halbierenden. Wie ich bewiesen habe [1]), ergibt jedes Paar der Dreiecke $D_1, D_2, \ldots, D_{63}$ auf diese Art zwei neue Dreiecke mit rationalen Winkelhalbierenden. Ich erwähne hier nur sechs Beispiele, wo c die Basis, h die Höhe bezeichnet.

	a	b	c	$^1/_{24}\,h$	$\cos {}^1/_2\,A$	$\sin {}^1/_2\,A$	$\cos {}^1/_2\,B$	$\sin {}^1/_2\,B$	$\cos {}^1/_2\,C$	$\sin {}^1/_2\,C$
$D_2 - 5D_1$	125	169	84	5	12 : 13	5 : 13	3 : 5	4 : 5	63 : 65	16 : 65
$D_2 + 5D_1$	125	169	154	5	12 : 13	5 : 13	4 : 5	3 : 5	56 : 65	33 : 65
$2D_2 - D_3$	289	338	77	10	12 : 13	5 : 13	8 : 17	15 : 17	220 : 221	21 : 221
$2D_2 + D_3$	289	338	399	10	12 : 13	5 : 13	15 : 17	8 : 17	171 : 221	140 : 221
$D_7 - 3D_3$	867	1681	1036	30	40 : 41	9 : 41	8 : 17	15 : 17	672 : 697	185 : 697
$D_7 + 3D_3$	867	1681	2002	30	40 : 41	9 : 41	15 : 17	8 : 17	455 : 697	528 : 697

[1]) Diese Proceedings, **34**, S. 1394.

Mathematics. — *On the Solution of the Matrix Equation $AX + XB = C$.*

By D. E. RUTHERFORD. (Communicated by Prof. R. WEITZENBÖCK).

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I. In this problem A, B and C are given matrices and it is required to find X, or rather, to find the elements of X in terms of the elements of A, B and C. A solution is possible only if A and B are square matrices, let us say of orders n and m respectively, and when C is a conformable matrix of n rows and m columns. It follows that X also must have n rows and m columns.

When $PX = XQ$, X is called a *commutant* of P and Q, and is often written $X = (P, Q)$. It is a fundamental fact that this commutant can only be the null matrix, unless the matrices P and Q have at least one latent root in common. When common latent roots appear, then the general X is nonzero and contains arbitrary parameters. (Cf. e. g. TURNBULL and AITKEN, *Canonical Matrices*, (Glasgow, 1932) Chap. X). As may be suspected, our problem presents similar features. If $C = 0$, then evidently $X = (A, -B)$ is the commutant of A and $-B$, two matrices whose latent roots will be denoted by λ_i, $-\mu_j$. Uniqueness or otherwise of the solution X will depend on whether λ_i is equal to $-\mu_j$ or not; in the case of uniqueness (IV below) when $\lambda_i + \mu_j = 0$, X will however, not be zero.

We shall first consider the case where A and B are in the classical canonical form. We therefore write

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_q \end{bmatrix};$$

$$\text{where } A_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \cdot & 1 \\ & & & & \lambda_i \end{bmatrix} \text{ and } B_j = \begin{bmatrix} \mu_j & 1 & & & \\ & \mu_j & 1 & & \\ & & \ddots & \ddots & \\ & & & \cdot & 1 \\ & & & & \mu_j \end{bmatrix},$$

all elements not indicated being zero, and where neither $\lambda_1, \lambda_2, \ldots, \lambda_p$ nor $\mu_1, \mu_2, \ldots, \mu_q$ are necessarily all distinct. We now split up C and X in the following manner,

$$X = \begin{bmatrix} X_{11}, X_{12}, \ldots, X_{1q} \\ X_{21}, X_{22}, \ldots, X_{2q} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ X_{p1}, X_{p2}, \ldots, X_{pq} \end{bmatrix} \text{ and } C = \begin{bmatrix} C_{11}, C_{12}, \ldots, C_{1q} \\ C_{21}, C_{22}, \ldots, C_{2q} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ C_{p1}, C_{p2}, \ldots, C_{pq} \end{bmatrix}$$

where the submatrices X_{ij} and C_{ij} have the same number of rows as A_i, and the same number of columns as B_j.

