Hydrodynamics. — *A remark on a formula for the resistance experienced by a body in a fluid, given by* OSEEN *and* ZEILON. By J. M. BURGERS. (Mededeeling N°. 15 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft). (Communicated by Prof. P. EHRENFEST).

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§ 1. One of the most interesting results which can be deduced from OSEEN's hydrodynamical theory is the relation between the resistance W experienced by a body in a current of fluid of velocity V, and the inflow Q in the wake behind it:

$$W = \varrho\, V\, Q \qquad (1)$$

(ϱ being the density of the fluid). [1]) This relation is a consequence of the circumstance that the solution of OSEEN's equations:

$$\left.\begin{aligned} \varrho\left(\nu \triangle u_i - V\frac{\partial u_i}{\partial x_1}\right) &= \frac{\partial q}{\partial x_i} - Y_i \\ \sum_j \frac{\partial u_j}{\partial x_j} &= 0 \end{aligned}\right\} \qquad (2)$$

can be separated into two parts as follows: [2])

$$u_i = v_i + \frac{\partial \varphi}{\partial x_i} \qquad (3)$$

Here φ is a potential (which, however, is not a solution of LAPLACE's equation), whereas the v_i represent a motion with vorticity. The v_i exist only in the wake behind the body, and are determined by the formulae:

$$v_i = \frac{1}{4\pi\varrho\nu}\iiint d\xi_1\, d\xi_2\, d\xi_3\, \frac{e^{-s}}{r}\, Y_i \qquad (4)$$

[1]) The theorem has been enunciated by various authors, see f.i. N. FILON, Proc. Roy. Soc. (A) 118, p. 7, 1926; further J. M. BURGERS, Proc. Acad. Amst. **31**, p. 433, 1928 and especially S. GOLDSTEIN, Proc. Roy. Soc. A **123**, p. 216, 1929 and a subsequent paper by the same author, to be published soon. The present note has come out of a correspondence with Dr. GOLDSTEIN on this subject.

[2]) Comp. C. W. OSEEN, Hydrodynamik (Leipzig, 1927). — The notation in the present paper is the same as that used in a former one, Proc. Acad. Amst. **32**, p. 1278, 1929. The variables u_i denote the components of the velocity of the field that by the action of the "forces" Y_i is superposed on the general motion of the fluid with the velocity V parallel to the x_1-axis.

In this expression ν represents the kinematical viscosity; further:

$$s = k(r - x_1 + \xi_1), \quad \text{where} \quad k = V/2\nu, \quad r = \sqrt{\Sigma (x_j - \xi_j)^2};$$

the Y_i are to be considered as functions of the ξ_i .

When we suppose that the Y_i are zero everywhere outside of a region of limited extension, then we have for values of x_1, lying sufficiently far downstream of this region:

$$\iint dx_2\, dx_3\, v_i = \frac{1}{\varrho V} \iiint d\xi_1\, d\xi_2\, d\xi_3\, Y_i \quad . \quad . \quad . \quad . \quad (5)$$

When we consider the case $i = 1$, then the integral on the left hand side of this equation represents the inflow in the wake, taken with the opposite sign.

Now in eq. (2) the Y_i represent the sum of the real or exterior forces X_i, acting on the fluid, and the "apparent" or "interior" forces y_i, which express the influence of the quadratic terms of the equations of hydrodynamics:

$$Y_i = X_i + y_i = X_i - \varrho \sum_j u_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \quad . \quad . \quad . \quad (6)$$

As we may write:

$$y_i = -\varrho \sum_j \frac{\partial}{\partial x_j} (u_j\, u_i) + \frac{1}{2} \varrho \frac{\partial}{\partial x_i} (\sum_j u_j^2) \quad . \quad . \quad . \quad . \quad (6^a)$$

we have:

$$\iiint d\xi_1\, d\xi_2\, d\xi_3\, y_i = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

(supposing that the quantities y_i tend to zero sufficiently fastly, when we go away to great distances), and so:

$$\iiint Y_i\, d\xi_1\, d\xi_2\, d\xi_3 = \iiint X_i\, d\xi_1\, d\xi_2\, d\xi_3 = F_i \quad . \quad . \quad . \quad (8)$$

where the F_i represent the components of the resultant of the exterior forces.

