2 C_{\lambda\mu} \}, \end{aligned} \right\} \quad (97) [i, k]$$

$$\left. \begin{aligned} M(\Delta s_i^2) &= \frac{\varphi_0}{(N-n)(V_1 C)^3} \left(\left(1 + \varphi_0 \frac{UC}{VC} \right) \{ (V_1 C)^2 \cdot UC_{ii} - \right. \\ &\quad \left. - V_1 C \cdot \sum_{\lambda} U(C_{i\lambda}^2) + \frac{C_{ii}}{2} \sum_{\lambda} \sum_{\mu} U(C_{\lambda\mu}^2) \} \right) - \varphi_0 \{ (V_1 C)^2 \cdot U^2 C_{ii} - \\ &\quad - 2 V_1 C \cdot \sum_{\lambda} C_{i\lambda} U^2 C_{i\lambda} + C_{ii} \sum_{\lambda} \sum_{\mu} C_{\lambda\mu} U^2 C_{\lambda\mu} \}, \end{aligned} \right\} \quad (97) [i, i]$$

$$\text{II. } e_{ij} = a_{ij}, \quad \epsilon = a, \quad VC = UC, \quad t_i = \frac{p_i}{\sqrt{a}} = \sqrt{\frac{C_{ii}}{UC}}. \quad . . . \quad (98)$$

$$M(\Delta t_i \Delta t_k) = \frac{\varphi_0}{(N-n)(UC)^2} \left\{ \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) (UC \cdot UC_{ik} - \right. \right. \\ \left. \left. - \frac{1}{2} C_{ik} U^2 C) - \varphi_0 (UC \cdot U^2 C_{ik} - \frac{1}{3} C_{ik} U^3 C) \right\} \quad (99) [i, k]$$

$$M(\Delta t_i^2) = \frac{\varphi_0}{(N-n)(UC)^2} \left\{ \left(1 + \varphi_0 \frac{U^2 C}{UC} \right) (UC \cdot UC_{ii} - \frac{1}{2} C_{ii} U^2 C) - \right. \right. \\ \left. \left. - \varphi_0 (UC \cdot U^2 C_{ii} - \frac{1}{3} C_{ii} U^3 C) \right\} \quad (99) [i, i]$$

$$M(\Delta t_0^2) = \frac{\varphi_0}{N-n} + \sum_{\lambda} \sum_{\mu} \bar{x}_{\lambda} \bar{x}_{\mu} M(\Delta t_{\lambda} \Delta t_{\mu}), \quad . . . \quad (100)$$

$$M(\Delta t_i \Delta \varphi_0) = 0. \quad (i = 0, 1, \dots, n) \quad . . . \quad (101), (104)$$

$$t_i = \frac{s_i}{d} \left(d = \text{distance of } O \text{ from the tangent hyperplane of } \right. \\ \left. \sum_{\lambda} \sum_{\mu} f_{\lambda\mu} x_{\lambda} x_{\mu} = 1 \text{ parallel to } \tau \right). \quad (110)$$