II. The submatrix X_{ij} is now given by the equation

$$A_i X_{ij} + X_{ij} B_j = C_{ij}.$$

Let A_i and B_j be of orders s and t respectively; then we write

$$X_{ij} = \begin{bmatrix} x_{11}, x_{12}, \ldots, x_{1s} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_{t1}, x_{t2}, \ldots, x_{ts} \end{bmatrix} \text{ and } C_{ij} = \begin{bmatrix} c_{11}, c_{12}, \ldots, c_{1s} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ c_{t1}, c_{t2}, \ldots, c_{ts} \end{bmatrix}.$$

The equations for the elements x are easily shown to be

$$\left.\begin{array}{ll} \lambda_i x_{k.l} + x_{k+1.l} + x_{k.l-1} + \mu_j x_{k.l} = c_{k.l}, & (k < t;\ l > 1) \\ \lambda_i x_{k.1} + x_{k+1.1} + \mu_j x_{k.1} = c_{k.1}, & (k < t) \\ \lambda_i x_{t.l} + x_{t.l-1} + \mu_j x_{t.l} = c_{t.l}, & (l > 1) \\ \lambda_i x_{t.1} + \mu_j x_{t.1} = c_{t.1}. & \end{array}\right\} \quad . \quad . \quad (1)$$

These may be rewritten as

$$x_{t.1} = \frac{c_{t.1}}{\lambda_i + \mu_j},$$

$$x_{t.l} = \frac{c_{t.l} - x_{t.l-1}}{\lambda_i + \mu_j}, \qquad (l > 1),$$

$$x_{k.1} = \frac{c_{k.1} - x_{k+1.1}}{\lambda_i + \mu_j}, \qquad (k < t),$$

$$x_{k.l} = \frac{c_{k.l} - x_{k+1.l} - x_{k.l-1}}{\lambda_i + \mu_j}, (k < t; l > 1).$$

III. We shall first consider the case where $\lambda_i + \mu_j \neq 0$. In this case every element of X_{ij} is exactly determined from the last equations. Indeed any element x is equal to the corresponding c, minus the x element immediately on the left, minus the x element immediately below, all divided by $\lambda_i + \mu_j$. Hence any letter $c_{k.l}$ will appear in the kl th place in X_{ij} and also in every place to the right of, and above the kl th place; also the coefficient of $c_{k.l}$ in the uv th place of X_{ij} $(u < k, v > l)$ is

$$(-1)^{u+v-k-l} \binom{u+v-k-l}{u-k} (\lambda_i + \mu_j)^{k+l-u-v-1}.$$

IV. Let us now consider the case were $\lambda_i + \mu_j = 0$. If $c_{t.1} \neq 0$, then $x_{t.1}$ and every other element x of X_{ij} can easily be shown to be infinite. Similarly, if any $x_{r.w}$ of X_{ij} be infinite, every element above and to the right of $x_{r.w}$ must be infinite. If, however, $c_{t.1} = 0$ a finite solution may exist, for then will $x_{t.1}$ be arbitrary; but $x_{t.2}$ will be infinite unless $c_{t.2} - x_{t.1} = 0$, so that if we take $x_{t.1} = c_{t.2}$, then $x_{t.2}$ is arbitrary.

Now if $t < s$, we may proceed in this manner and obtain $x_{t.s}$ arbitrary by putting $x_{t.l} = c_{t.l+1}$ where $l < s$. If $x_{t-1.1}$ be not infinite, then must

$$c_{t-1.1} - x_{t.1} = c_{t-1.1} - c_{t-1.2} = 0,$$

and, proceeding as before, we have $x_{t-1.s}$ arbitrary by putting

$$x_{t-1.l} = c_{t-1.l+1} - x_{t.l+1}.$$

In the same way, if $x_{t-h.1}$ be not infinite then must

$$c_{t-h.1} - c_{t-h+1.2} + c_{t-h+2.3} - \ldots (-)^h c_{t.h+1} = 0 \quad (h < t < s) \, . \quad . \quad (2)$$

Also we have $x_{t-h.s}$ arbitrary by taking

$$x_{t-h.l} = c_{t-h.l+1} - x_{t-h+1.l+1} \quad (l < s).$$

In this way we can obtain every element x in terms of the c_{ij} and of t arbitrary parameters.

