Mathematics. — *On the Fundamental Theorems of Invariant-theory for the Unitary Group.* By H. W. TURNBULL. (Communicated by Prof. R. WEITZENBÖCK).

(Communicated at the meeting of March 28, 1931).

Introduction.

The present work is designed to show that the properties of linear transformations of n variables, within the *unitary* group, are intimately related to the classical projective invariant theory. In an interesting series of publications [1]) R. WEITZENBÖCK has proved that, by adjoining a particular linear or quadratic form to a set of given groundforms, and by discussing the projective invariant theory of this augmented set, it is possible to give a complete account of the invariants belonging to certain subgroups: e.g. the affine, the orthogonal, and so on. For the affine subgroup a linear groundform is adjoined; for the orthogonal, a quadratic $(x \mid x) \equiv \sum_{i=1}^{n} x_i^2$ is adjoined.

I propose to shew that the unitary subgroup can be dealt with in the same way, and that it needs a *bilinear* adjoint groundform. The necessity for this is fairly obvious; the sufficiency is not so obvious, but follows by extending to the unitary case a device which has been successfully used in the orthogonal case [2]).

I. *Unitary transformations.*

Let

$$x = P\xi, \quad x_i = \sum_{k=1}^{n} p_{ik}\, \xi_k, \quad i = 1, 2, \ldots n, \quad . \quad . \quad . \quad . \quad (1)$$

denote a linear transformation of variables x_i to ξ_i, with a matrix $P = (p_{ik})$ whose elements p_{ik} are complex numbers. Let $\bar{x} = \bar{P}\bar{\xi}$ denote the conjugate complex transformation, with a matrix $\bar{P} = (\bar{p}_{ik})$ such that $p_{ik}, \bar{p}_{ik}$ are conjugate complex numbers, and so also are x_i, $\bar{x}_i$ and ξ_i, $\bar{\xi}_i$.

---

[1]) Sitzungsberichte der Wiener Akademie **122** (1913) — **132** (1924); cf. R. WEITZENBÖCK Invariantentheorie (Groningen 1923), 223—301.

[2]) Cf. E. STUDY, Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vektorenrechnung (Braunschweig 1923) **23**.

The unitary transformation is defined by the condition

$$\bar{P}'\,P = (\delta_{ik}) = E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\bar{P}' = (\bar{P}_{ki})$, and $\delta_{ik}=0,\ i \neq k;\ \delta_{ik}=1,\ i=k$. Thus the product of the matrices $\bar{P}', P$ is the unit matrix E. The necessary and sufficient condition for (2) to hold is that

$$(\bar{x}\,|\,x) = \bar{x}_1\,x_1 + \ldots + \bar{x}_n\,x_n = \bar{\xi}_1\,\xi_1 + \ldots + \bar{\xi}_n\,\xi_n = (\bar{\xi}\,|\,\xi) \quad . \quad . \quad (3)$$

namely, the quadric $(\bar{x}\,|\,x)$ is *latent* in the group Γ of these transformations.

If a, b denote further vectors cogredient with x (so that $\bar{a}, \bar{b}$ are cogredient with $\bar{x}$) then the inner products $(\bar{a}\,|\,a)$, $(\bar{a}\,|\,b)$, are absolute invariants of the group Γ, as is easily verified. The outer products

$$(ab\ldots hk), \quad (\bar{a}\bar{b}\ldots\bar{h}\bar{k}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which involve n cogredient vectors $a, b, \ldots, h, k$, (or their n conjugates) are relative invariants. After transformation they are multiplied by the respective determinants $\triangle = |p_{ik}|$, $\bar{\triangle} = |\bar{p}_{ik}|$.

Since it follows from (2) that $\bar{\triangle}\triangle = 1$, neither matrix P nor $\bar{P}$ is singular. Hence both transformations are reversible, namely $\xi = P^{-1}\,x$, $\bar{\xi} = \bar{P}^{-1}\,\bar{x}$. Further as is well known, if $\xi = Q\eta$ is another unitary transformation, the product $P\,Q$ gives a resultant unitary transformation, $x = P\,Q\,\eta$. In fact $(\bar{P}\bar{Q})'\,(PQ) = \bar{Q}'\,\bar{P}'\,PQ = E$. Relation (3) shews that the vectors $x, a, b, \ldots, k$ (which are cogredient) are contragredient to their conjugates $\bar{x}, \bar{a}, \bar{b}, \ldots, \bar{k}$.

