**Applied Geology.** — *The equation of flow of oil and gas to a well after dynamic equilibrium has been established.* By J. VERSLUYS.

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SYMBOLS USED IN TEXT.

$p$ Pressure at an arbitrary distance from the axis of the well, in atmospheres.
$p_r$ Pressure at the bottom of the well.
$P$ Pressure at which oil would be saturated with the coexisting gas.
$P_1$ Reservoir pressure.
$a$ The force exerted by atmospheric pressure per unit of area.
$q$ Volume of liquid flowing per unit of time through the unit of area in the cross section.
$Q$ Volume of liquid flowing per unit of time through the cross section which is considered.
$v$ Volume of gas measured at atmospheric pressure per unit of time flowing through the unit of area in the cross section.
$u$ Same measured at the prevailing pressure $p$.
$R$ Depletion radius.
$K$ Darcy's coefficient; this is the volume of a liquid which has the unit coefficient of internal friction, flowing per unit of time through the unit of area of the cross section, when the drop of pressure head is equal to the unit of force per unit of area, per unit of length.
$\eta_0$ Coefficient of internal friction (in text called viscosity) of oil.
$\eta_g$ Same for gas.
$\alpha$ Absorption coefficient of gas and oil.
$\varrho$ Horizontal distance of the point under consideration to the axis of the well.
$\varepsilon$ Thickness of the oil bearing layer.
$W_1$, $W_2$, etc. Amounts of work.

The ideal condition of the flow of oil and gas to a well is the following: The thickness of the oil bearing layer is unvaried over a great area, the layer lies nearly horizontally and the well is perforated or open opposite the whole thickness of the sand. The sand slopes at a small rate in all directions from the well to a great distance and the line of the edgewater is at a great distance from the well. The oil contains dissolved gas but pressure is at the beginning so great that the oil is not saturated with gas at this prevailing pressure; so there is no free gas before the well is opened.

Under these conditions, the motion of oil and gas may be estimated to be horizontal and so slow that kinetic energy may be neglected. So all positive work, which is performed, is counterbalanced by the work of the internal friction of liquid and gas and the former in this way is converted to heat. In the oil where work is performed by pressure differentials, some heat will be developed and this heat will tend to cause a rise of temperature. The work performed by the gas is a consequence of expansion due to which the temperature would decrease. All this work is exhausted by internal friction of the gas and consequently, in the gas the temperature would remain unaltered as long as BOYLE's law would apply and the temperature would be unaltered. Where gas is liberated from the oil, however, work will be performed which is smaller than the equivalent of the heat which is bound at its liberation. The first energy is converted to heat but this heat is not sufficient to prevent cooling caused by liberation of gas. Also, the friction in the oil does not yield sufficient heat to prevent lowering of temperature. Consequently, as a well produces oil and gas, temperature in the sand will decrease due to the liberation of gas. As soon as the edgewater approaches the well, the cooling will stop and make place for a rise of temperature due to the fact that less gas is dissolved in the edgewater than in the oil. Changes of temperature will be disregarded in the mathematical development, for sake of simplicity.

As the well is opened, pressure therein is decreased and oil and gas flow towards the well; at the same time edgewater moves towards the well. The pressure in the well, however, is so much decreased that within a certain radius $R$, a part of the dissolved gas is set free. Consequently, within this radius $R$, oil and the dissolved gas but also free gas flow to the well; outside this radius oil with only dissolved gas flow toward the well. Thus, within the radius $R$, depletion takes place which means that a portion of the pores of the sand no longer are filled with oil but with gas. This radius will be called the "*depletion radius*". The sand, as it has been stated before, is supposed to be not very thick and it will be presumed that the paths of the gas and the oil molecules are horizontal so that gravity exerts no work by its action on these molecules.

As it has been stated before, within the depletion radius, oil is partly displaced by gas and this oil flows to the well. This is a part of the production of the well. Consequently, the production of the well should be variable with time. If this were taken into regard, our problem would be too complicated and it will be at first supposed that a dynamic equilibrium has been established. Afterwards it can be attempted to take the influence of time into regard by means of an approximation. Because a condition of dynamic equilibrium is considered, we have assumed that the condition is independent of the time. So we assume that constant volumes of oil and gas flow per unit of time to the well. The velocity of oil and gas in the sand varies with respect to the distance to the well. According to our assumption of the case that the depletion radius does no more increase, however, pressure does not vary with respect to time. The depletion radius $R$ is unvariable. Under these circumstances, the pressure at a distance $R$ from the axis of the well is so great, that oil at that distance is saturated with the coexistent gas. Within this radius pressure is smaller and a part of the coexistent gas must be set free with the consequence that depletion takes place. At a distance greater than this depletion radius $R$ from the well, pressure is so great that all coexisting gas remains dissolved. Thus, within the depletion zone, a mixture of oil and gas flows to the well while at a distance greater than this radius merely liquid with dissolved gas, which behaves as an ordinary liquid, flows to the well. Through every cylindrical surface generated by the evolution of a vertical line about the axis of the well, the same volume of oil flows to the well per unit of time and also the same amount of gas. Within the distance of the depletion radius, two conditions may occur:

*first,* oil and gas flow at the same speed at any distance from the well, and *second,* free gas and oil have different velocities at any distance from the well.