If $t > s$ then we proceed in a similar manner, but we take the s arbitrary elements $x_{1,1}, x_{1,2}, \ldots, x_{1,s}$ instead of $x_{1,s}, x_{2,s}, \ldots, x_{t,s}$.

The same relations between the elements c will hold, but in equation (2) the values of h are given by $h < s < t$. The equations for the elements x in this case will be

$$x_{k,1} = c_{k-1,1}\,; \qquad x_{k,l} = c_{k-1,l} - x_{k-1,l-1} \quad (k > 1\,;\, l > 1).$$

Thus, if $\lambda_i + \mu_j = 0$, a finite solution exists if, and only if, the following relations hold for the elements c;

$$c_{t,1} = 0,$$
$$c_{t-1,1} - c_{t,2} = 0,$$
$$c_{t-2,1} - c_{t-1,2} + c_{t,3} = 0,$$
$$\cdots\cdots\cdots$$

the last equation of this series being

$$c_{1,1} - c_{2,2} + c_{3,3} - \ldots (-)^{t-1}\, c_{t\,t} = 0, \text{ if } s > t,$$

or

$$c_{t-s+1,1} - c_{t-s+2,2} - \ldots (-)^{s-1}\, c_{t,s} = 0, \text{ if } s < t. \qquad \ldots (3)$$

The solution is then of the form

$$\begin{bmatrix} c_{1,2} - c_{2,3} + \ldots (-)^{t-1}\, c_{t,t+1}, & \ldots, & c_{1,s-1} - c_{2,s} + a_{t-2}, & c_{1,s} - a_{t-1}, & a_t \\ c_{2\,2} - c_{3,3} + \ldots (-)^{t-2}\, c_{t,t} & \ldots, & c_{2,s-1} - c_{3,s} + a_{t-3}, & c_{2s} - a_{t-2}, & a_{t-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{t-1,2} - c_{t,3} & \ldots, & c_{t-1,s-1} - c_{t,s}, & c_{t-1,s} - a_1, & a_2 \\ c_{t,2} & \ldots, & c_{t,s-1}, & c_{t,s}, & a_1 \end{bmatrix}$$

or

$$\begin{bmatrix} b_1, & b_2 & \ldots, & b_{s-1}, & b_s \\ c_{1,1}, & c_{1,2} - b_1 & \ldots, & c_{1,s-1} - b_{s-2}, & c_{1,s} - b_{s-1} \\ c_{2,1}, & c_{2,2} - c_{1,1} & \ldots, & c_{2,s-1} - c_{1,s-2} + b_{s-3}, & c_{2,s} - c_{1,s-1} + b_{s-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{t-1,1}, & c_{t-1,2} - c_{t-2,1} & \ldots, & c_{t-1,s-1} - \ldots (-)^{s-2}_{t-s+1,1}, & c_{t-1,s} - c_{t-2,s-1} + \ldots (-)^{s-1}\, c_{t-s,1} \end{bmatrix} \qquad (4)$$

according as s is greater or less than t, where the elements a and b are arbitrary parameters.

V. We have now obtained a solution of the equation

$$A_i X_{ij} + X_{ij} B_j = C_{ij}.$$

Solving this equation for all values of i and j we obtain every submatrix X_{ij}; and building these together we arrive at the solution of $AX + XB = C$, where A and B are in the classical form. If A and B be not already in the classical canonical form, then let $A = H^{-1} R H$ and let $B = K^{-1} S K$, where H and K are nonsingular and where R and S are in the classical form, so that $H A H^{-1} = R$ and $K B K^{-1} = S$.