By an *invariant* I is meant a polynomial unitary invariant, namely a polynomial in the components a_i of any number of cogredient or contragredient vectors $a, b, \ldots, \bar{a}, \bar{b}, \ldots$, which are transformed by (I) to $\alpha, \beta, \ldots, \bar{\alpha}, \bar{\beta}, \ldots$ such that

$$I \equiv I\,(a_i\ldots) = \Phi\,. \quad I\,(\alpha_i\ldots) \not\equiv 0\,,$$

where Φ is a polynomial in the coefficients p_{ik}, $\bar{p}_{ik}$ of the transformation. The vectors $\alpha, \bar{\alpha}, \ldots$ satisfy $a = P\alpha$, $\bar{a} = \bar{P}\,\bar{\alpha}$, $b = P\beta, \ldots$ This is the ordinary definition of a rational integral invariant of a group, adapted to the present case of the *unitary group* Γ.

I shall prove the fundamental theorem that every such invariant is expressible as a polynomial in the arguments θ of the three types:

$$\theta_1 = (ab\ldots hk), \quad \theta_2 = (\bar{a}\,|\,b), \quad \theta_3 = (\bar{a}\bar{b}\ldots\bar{h}\bar{k})\,.$$

2. *Preliminary Lemmas.*

As in the usual theory it is only necessary to consider invariants which are homogeneous in their several components. The theorem needs two preliminary lemmas.

LEMMA I: *A unitary transformation exists which transforms a given normalised vector p into another such vector q.*

Proof: Consider the transformation

$$\frac{2\,(x\,|\,\bar{p}+\bar{q})}{(p+q\,|\,\bar{p}+\bar{q})}\,(p_i+q_i)-x_i=\xi_i,\quad (i=1,2,\dots n)\quad . \quad . \quad . \quad (1)$$

where $(p+q\,|\,\bar{p}+\bar{q})=\sum_i (p_i+q_i)(\bar{p}_i+\bar{q}_i)\neq 0$, and p, q are at present arbitrary constant vectors. This is a linear transformation from ξ to x, which is unitary. For on forming this relation, by means of the conjugate value of $\bar{\xi}$ in terms of $\bar{x}, p, q, \bar{p}, \bar{q}$. it is found that $(\bar{\xi}\,|\,\xi)=(\bar{x}\,|\,x)$.

Further, suppose that

$$(\bar{p}\,|\,p)=(\bar{q}\,|\,q),\quad (\bar{p}\,|\,q)=(\bar{q}\,|\,p),\quad p+q\neq 0\quad . \quad . \quad . \quad . \quad (2)$$

Now let p and q be any two normalised vectors, which means that $(\bar{p}\,|\,p)=(\bar{q}\,|\,q)=1$. This secures the first relation (2). If π_x denote the whole coefficient of p_i+q_i in (I), we may write

$$\pi_x\,(p+q)-x=\xi,\quad \pi_{\bar{x}}\,(\bar{p}+\bar{q})-\bar{x}=\bar{\xi},\quad . \quad . \quad . \quad . \quad (3)$$

and it is found that $\pi_p=\pi_q=\pi_{\bar{p}}=\pi_{\bar{q}}=1$, provided that (2) holds.

Substituting $x=p$ in (3), then $\bar{x}=\bar{p}$, $\xi=q$, $\bar{\xi}=\bar{q}$, which proves the lemma if (2) holds. If $(\bar{p}\,|\,q)\neq(\bar{q}\,|\,p)$, choose a third normalised vector $r=\{0,0,\dots,0,1\}$. If $p_n=\varrho e^{i\alpha}$, $|\,p_n\,|=\varrho\geqslant 0$, the unitary transformation, of matrix $e^{-i\alpha}E$, changes p_n to ϱ, and $\bar{p}_n$ also to ϱ. Conditions (2) now hold for p, r: and similarly for q, r. By transforming p first to r and r next to q, p is transformed to q and the lemma is proved. Simultaneously $\bar{p}$ is also transformed to $\bar{q}$.