All these remarks refer to the case of a field of unlimited extension. In order to apply them to the case of the motion along a body, we shall imagine the body to be taken out of the field, and to be replaced by fluid at rest. At the same time we introduce a system of forces, acting in the points that were formerly occupied by the surface of the body, and chosen in such a way, that they exactly represent the influence which the body exerted on the fluid. When the motion of the fluid in the neighbourhood of the body is given, such a system of forces can be determined by means of equations (2) from the discontinuities, which the motion of the fluid shows at this surface. We shall not inquire into the nature of this system of forces, as all that we want is that the resultant of these forces is equal and opposite to the resistance experienced by the body. Then formula (5), combined with (8), at once leads to formula (1).

§ 2. Now OSEEN and ZEILON have given a formula for the resistance which applies to the case of very small viscosity (very high REYNOLDS' numbers), and which shows a different form: [1])

$$W=\frac{1}{2}\varrho\iint dx_2\,dx_3\,v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Here v represents the value of v_1 in the cylindrical part of the wake which extends immediately behind the body; it is a function of x_2 and x_3 only. The formula is demonstrated for the case of two-dimensional motion, but the demonstration can be extended to the case of a body having rotational symmetry about the x_1-axis. As the inflow in the wake is given by:

$$Q=-\iint dx_2\,dx_3\,v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

formula (1) seems to be not applicable to this case.

The object of the present note is to show that the origin of this discrepancy is to be found in the circumstance, that formula (9) is derived from an integration of the pressure p over the surface of the body, and that in the calculation of p from the quantity q, which occurs in equations (2), terms of the second degree in the u_i have been taken into account. In fact leaving aside the hydrostatical part of the pressure, we have:

$$p=q-\frac{1}{2}\varrho\,\Sigma\,u_j^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The determination of the field of flow on the other hand is based wholly upon the application of the reduced equations, from which all quadratic terms have been omitted.

In order to study this point more closely we may investigate by which system of forces the motion, calculated by ZEILON, could be held up when the body was absent and was replaced by fluid at rest.

We shall restrict ourselves to the case of two-dimensional motion. Then the solution of equations (2) may be written as follows: [2])

$$\left.\begin{aligned} u_1&=\frac{1}{2\pi\varrho V}\iint d\xi_1\,d\xi_2\left[Y_1\left\{\frac{V}{\nu}\,\varepsilon-\frac{\partial}{\partial x_1}(lg\,r+\varepsilon)\right\}-Y_2\frac{\partial}{\partial x_2}(lg\,r+\varepsilon)\right] \\ u_2&=\frac{1}{2\pi\varrho V}\iint d\xi_1\,d\xi_2\left[-Y_1\frac{\partial}{\partial x_2}(lg\,r+\varepsilon)+Y_2\frac{\partial}{\partial x_1}(lg\,r+\varepsilon)\right] \end{aligned}\right\}\quad(12)$$

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[1]) N. ZEILON, K. Sv. Vet. Akad. Handl. Ser. 3, Bd. 1, 1923; C. W. OSEEN, Hydrodynamik, p. 294 (the formula mentioned at p. 316 has to be corrected).

[2]) C. W. OSEEN, Hydrodynamik, p. 37. $K_0(x)$ is a Bessel function of the second kind, which also is denoted by $\frac{i\pi}{2}\,H'_0\,(ix)$.

where:

$$r = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2} \quad , \quad \varepsilon = e^{k(x_1 - \xi_1)} K_0(kr), \quad k = V/2\nu.$$

When the value of REYNOLDS' number $R = VD/\nu$ (D being the diameter of the body) is sufficiently high, we may replace these formulae by the following approximations: [1])

$$u_1 = \frac{\partial \varphi}{\partial x_1} + v, \qquad u_2 = \frac{\partial \varphi}{\partial x_2} \quad \ldots \ldots \quad (13)$$

where the potential φ is determined by:

$$\varphi = \frac{1}{2\pi\varrho V} \iint d\xi_1 \, d\xi_2 \left[- Y_1 \lg r + Y_2 \operatorname{arctg} \frac{x_2 - \xi_2}{x_1 - \xi_1} \right] \quad . \; . \quad (14)$$

while v has the value:

$$v = \frac{1}{\varrho V} \int_{-\infty}^{x_1} d\xi_1 \, Y_1 \quad \ldots \ldots \ldots \quad (15)$$

These expressions can be used in all parts of the field, except at distances downstream of the region where the forces Y_i are supposed to act, of about the order $10^{-5}\, RD$ and greater, as at such distances the diffusion of the vorticity in consequence of the viscosity may no longer be neglected. We shall come back to this point afterwards; presently we shall consider only those parts of the field that are not lying at such distances downstream.