According as the oil approaches the well, pressure decreases and more gas is set free but at the same time gas which is already set free expands. So work is performed not only by the difference of pressures in the well and at a distance $R$ from the well but at the same time work if performed by the gas at the moment it is set free and by its expansion (1). At the motion of oil and gas through the pores, internal friction causes resistance and this resistance performs negative work. The velocities of oil and gas, as it has been explained before, are so small that they may be neglected and practically no potential energy is converted to kinetic energy. Moreover, it has been stated that the motion will be considered to be horizontal and consequently the action of gravity on the oil performs no work.

Equations of the equilibrium will be deduced for the various quantities of work which are performed per unit of time in the space which is enclosed by the cylindrical surfaces with radii $\varrho$ and $\varrho + d\varrho$ about the axis of the well. It will first be assumed that the first of the above mentioned conditions prevails, viz. oil and gas flow at the same speed.

Flow of an uncompressible liquid like oil is determined by DARCY's (2 and 4) law which can be written as follows:

$$q = \frac{Ka}{\eta_0} \frac{\partial p}{\partial x} \qquad \text{(1)}$$

if $x$ is the distance measured in a direction opposite to the motion. The meaning of the other symbols used in this article is explained in the list at the beginning of this article.

As it has been explained in a former paper (3, page 60) for a gas this equation can be written as follows:

$$u = -\frac{Ka}{\eta_g} \frac{\partial p}{\partial x} \qquad \text{(2)}$$

as it has been presumed, that BOYLE's law applies to the gas. If in this equation $\frac{V}{p}$ is substituted to $u$, this equation is converted to:

$$v = \frac{K\alpha}{\eta_g} p \frac{\partial p}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Outside the radius $R$, according to HENRY's law which is supposed to hold good, the amount of gas dissolved in the unit of volume of oil would occupy a volume:

$$aP \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

at atmospheric pressure. Within this radius pressure is $p < P$, at a distance $\varrho$ from the axis of the well and consequently at that distance a quantity of gas which occupies a volume:

$$ap \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

at atmospheric pressure would be dissolved per unit of volume of oil. The gas which is set free per unit of volume of oil would occupy a volume:

$$a(P-p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

at atmospheric pressure and:

$$a\frac{P-p}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \quad (7)$$

at the prevailing pressure.

As long as the dynamic equilibrium, exists, a constant volume of oil $Q$ flows per unit of time through every cylinder and this volume of oil is accompanied by a volume of gas which, according to equation (7), would occupy a volume:

$$Qa\frac{P-p}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

at the prevailing pressure. It has been assumed that gas and oil have formed an intimate mixture. The pressure on the cylinder with radius $\varrho$ pushes a volume of oil $Q$ and a volume gas as expressed by formula (8) per unit of time through this cylinder. The work performed by the pressure on the oil is:

$$a\,Qp \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and the work performed by the pressure on the gas is in the same way:

$$a\,Qa\,(P-p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The differential of the two amounts of work performed by the pressure on the portion of the sand between the two cylinders with radii $\varrho$ and $\varrho + d\varrho$ is the sum of the differentials of (9) and (10). So for this work may be written:

$$dW_1 = a\,Q\,(1-a)\,dp \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Within the space between the two cylinders we considered, part of the gas is dissolved and a part of the gas is free. As it enters this space, gas and liquid have a pressure $p + dp$ and as they leave this space the pressure is $p$. In a former paper (1), it has been proved that the amount of work performed does not change in case a part of the gas is dissolved in liquid and saturates same. This has been deduced as follows: If the considered liquid is saturated at a pressure $p + dp$ with an amount of gas which at atmospheric pressure occupies volume $V_1$, and if the amount of gas which is free would occupy the atmospheric pressure volume $V_2$, the increment of volume of the free gas is: $\frac{V_2}{p} - \frac{V_2}{p + dp} = V_2 \frac{dp}{p}$ according as pressure decreases to $p$. At a pressure $p$ the volume of this amount of gas would have been $\frac{V_2}{p} dp$. The volume of the dissolved gas which would have been $V_1$ at a pressure $p + dp$ would have been reduced to a volume $V_1 \frac{p}{p+dp}$ as pressure would have decreased to $p$, both volumes measured at atmospheric pressure. So as the pressure decreases from $p + dp$ to $p$ an amount of gas is set free, the volume of which would be $V_1 — V_1 \frac{p}{p+dp} = \frac{V_1}{p} dp$ and at the prevailing pressure the volume would be $\frac{V_1}{p^2} dp$. As a consequence of the decrease of pressure, the volume of liquid and the coexistent gas has increased $(V_1 + V_2) \frac{dp}{p^2}$. The work performed at this increase of volume is $(V_1 + V_2) \frac{dp}{p}$. So the work performed is independent of the presence of liquid in which a part of the gas is dissolved at the prevailing pressures, as the sum $V_1 + V_2$ is not changed by the presence of the liquid. So the work performed at the liberation and the expansion of gas within the space we considered is:

$$dW_2 = a\,\alpha P Q p \left(\frac{1}{p + dp} - \frac{1}{p}\right) = a\,\alpha P Q \frac{dp}{p} \quad . \quad . \quad . \quad (12)$$

The work which is performed per unit of time by the friction resistance in the space we considered can be expressed as follows: The total volume which flows per unit of time through the inner cylinder is the sum of the volumes of oil and gas expressed by (8):

$$Q + Q\alpha \frac{P - p}{p} = Q \frac{p + \alpha P - \alpha p}{p} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Per unit of area, this volume is:

$$\frac{Q}{2\,\pi\varrho\varepsilon} \frac{p + \alpha P - \alpha p}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In case $q$ units of volume of oil would flow through the unit of area in the cross section and no gas, the resistance in this space would be:

$$-\frac{q\eta_0}{K}\,d\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and the total force exerted by this resistance would be:

$$-2\,\pi\varrho\varepsilon\frac{q\eta_0}{K}\,d\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This force would exert an energy per unit of time:

$$-2\,\pi\varrho\varepsilon\frac{q^2\eta_0}{K}\,d\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If instead of $q$ the value of formula (14) is substituted, formula (17) is converted to:

$$-\frac{Q^2}{2\,\pi\varrho\varepsilon}\frac{\eta_0}{K}\left\{\frac{p+aP-ap}{p}\right\}^2 d\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In the same way for the work of resistance in the gas the following equation can be deduced:

$$-\frac{Q^2}{2\,\pi\varrho\varepsilon}\frac{\eta_g}{K}\left\{\frac{p+aP-ap}{p}\right\}^2 d\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The volume in which the pores are filled with oil in the solid we considered is expressed by:

$$2\,\pi\varrho\,d\varrho\,\frac{p}{p+aP-ap} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and the volume in which the pores are filled with gas is:

$$2\,\pi\varrho\,d\varrho\,\frac{a\,(P-p)}{p+aP-ap} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

So the total work performed per unit of time by friction resistance in the solid we considered is:

$$dW_3 = -\frac{Q^2}{2\pi K\varepsilon}\frac{(p+aP-ap)\,(\eta_0\,p+\eta_g\,aP-\eta_g\,ap)}{p^2}\frac{d\varrho}{\varrho} \quad . \quad (22)$$

The algebraic sum of the various amounts of work must be O, hence, we may write:

$$dW_1 + dW_2 + dW_3 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

or:

$$\left.\begin{aligned} &aQ\,(1-a)\,dp + \alpha\,aPQ\,\frac{dp}{p} - \\ -\frac{Q^2}{2\pi\varepsilon K}&\frac{(p+aP-ap)\,(\eta_0\,p+\eta_g\,aP-\eta_y\,ap)}{p^2}\frac{d\varrho}{\varrho} = 0 \end{aligned}\right\} \quad . \quad (24)$$

The variables can be separated as follows:

$$\frac{Q}{2\,\pi\varepsilon Ka}\,\frac{d\varrho}{\varrho} = \frac{pdp}{\eta_0\,p - \eta_g\,ap + \eta_g\,aP} \quad \ldots\ldots \quad (25)$$

The second of the above mentioned conditions: oil and gas flow at different velocities will also be considered. In order to solve this problem, two equations of equilibrium will be deduced, one for the work performed in the oil and one for the work performed in the gas. It will be presumed that in the cylindrical surface about the axis of the well with the radius $\varrho$ which has an area $2\pi\varepsilon\varrho$ within the sand in a portion $2\pi\varrho y$ the pores are filled with oil and in a portion $2\pi\varrho\ (\varepsilon - y)$ with gas. For a dynamic equilibrium, it is necessary that at the same distance $\varrho$ pressure in oil and gas are the same. The pressure differential caused by the height of the column of liquid will be neglected as it has been done before.