LEMMA II: *The following identities emerge by use of the operator*

$$\left(\frac{\partial}{\partial q}\,\middle|\,\frac{\partial}{\partial \bar{q}}\right)=\sum_{i=1}^{n}\frac{\partial^2}{\partial q_i\,\partial \bar{q}_i}=\Omega .$$

$$\Omega\,(\bar{q}\,|\,q)^{\lambda}=\lambda\,(n+\lambda-1)\,(\bar{q}\,|\,q)^{\lambda-1},\quad \Omega\,(\bar{q}\,|\,a)\,(q\,|\,\bar{b})=(a\,|\,\bar{b}),$$

$$\Omega\,(\bar{q}\,|\,k)\,(ab\dots hq)=(ab\dots hk),\quad \Omega(q\,|\,\bar{k})\,(\bar{a}\bar{b}\dots\bar{h}\bar{q})=(\bar{a}\bar{b}\dots\bar{h}\bar{k}),$$

$$\Omega\,(\bar{q}\,|\,q)\,(\tilde{q}t)=(n+1)(\tilde{q}t),\quad t=\bar{a}\ \text{or}\ (b\dots hk)_{n-1},\quad \tilde{q}=q\ \text{or}\ \bar{q}\,.$$

The proofs are straightforward.

The above proof of Lemma I is suggested by a somewhat similar theorem given by HILTON [1]).

[1]) *Linear Substitutions* (Oxford 1914), 42.

3. *Proof of the Fundamental Theorem.*

Every polynomial invariant of vectors $a, \bar{a}, b, \bar{b}, \ldots$ *is expressible as a polynomial in arguments* θ *of the three types* $\theta_1\, \theta_2\, \theta_3$.

This follows the orthogonal proof given by E. STUDY in the above reference [1]). If $n=1$, the matrix P can be written as $e^{i\varphi}$, $\bar{P}$ is $e^{-i\varphi}$, and $x=P\xi$ is an ordinary scalar product. The truth of the theorem is obvious.

By assuming it true for $m=n-1$ we infer by induction, its truth for n. If P_m denote the minor matrix of P obtained by erasing the last row and column, we write

$$P_0=\begin{pmatrix} P_m & \cdot \\ \cdot & 1 \end{pmatrix},\quad P_m=\begin{pmatrix} p_{11} & \ldots & p_{1m} \\ \vdots & & \\ p_{m1} & \ldots & p_{mm} \end{pmatrix},\quad \bar{P}_0=\begin{pmatrix} \bar{P}_m & \cdot \\ \cdot & 1 \end{pmatrix}. \quad . \quad . \quad (1)$$

In P_0 the last row and column each contains $n-1$ zeroes. If P_m is unitary so also is P_0. Now P_0 belongs to the subgroup γ which leaves the vector $q=\{0.0, \ldots, 0,1\}$ latent: and this q is normalised since $(\bar{q} \mid q)=1$. Also in the subgroup γ every component a_i, $(i=1, 2, 3, \ldots m)$ can vary, but a_n is fixed.

Let $(ab \ldots h)_m = |\, a_1\, b_2 \ldots h_m \,|$ be a determinant in the first m components of the vectors: also let $(\bar{a} \mid b)_m = \sum_{i=1}^{m} \bar{a}_i\, b_i$. If I is an invariant of the group Γ, it can be expressed in simplified form as $\Sigma\, k_i\, g_i$, where the g_i are functions solely of the n components a_n, $\bar{a}_n \ldots$, and where k_i are free from the same. By hypothesis any invariant, of a field lower than n, is expressible in the suitable types. We infer that, as in the orthogonal case, each k_i can be expressed in terms of $(ab \ldots h)_m$, $(\bar{a} \mid b)_m$ $(\bar{a}\,\bar{b} \ldots \bar{h})_m$, within the subgroup.

By using the vector $q=\{0.0, \ldots, 0,1\}$ we find that

$$(ab \ldots h)_m=\frac{(ab \ldots hq)}{\sqrt{\bar{q} \mid q}},\ (\bar{a} \mid b)_m=\frac{(\bar{q} \mid q)(\bar{a} \mid b)-(q \mid \bar{a})(\bar{q} \mid b)}{\sqrt{(\bar{q} \mid q)}}. \quad . \quad (2)$$

$a_n=\dfrac{(\bar{q} \mid a)}{\sqrt{(\bar{q} \mid q)}}$, etc., where $\sqrt{(\bar{q} \mid q)}$ is the positive square root of $(\bar{q} \mid q)$.