Then we may say that, as far as the potential φ is concerned, every force Y_1 may be considered as the seat of a (negative) source of strength $- Y_1/\varrho V$, and every force Y_2 as the seat of a vortex of strength $+ Y_2/\varrho V$. Formula (15) moreover shows that downstream from the region where the forces are applied, v is independent of x_1; upstream of this region it is zero.

§ 3. Now the solution obtained by OSEEN for the motion along a

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[1]) In order to justify these approximations we remark that for positive values of x_1 and for $|x_2 - \xi_2| > D\sqrt{4Mx_1/RD}$ (or in the case of very small values of x_1, when $|x_2 - \xi_2| > 2\,MD/R$) the function ε vanishes as e^{-M}. Hence, when x_1 lies below the limit mentioned in the text, we may write for any function $f(\xi_2)$, which is sufficiently continuous:

$$\int d\xi_2 \, f(\xi_2) \, \varepsilon \cong f(x_2) \int_{-\infty}^{+\infty} d\xi_2 \, \varepsilon = \frac{2\pi\nu}{V} f(x_2).$$

Hence from all the terms containing the function ε only that one has to be taken into account which is multiplied by V/ν. For negative values of x_1 the function ε vanishes very fastly.

body in the case of high REYNOLDS' numbers has just the same form (13), were φ and v are determined by the conditions: [1])

$$\triangle \varphi = 0 \quad \text{in the whole field} \quad \ldots \ldots \quad (16^a)$$

$$\frac{\partial \varphi}{\partial n} + V \cos(n\, x_1) = 0 \quad \begin{array}{l}\text{at the front surface}\\ \text{of the body}\end{array} \quad \ldots \quad (16^b)$$

$$\left.\begin{array}{r} V + \dfrac{\partial \varphi}{\partial x_1} + v = 0 \\ \dfrac{\partial \varphi}{\partial x_2} \qquad = 0 \end{array}\right\} \text{at the back of the body} \ \ldots \ \left\{\begin{array}{l}(16^c)\\ (16^d)\end{array}\right.$$

$$\varphi = 0 \quad \text{at infinity} \quad \ldots \ldots \ldots \quad (16^e)$$

In order to determine the magnitude of the forces Y_i necessary to produce this motion, we shall consider the field of flow determined by the potential φ only, and completed in the interior of the body by a motion with the velocity $-V$ parallel to the x_1-axis. (It has to be kept in mind, that in our equations the quantities u_i represent the motion that has to be superposed on the general velocity V of the fluid in the positive direction of the x_1-axis; when we leave out this velocity everywhere, we get the velocity $-V$ in the interior of the space occupied by the body).

It is to be seen from (16^b), (16^c), (16^d) that along the front surface of the body the field so determined presents a discontinuity of the tangential component of the velocity, but not of the normal component, whereas along the back surface both components are discontinuous. We thus may regard this surface as the seat of a system of vortices and of sources. Calling β the strength of the sources per unit length, γ the vorticity per unit length, and denoting an element of the circumference by ds (measured in the counterclockwise direction), we have the following relations:

$$\text{along the front surface:} \quad \beta = \frac{\partial \varphi}{\partial n} + V \cos(nx_1) = 0 \quad . \quad (17^a)$$

$$\gamma = \frac{\partial \varphi}{\partial s} - V \sin(nx_1) \quad \ldots \quad (17^b)$$

$$\text{along the back surface:} \quad \beta = - v \cos(nx_1) \ldots \ldots \quad (17^c)$$

$$\gamma = + v \sin(nx_1) \ldots \ldots \quad (17^d)$$

In consequence of what has been said in § 2 we see that the motion in the absence of the body could be obtained by applying a system of forces, acting in the points of the surface and determined by the equations:

$$Y_1 = -\beta \varrho V, \qquad Y_2 = +\gamma \varrho V \quad \ldots \ldots \quad (18)$$

where the intensities Y_1, Y_2 are referred to unit length.

[1]) C. W. OSEEN, Hydrodynamik, p. 271. See also Proc. Acad. Amst. 31, p. 433, 1928.

The resultant of this system of forces, taken along the axis of x_1, has the value:

$$-W = \int ds\, Y_1 = \varrho V \int ds\, v \cos(nx_1) = -\varrho V Q \quad . \quad . \quad . \quad (19)$$

in accordance with formula (1).