But we can find a solution Y of $R Y + Y S = D$

$$\text{or of } H A H^{-1} Y + Y K B K^{-1} = D$$

$$\text{or of } A H^{-1} Y K + H^{-1} Y K B = H^{-1} D K.$$

Hence if we choose $H^{-1} D K = C$ or $D = H C K^{-1}$, then $X = H^{-1} Y K$ is solution of $A X + X B = C$. Thus a finite solution to this equation always exists provided that no latent root of A is equal to the negative of a latent root of B; or if this be so, provided that the relations (3) hold. In particular, no finite solution will exist if both the matrices A and B be singular.

VI. If every element of C_{ij} is zero, then so is every element of X_{ij} unless $\lambda_i + \mu_j = 0$. If this be the case then all the equations (2) are satisfied and hence a finite solution exists. By deleting all the elements c from the matrices (4), we obtain the solution of

$$A_i X_{ij} + X_{ij} B_j = 0 \quad (\lambda_i + \mu_j = 0).$$

Building up the submatrices X_{ij} in the same way as before, we have the solution of

$$A X + X B = 0.$$

VII. We shall conclude by giving a simple illustration of the foregoing methods. We wish to find the value of X for which $AX + XB = C$ where

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & -3 & 0 & 0 \\ -12 & -5 & -4 & 4 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now $A = H R H^{-1}$ and $B = K S K^{-1}$ where

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix},$$

and $S = \left[\begin{array}{cc:cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$; also $H^{-1}CK = \left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ \hdashline 4 & 3 & 2 & 1 \end{array}\right] = D$

Here the latent roots of A are 2, 2, 2, 2 and those of $-B$ are $-2, -2, -1, -1$. Hence none are common and X is therefore unique. The matrices R, S and D are now partitioned into submatrices in the appropriate manner, as is shown by means of the dotted lines: and we proceed to solve the equation $RY + YS = D$. Let Y be $\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$, then we must have

$$Y_{11} = \begin{bmatrix} -\frac{1}{3^2} - \frac{1}{3^3}, & \frac{1}{3} + \frac{2}{3^3} + \frac{3}{3^4} \\ \frac{1}{3} + \frac{1}{3^2}, & -\frac{1}{3^2} - \frac{2}{3^3} \\ -\frac{1}{3}, & +\frac{1}{3^2} \end{bmatrix}, \quad Y_{12} = \begin{bmatrix} -\frac{1}{4^3}, & +\frac{3}{4^4} \\ +\frac{1}{4^2}, & -\frac{2}{4^3} \\ -\frac{1}{4}, & +\frac{1}{4^2} \end{bmatrix},$$

$$Y_{21} = \left[4\frac{1}{3},\ 3\frac{1}{3} - 4\frac{1}{3^2} \right], \qquad Y_{22} = \left[2\frac{1}{4}, \frac{1}{4} - 2\frac{1}{4^2} \right].$$

Hence the solution is $X = HYK^{-1}$ or

$$X = \begin{bmatrix} \frac{1}{3} + \frac{1}{3^2} + \frac{2}{4^2}, & -\frac{1}{3^2} - \frac{2}{3^3} + \frac{6}{4^3}, & \frac{2}{4^2}, & -\frac{2}{4^3} \\ -\frac{1}{3^2} - \frac{1}{3^3} - \frac{3}{4^3}, & \frac{1}{3} + \frac{1}{3^2} + \frac{9}{4^4}, & -\frac{1}{4^3} - \frac{6}{4^4}, & \frac{3}{4^4} \\ -\frac{1}{3^2} + \frac{2}{4^2}, & \frac{2}{3^3} + \frac{3}{4^2} - \frac{6}{4^3}, & \frac{1}{4}, & -\frac{1}{4^2} + \frac{2}{4^3} \\ \frac{4}{3} - \frac{1}{2}, & 1 - \frac{4}{3^2} - \frac{3}{4} + \frac{6}{4^2}, & \frac{1}{4}, & \frac{1}{4} - \frac{2}{4^2} \end{bmatrix}.$$