The work performed by pressure is:

$$dW_1 = aQ\,dp \quad \ldots\ldots\ldots \quad (26)$$

and the work of the resistance:

$$dW_3 = -\frac{Q^2\,\eta_0}{2\,\pi\varrho\,yK}\,d\varrho \quad \ldots\ldots\ldots \quad (27)$$

The equilibrium is expressed by:

$$dW_1 + dW_3 = 0 \quad \ldots\ldots\ldots \quad (28)$$

or:

$$aQ\,dp - \frac{Q^2\,\eta_0}{2\,\pi\varrho\,yK}\,d\varrho = 0 \quad \ldots\ldots\ldots \quad (29)$$

This equation can be transformed as follows:

$$2\,\pi\,Ka\,ydp = Q\,\eta_0\,\frac{d\varrho}{\varrho} \quad \ldots\ldots\ldots \quad (30)$$

This is the equation of the equilibrium of the work per unit of time performed in the oil. For the gas the following deduction can be made.

The work of pressure is:

$$dW_1 = -\,a\,\alpha Q\,dp \quad \ldots\ldots\ldots \quad (31)$$

The work performed by the gas at its liberation and its expansion is again expressed by equation (12). The work of the resistance is expressed by equation (28), if we substitute $\varepsilon - y$ for $y$, $\eta_g$ **for** $\eta_0$ and if Q is replaced by the volume of the free gas at the pressure $\varrho$. This gas volume is expressed by equation (7). So we may write:

$$dW_3 = -\left(\frac{P-p}{p}\right)^2 Q^2\,\frac{\eta_g\,a^2}{2\,\pi\varrho\,(\varepsilon - y)\,K}\,d\varrho \quad \ldots\ldots \quad (32)$$

An equation of equilibrium can be deduced from the equations (32), (12) and (33):

$$dW_1 + dW_2 + dW_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

or:

$$- a\,\alpha Q\,dp + a\,\alpha PQ\frac{dp}{p} - \left(\frac{P-p}{p}\right)^2 \frac{Q^2\,\eta_g\,\alpha^2}{2\,\pi\,K(\varepsilon - y)}\frac{d\varrho}{\varrho} = 0 \quad . \quad (34)$$

or:

$$2\,\pi\,Ka\,(\varepsilon - y)\,dp = Q\,\eta_q\,\alpha\,\frac{P-p}{p}\frac{d\varrho}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

This is the equation of the equilibrium of gas which combined with the equation (30) for oil will yield the solution of the problem. The equations (30) and (35) are two simultaneous differential equations in which the three variables $\varrho$, $p$ and $y$ enter and the differentials of $\varrho$ and $p$. From these two differential equations $y$ may be eliminated and again equation (25) is obtained:

If we integrate this equation between two arbitrary limits of each variable, respectively, $\varrho$ and $\varrho_1$, $p$ and $p_1$, this equation gives the following solution:

$$\frac{2\,\pi\,Ka\varepsilon}{(\eta_0 - \eta_g\alpha)^2 Q}\left[(\eta_0 - \eta_g\,\alpha)(p - p_1) - \eta_g\,\alpha P \lg\frac{(\eta_0 - \eta_q\,\alpha)p + \eta_g\,\alpha P}{(\eta_0 - \eta_g\,\alpha)p_1 + \eta_g\,\alpha P}\right] = \lg\frac{\varrho}{\varrho_1} \quad (36)$$

If equation (25) is integrated between the limits $r$ and $R$, respectively, $p_r$ and $P$, the following equation is obtained:

$$\left.\begin{aligned} &\frac{Q}{2\pi\varepsilon Ka}\lg\frac{R}{r} = \\ &= \frac{1}{(\eta_0 - \eta_g\,\alpha)^2}\left\{(\eta_0 - \eta_g\,\alpha)(P - p_r) - \eta_g\,\alpha P \lg\frac{\eta_0 P}{(\eta_0 - \eta_g\,\alpha)\,p_r + \eta_g\,\alpha P}\right\}\end{aligned}\right\} \quad (37)$$

It has been presumed in the foregoing deductions that $\eta_0$ is independent of the pressure. This is not exact, but as now no data concerning the variation of the viscosity of oil with the amount of gas dissolved are known.

Another point has been neglected in the above formulae. The volume of the liquid also depends on the amount of gas dissolved in the oil. For the moment no data re this question are available.

Furthermore it has been accepted that a dynamic equilibrium has been established.

If in the above formulae (25, 36 and 37) we substitute $\alpha = 0$ which means that gas is insoluble, we obtain:

$$\frac{Q}{2\,\pi\varepsilon\,Ka}\frac{d\varrho}{\varrho} = \eta_0\,dp, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$Q = \frac{2\,\pi\,Ka\,\varepsilon}{\eta_0}\frac{p - p_1}{\lg\varrho - \lg\varrho_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and

$$Q = \frac{2\,\pi\,Ka\,\varepsilon}{\eta_0}\frac{P - p_r}{\lg R - \lg r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Supposing that gas is insoluble, means that no gas is present, and the pores are wholly filled with oil. There would not be a material depletion. Formula (40) is practically J. DUPUIT's (5 and 6) formula for the flow of water to an artesian well.

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