§ 4. Thus far we have limited ourselves to the linear equations. Now we shall pass over to a consideration of the terms of the second degree. In order to obtain these terms we have to introduce the "interior" forces y_i, the components of which in our case are given by the expressions:

$$y_1 = +\varrho u_2 w, \qquad y_2 = -\varrho u_1 w. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where w denotes the vorticity, present in the real motion (that is, in the motion which is obtained, when to the field described in § 3, is added the velocity v in the wake).

Vortex motion is present (a) at the surface of the body, (b) in the wake.

We begin with the "interior" forces arising from the vortex layer at the surface of the body. At the front side the intensity of this layer is determined by the quantity γ, given in eq. (17^b); at the back side the intensity is zero, as in the motion that results from the combination of the field determined by φ with the velocity v, no discontinuities occur at the back side. So we have to consider the front side only. When we integrate the expressions (20) over the thickness of this layer, we get for the intensities of the forces per unit length:

$$\bar{y}_1 = \frac{1}{2}\varrho\gamma \left\{ (u_2)_e + (u_2)_i \right\} = \frac{1}{2}\varrho\gamma \frac{\partial\varphi}{\partial x_2} \quad . \quad . \quad . \quad . \quad . \quad (21^a)$$

$$\bar{y}_2 = -\frac{1}{2}\varrho\gamma \left\{ (u_1)_e + (u_1)_i \right\} = \frac{1}{2}\varrho\gamma \left(V - \frac{\partial\varphi}{\partial x_1} \right) \quad . \quad . \quad (21^b)$$

(the indices e, i respectively refer to the exterior and to the interior side of the surface of discontinuity).

The interior forces exert just the same influence on the field as the real or exterior forces do. Now as we have said, we may consider the Y_i as representing the total forces; hence when we write:

$$Y_1 = X_1 + \bar{y}_1, \qquad Y_2 = X_2 + \bar{y}_2 \quad . \quad . \quad . \quad . \quad . \quad (22)$$

the quantities X_1, X_2 will represent the components of the forces that have to be applied by exterior means.

We calculate the resultant of the forces $\bar{y}_1$. Making use of (17^b) and (16^b), we obtain:

$$\int \bar{y}_1\, ds = \frac{1}{2}\varrho \int ds \left[\left(V + \frac{\partial\varphi}{\partial x_1} \right)^2 + \left(\frac{\partial\varphi}{\partial x_2} \right)^2 \right] \cos(nx_1).$$

This integral can be calculated as follows.[1]) Firstly we extend it along

[1]) Comp. N. ZEILON, l.c. p. 17, and C. W. OSEEN, Hydrodynamik, p. 295.

the whole circumference of the body, which adds to it the amount: $\frac{1}{2}\varrho\int dx_2\, v^2$. When we write: $\Phi = Vx_1 + \varphi$, our integral takes the form:

$$\frac{1}{2}\varrho\int dx_2\left[\left(\frac{\partial\Phi}{\partial x_1}\right)^2+\left(\frac{\partial\Phi}{\partial x_2}\right)^2\right],$$

where the integration still has to be performed in the counterclockwise direction. To this integral we add the quantity:

$$-\varrho\int\left[dx_2\left(\frac{\partial\Phi}{\partial x_1}\right)^2 - dx_1\frac{\partial\Phi}{\partial x_1}\frac{\partial\Phi}{\partial x_2}\right] = -\varrho\int dx_2\, v^2.$$

Our integral now becomes:

$$-\frac{1}{2}\varrho\int\left[dx_2\left\{\left(\frac{\partial\Phi}{\partial x_1}\right)^2-\left(\frac{\partial\Phi}{\partial x_2}\right)^2\right\} - 2\,dx_1\frac{\partial\Phi}{\partial x_1}\frac{\partial\Phi}{\partial x_2}\right].$$

The latter expression represents the real part of the integral:

$$\frac{1}{2}\,i\,\varrho\int dz\left(\frac{d\chi}{dz}\right)^2,$$

where $z = x_1 + i\,x_2$ and χ denotes the complex potential related to Φ. As at infinity we have the expansion:

$$\chi = Vz + \frac{Q}{2\pi}\,lg\, z + \ldots$$

the value of the complex integral becomes: $-\varrho V Q$. From this we easily obtain:

$$\int \overline{y}_1\, ds = -\varrho VQ + \frac{1}{2}\varrho\int dx_2\, v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Now we can calculate the resultant of the exterior forces X_1. Using (19) and (23) we obtain:

$$-W' = \int ds\, X_1 = -\frac{1}{2}\varrho\int dx_2\, v^2 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This is ZEILON's result, quoted before (form. (9)).