Now for all values of q, the vector $q/\sqrt{(\bar{q} \mid q)}$ is normalised, and so is its conjugate. Also we may transform this q to any other normalised vector p by Lemma I. Such a transformation replaces the subgroup γ by a *similar* subgroup $\gamma^{\star}$, and an invariant of all such subgroups is an invariant of the whole group Γ. This change from γ to $\gamma^{\star}$ leaves the expressions $(ab \ldots hq)$ etc. in (2) unaltered in type. In fact q becomes p, by the Lemma; the inner product type θ_2 is unchanged: and we have

---

[1]) Cf. TURNBULL, *Theory of Determinants, Matrices and Invariants* (Glasgow 1928), **318**.

only to consider the change in the outer product. Using § 2 (3), let a become α etc., then

$$(\alpha\beta\ldots\varrho p) = (\pi_a(p+q) - a, \pi_b(p+q) - b, \ldots, \pi_q(p+q) - q)\,.$$

The right side denotes a determinant of n columns, where in the last column

$$\pi_q(p+q) - q = p.$$

On expansion we have a sum of determinants $(lm\ldots q)$, where $l = p, q, a;\ m = p, q, b; \ldots$ The nonzero terms of this sum are given by

$$(-)^n\,[\pi_a(qbc\ldots hp) + \pi_b(aqc\ldots hp) + \ldots + \pi_h(ab\ldots qp) - (abc\ldots hp)]$$
$$= (-)\ [\pi_q(ab\ldots hp)\ - \pi_p(ab\ldots hq)\ - (ab\ldots hp)]$$

by a fundamental identity. Since $\pi_p = \pi_q = 1$, we finally have

$$(\alpha\beta\ldots\varrho p) = (-)^{n-1}(ab\ldots hq),$$

so that the outer product is unaltered in type, but for sign: and similarly for $(\bar{a}\bar{b}\ldots\bar{h}\bar{q})$. The argument is unaffected if, as before, an auxiliary vector r is needed. The steps $p\to r$, $r\to q$ can be combined.

Summing up, so far, we infer that every I belonging to the group is expressible as

$$I = (G_1 + G_2\sqrt{(\bar{q}\,|\,q)})\,/\,(\sqrt{(\bar{q}\,|\,q)})^k,$$

where each of G_1 and G_2 is a polynomial of the type

$$G\{\bar{q}\,|\,q), (\bar{q}\,|\,a), (q\,|\,\bar{a}), \ldots, (ab\ldots hq), (\bar{a}\bar{b}\ldots\bar{h}\bar{q}), \ldots (\bar{a}\,|\,b), \ldots\}\,,$$

and where k is necessarily a positive integer, since only one q enters every type (2). The present q is an arbitrary normalised vector.

Now $G_1 \neq 0$, else we could cancel out $\sqrt{(\bar{q}\,|\,q)}$ and treat G_2 as a new G_1. This being so, G_2 must vanish; for otherwise we could always express $\sqrt{(\bar{q}\,|\,q)}$ *rationally* in terms of the other rational quantities involved in this relation: which is impossible even in the case $q = \{0.0, \ldots, 1.1\}$. Hence $G_2 = 0$. Clearing fractions we have

$$(\bar{q}\,|\,q)^\lambda I = G\{(\bar{q}\,|\,q), \ldots, (ab\ldots q), \ldots\}$$

where λ can only be a positive integer, to make the left side agree with the rational function of q on the right. This is an identity for all q. By operating λ times with Ω and using Lemma II, the left hand member becomes μI, where μ is a positive integer, and the right loses all its q and $\bar{q}$, becoming a polynomial in the $\theta_1, \theta_2, \theta_3$. This proves the theorem.

4. *Applications of the Theorem.*

Since the above proof holds for cogredient and contragredient vectors

$a, \alpha, b, \beta, \ldots,$ it also holds for general groundforms which can be symbolised by one or more such symbolic vectors. Again *all compound matrices of P and $\overline{P}$ are unitary;* for example the second compound of the matrix $R = [r_1, r_2, \ldots r_n]$ where r_j is a vector of elements r_{ij} $(i = 1.2, \ldots n)$ is.

$$R^{(2)} = [(r_{j_1}, r_{j_2})_{k_1 k_2}], \quad \begin{pmatrix} j_1, j_2 = 1, 2, \ldots, n, & j_1 \neq j_2 \\ k_1, k_2 = 1, 2, \ldots, n, & k_1 \neq k_2 \end{pmatrix}$$

Now $\overline{R}^{(2)} R^{(2)} = (\overline{r}_{i_1} \overline{r}_{i_2} \mid r_{j_1} r_{j_2}) = E^{(2)}$, provided that $(\overline{r}_i \mid r_i) = 1$, $(\overline{r}_i \mid r_j) = 0$, which is the case. Here $E^{(2)}$ is the unit matrix of $\binom{n}{2}$ rows and columns. Likewise for all compounds $R^{(s)}, \ldots, R^{(n-1)}$. It follows that groundforms of all types, involving variables x, $(x\,y)$, $(x\,y\,z), \ldots,$ $u = (x\,y \ldots)_m$, (where $m = n-1$), can be treated within the unitary group Γ, exactly as in the usual projective group. Symbols can be compounded or resolved, the ARONHOLD process avails, and polarisation also [1]). Thus a compound of $n-1$ symbols $a, b, \ldots, h$ may be written with the accent, $a' = (a\,b \ldots h)_m$ and it is contragredient to a but is cogredient with $\overline{a}$.