It is not difficult to ascertain that the resultant of the forces X_1 and X_2 at every point stands perpendicular to the surface of the body, and represents the pressure, calculated from formula (11), supposing the additive constant that occurs in the expressions for q and p is chosen in such a way, that the pressure at infinity has the value $-\frac{1}{2}\varrho V^2$. [1])

§ 5. We have seen how ZEILON's formula can be obtained by a

[1]) This also applies to the back side of the body, where $X_1 = Y_1$, $X_2 = Y_2$.

consideration of the "interior" (or second order) forces, acting in the points of the surface of the body.

However, forces of just the same kind arise from the vorticity that is present in the wake. These forces will have some influence on the whole field of motion. Especially the components y_1 may be considered as producing a system of (negative) sources, and hence the outflow Q at great distances from the body will have to be diminished by the amount:

$$\iint dx_1\, dx_2\, \frac{y_1}{\varrho V} = \frac{1}{V} \iint dx_1\, dx_2\, u_2\, w \quad . \quad . \quad . \quad . \quad (25)$$

where the integration extends over the wake.

From equations (12) we obtain the following expression for the vorticity:

$$w = \frac{1}{2\, \pi\, \varrho\, \nu} \iint d\xi_1\, d\xi_2 \left[- Y_1 \frac{\partial \varepsilon}{\partial x_2} + Y_2 \frac{\partial \varepsilon}{\partial x_1} \right] \quad . \quad . \quad . \quad (26)$$

At the same time the full expression for u_2 may be written as follows:

$$u_2 = \frac{\partial \varphi}{\partial x_2} + \frac{\nu}{V} w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

A consideration of the orders of magnitude of the two terms, occurring in the expression for w, shows us that the first term is preponderant, and that the second one may be neglected.

When we go away from the body in the direction of the positive axis of x_1, the derivative $\partial\varphi/\partial x_2$ decreases as r^{-2}. Hence for sufficiently great values of REYNOLDS' number the integral:

$$\frac{1}{V} \iint dx_1\, dx_2\, \frac{\partial \varphi}{\partial x_2}\, w,$$

which represents the first part of (25), may be limited to that part of the wake, in which the diffusion of the vorticity is not yet appreciable. Then we may write:

$$w = - \frac{dv}{dx_2},$$

and so we get:

$$- \frac{1}{V} \iint dx_1\, dx_2\, \frac{\partial \varphi}{\partial x_2} \frac{dv}{dx_2} = \frac{1}{V} \int dx_2\, v\, \frac{d}{dx_2} \left(\int dx_1\, \frac{\partial \varphi}{\partial x_2} \right) = \frac{1}{V} \int dx_2\, v\, \frac{\partial \varphi}{\partial x_1},$$

where $\partial\varphi/\partial x_1$ has to be taken at the back of the body. Using (16^c) we find for the latter expression:

$$- \frac{1}{V} \int dx_2\, v\, (V + v) = Q - \frac{1}{V} \int dx_2\, v^2 \quad . \quad . \quad . \quad . \quad (28)$$

§ 6. In order to calculate the second part of (25), that is $\frac{\nu}{V^2}\iint dx_1\, dx_2\, w^2$, we make use of the approximation:

$$\varepsilon \cong \left(\frac{\pi}{2\,k\,x_1}\right)^{1/2} e^{-\frac{k\,(x_2-\xi_2)^2}{2x_1}},$$

and write (compare (15) and the end of § 2):

$$\int d\xi_1\, Y_1 = \varrho\, V\, v\,(\xi_2).$$

Then we obtain for the vorticity:

$$w = \frac{V}{2\,\pi\,\nu}\int d\xi_2\, v\,(\xi_2)\,\frac{\pi^{1/2}\,k^{1/2}\,(x_2-\xi_2)}{2^{1/2}\,x_1^{3/2}}\, e^{-\frac{k\,(x_2-\xi_2)^2}{2x_1}},$$

or by means of a partial integration (writing $v'(\xi_2)$ for $dv/d\xi_2$):

$$w = -\frac{V}{2\,\pi\,\nu}\int d\xi_2\, v'\,(\xi_2)\left(\frac{\pi}{2\,k\,x_1}\right)^{1/2} e^{-\frac{k\,(x_2-\xi_2)^2}{2x_1}} \quad . \quad . \quad . \quad (29)$$

Now we have to calculate:

$$\frac{1}{4\pi V}\int d\xi_2\, v'(\xi_2)\int d\xi'_2\, v'(\xi'_2)\int dx_1\int dx_2\,\frac{1}{x_1}\, e^{-\frac{k}{2x_1}[(x_2-\xi_2)^2+(x_2-\xi'_2)^2]}\ .$$

The integration with respect to x_2 gives:

$$\frac{1}{4\pi V}\sqrt{\frac{\pi}{k}}\int d\xi_2\, v'(\xi_2)\int d\xi'_2\, v'(\xi'_2)\int dx_1\,\frac{1}{\sqrt{x_1}}\, e^{-\frac{k}{x_1}\left(\frac{\xi_2-\xi'_2}{2}\right)^2}\ .$$

The integration with respect to x_1, considered separately, appears to be divergent when x_1 approaches ∞. The whole integral, however, is convergent on account of the circumstance that both:

$$\int d\xi_2\, v'\,(\xi_2) = 0, \quad \text{and:} \quad \int d\xi'_2\, v'(\xi'_2) = 0.$$

Therefore we shall write our integral:

$$\lim_{m=\infty}\int d\xi_2\, v'(\xi_2)\int d\xi'_2\, v'(\xi'_2)\int_0^m dx_1\,\frac{1}{\sqrt{x_1}}\, e^{-\frac{k}{x_1}\left(\frac{\xi_2-\xi'_2}{2}\right)^2} \quad . \quad . \quad . \quad (\alpha)$$

As we have:

$$\int d\xi_2\, v'(\xi_2)\int d\xi'_2\, v'(\xi'_2)\int_0^m dx_1\,\frac{1}{\sqrt{x_1}} = 0$$

for all values of m, we may write the expression (α) also in the form:

$$\lim_{m=\infty}\int d\xi_2\, v'(\xi_2)\int d\xi'_2\, v'(\xi'_2)\int_0^m dx_1\,\frac{1}{\sqrt{x_1}}\left[-1+e^{-\frac{k}{x_1}\left(\frac{\xi_2-\xi'_2}{2}\right)^2}\right] \quad . \quad (\beta)$$

In this form the integration with respect to x_1 is convergent; we have:

$$\int_0^\infty dx_1 \frac{1}{\sqrt{x_1}} \left[-1 + e^{-\frac{k}{x_1}\left(\frac{\xi_2-\xi'_2}{2}\right)^2} \right] = -\sqrt{\pi k} \,.\, | \xi_2 - \xi'_2 | \,.$$

Substituting this result in (β), and making use of some partial integrations, we finally obtain:

$$\frac{\nu}{V^2} \iint dx_1 \, dx_2 \, w^2 = \frac{1}{2V} \int d\xi_2 \, v^2 \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Adding together the amounts given in formulae (28) and (30), we get for the full value of the integral (25):

$$\frac{1}{V} \iint dx_1 \, dx_2 \, u_2 \, w = Q - \frac{1}{2V} \int dx_2 \, v^2 \quad . \quad . \quad . \quad . \quad (31)$$

This is exactly the opposite of the value (23), divided by ϱV, which of course is necessary, as we have proved in eq. (7) that the resultant of the whole system of "interior" forces always has the value zero.

Subtracting (31) from Q, we obtain for the corrected value of the outflow:

$$Q' = \frac{1}{2V} \int dx_2 \, v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Hence we have:

$$W' = \varrho V Q' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

so that the accordance with the general theorem again is established.

§ 7. One final remark has to be added to the former considerations. We have seen that in order to get ZEILON's formula for the resistance the influence of the second order terms had to be taken into account. However, we have considered their influence only partly: the "interior" forces y_1, y_2, acting in the wake, especially those acting at distances not far from the body, will cause certain velocities at the circumference of the latter, and thus will disturb the fulfilment of the boundary conditions. So the solution is incomplete, when not a correction for this effect has been calculated, which probably will also affect the resistance.

We may summarize our considerations by saying that as soon as formula (11) is used to determine the pressure p, we introduce terms of the second degree in the u_i, and it becomes necessary to calculate quite a new solution.

This calculation, however, is beyond the scope of the present note; perhaps it may be the subject of a separate paper.