For example, if $n = 3$, such types as

$$\theta_1 = (abc), \quad (ab\overline{c}'), \quad (a\overline{b}'\overline{c}'), \quad (\overline{a}'\overline{b}'\overline{c}')$$

will appear. The more complicated intermediate compounds, short of the $(n-1)^{\text{th}}$, involve further types $(a\,b \mid \overline{c}\,\overline{d})$, $(a\,b\,c \mid \overline{d}\,\overline{e}\,\overline{f}), \ldots,$ or else complex symbols.

The *real orthogonal* group provides a special case of the unitary group, in which all the elements are real, so that the distinction between a and $\overline{a}$ disappears. Instead of three fundamental types there are two, $(ab \ldots hk)$, $(a \mid b)$. This agrees with the original orthogonal theorems of STUDY, although the latter theorems apply equally for the orthogonal group in which the elements are complex numbers.

5. *The Group with a latent Hermitian bilinear form.*

By applying a general nonsingular linear transformation $x = My$, $\overline{x} = \overline{M}\,\overline{y}$, to the variables $x, \overline{x}$ we obtain a group Γ^* similar to the unitary Γ, as a subgroup of the projective group G. This replaces the bilinear form $(\overline{x} \mid x)$ by the form

$$\psi = \sum_{ik}^{n} \overline{y}_i\, h_{ik}\, y_k = \overline{y}'\, \overline{M}'\, M_y$$

whose matrix $H = \overline{M}'M = (h_{ik})$ is Hermitian. Here $\overline{y}'$ denotes a matrix of one row and n columns $[\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n]$. In the subgroup Γ^* this form ψ is latent.

[1]) Cf. R. WEITZENBÖCK, loc. cit., p. 225.

If we symbolise ψ by

$$u_\rho \, r_y \,, \; \{ \varrho_i r_k = h_{ik} \,, \; u = \bar{y}' \}$$

we may express the result of the above fundamental theorem as a *projective* invariant theorem in groundforms f^* of Γ^*, together with the *adjoined* groundform ψ. This transference is analogous to the corresponding orthogonal theory[1]) (when a nondegenerate quadric r_x^2 is latent). In the present case the inner product $(a \mid \bar{b})$ is now replaced by the elementary bilinear invariant type $(a \mid r)(\varrho \mid \bar{b})$. The outer-product-types are unchanged. This adjunction principle brings the unitary, and similar groups into line with previously discussed cases. It follows that the *Second Fundamental Theorem* of invariants can be asserted for the unitary case, as a corollary of the projective case. For example, with our original cogredient symbols $a, b, \ldots, k$, and their conjugates, the following types of fundamental identities are sufficient:

$$\Pi_1 \equiv \{ (ab \ldots hk)(l \ldots) = (lb \ldots hk)(a \ldots) + \ldots \},$$

$$\Pi_2 \equiv \{ (ab \ldots hk)(l \mid \bar{a}) = (lb \ldots hk)(a \mid \bar{a}) + \ldots \quad \},$$

$$\Pi_3 \equiv \left\{ (ab \ldots hk)(\bar{a}_1 \bar{b}_1 \ldots \bar{h}_1 \bar{k}_1) = \begin{vmatrix} (a \mid \bar{a}_1) & \ldots\ldots & (a \mid \bar{k}_1) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ (k \mid \bar{a}_1) & \ldots\ldots & (k \mid \bar{k}_1) \end{vmatrix} \right\},$$

and $\bar{\Pi}_1, \bar{\Pi}_2$ conjugates of the two former. It would be interesting to give a direct proof by induction, without recourse to the adjunction principle.

Identity Π_3 shews that the presence of both types of outer-product in one invariant, is unnecessary. It also shews that a rational integral unitary invariant, which is not merely relative but absolute, is expressible entirely by means of inner products $(a \mid \bar{b})$, for it contains equal amounts of the two conjugate types of symbol.

The argument throughout also holds for fields of elements a_i, in which $\bar{a}_i$ is conjugate with respect to a square root $\sqrt{r}$, where r is rational and not necessarily the same as -1.

[1]) R. WEITZENBÖCK, loc. cit., pp. 249, 262.