Le coefficient critique est:

$$K_{4d} = \frac{RT_k}{p_k\, v_k} = 3.395.$$

T_k = température critique absolue.
p_k = pression critique en atmosphères internationales, voir 34.529 atm.
v_k = volume critique.
R = constante des gaz.

4. Les déviations du diamètre ne sont pas aussi petites que celles pour l'hydrogène et pour le néon, car elles sont quelquefois de l'ordre de grandeur de 1 pour 100, comme nous les avons trouvé antérieurement pour l'azote et pour l'éthylène par exemple. Cependant on peut dire que l'oxyde de carbone obéit sensiblement à la loi du diamètre rectiligne.

Les déviations ont le caractère systématique, que nous avons déjà rencontré dans d'autres substances, par exemple l'argon et l'éthylène; c'est-à-dire que le diamètre expérimental est légèrement concave vers l'axe des températures au voisinage du point critique et légèrement convexe, au contraire, aux températures les plus basses.

Sommaire.

Les auteurs ont d'abord préparé une quantité suffisante de CO très pur. Puis ils ont mesuré les densités du liquide et de la vapeur saturée entre le point triple et le point critique. A l'aide de ces valeurs ils ont pu calculer le diamètre rectiligne de CAILLETET et MATHIAS et la densité critique.

Astronomy. — *On the structure and internal motion of the gaseous disc constituting the original state of the planetary system.* By H. P. BERLAGE Jr., Meteorological Observatory, Batavia. (Communicated by Prof. H. A. KRAMERS.)

(Communicated at the meeting of April 30, 1932.)

In this paper I endeavour to show not only that a thin gaseous disc, rotating with variable angular velocity about the sun, constitutes a possible embryo of the planetary system, but also that some present features of the system leave hardly any doubt that it really once evolved from such a gaseous disc.

Consider a perfectly gaseous atmosphere rotating non uniformly about an axis through the sun. Because the distribution of mass in the solar system is such that only one part in 700 is concentrated in the planets, ROCHE's model of a nucleous surrounded by a massless envelope is applicable, so that the potential at any point of the atmosphere is Newtonian.

Let r denote the distance of a volume element from the axis of rotation, h its height above or below the equatorial plane, p and ϱ gaspressure and density, M the mass of the sun, f the constant of gravitation, ω the angular velocity of an element of the nebula, then the two fundamental equations of equilibrium become

$$\frac{fMh}{(r^2+h^2)^{3/2}}+\frac{1}{\varrho}\frac{\partial p}{\partial h}=0 \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

$$\frac{fMr}{(r^2+h^2)^{3/2}}+\frac{1}{\varrho}\frac{\partial p}{\partial r}=\omega^2 r . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Introducing the gasequation

$$p = R\varrho T \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

and integrating (1), we obtain provisionally

$$lg\, p = lg\, p_e - \frac{fM}{R}\int\limits_0^h \frac{h dh}{T(r^2+h^2)^{3/2}} \; . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

A suffix e refers to the equatorial plane. From (2) and (3) follows

$$\frac{fMr}{(r^2+h^2)^{3/2}}+RT\frac{\partial\, lg\, p}{\partial r}=\omega^2 r . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Differentiating (4) with respect to r and substituting the resulting expression in (5), we get

$$\frac{T}{T_e}\frac{fM}{r^2}+RT\frac{d\, lg\, p_e}{dr}+fMT\int\limits_0^h \frac{\left(h\dfrac{\partial T}{\partial r}-r\dfrac{\partial T}{\partial h}\right)}{T^2\,(r^2+h^2)^{3/2}}\,dh=\omega^2 r . \quad . \qquad (6)$$

In order to simplify the problem, let us first suppose the temperature in the solar nebula to be a function of the distance from the centre only. In this case the third term on the left side of (6) reduces to 0, so that we remain with

$$\frac{T}{T_e}\frac{fM}{r^2}+RT\frac{d\, lg\, p_e}{dr}=\omega^2 r \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

This is a rather general case. Without sacrificing the most important aspects of the problem, we may examine the case of isothermy $T = T = T_i$. In this case we have, after elimination of p_e

$$\frac{fM}{r^2}+\frac{RT_i}{\varrho_e}\frac{d\varrho_e}{dr}=\omega^2 r \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

From the equations (7) and (8) we learn that only in the case of isothermy our nebula possesses the well known property of freely rotating masses, *that the angular velocity ω is a function of the distance r from the axis of rotation only.* From (4) we then get

$$p = p_e \, e^{-\frac{fM}{RT_i}\left[\frac{1}{r} - \frac{1}{\sqrt{r^2+h^2}}\right]} \quad \ldots\ldots\ldots \quad (9)$$

So we also have

$$\varrho = \varrho_e \, e^{-\frac{fM}{RT_i}\left[\frac{1}{r} - \frac{1}{\sqrt{r^2+h^2}}\right]} \quad \ldots\ldots\ldots \quad (10)$$

Considering (10) and its finite limiting value

$$\varrho = \varrho_e \, e^{-\frac{fM}{RT_i r}} \, (h = \infty) \quad \ldots\ldots\ldots \quad (11)$$

it is exceedingly interesting to remark that our solar nebula behaves morphologically like a whirl in infinite space filled with matter, as DESCARTES conceived it. Yet, KANT was perfectly right, when he believed his nebula, after concentration, to be shaped like a thin flat disc. To prove this, let us follow the indirect way, which is the simpler, and assume that we are practically free to take into account only that part of the mass, which is contained within very narrow limits of h and ignore the remaining part, although it is not negligible, but even infinite.

For $h << r$ (10) reduces to

$$\varrho = \varrho_e \, e^{-\frac{fM}{2\,RT_i}\frac{h^2}{r^3}} \quad \ldots\ldots\ldots \quad (12)$$

To fix the ideas, let us suppose our nebula to consist of air at a temperature $T_i = 23.2^\circ$ absolute. Then, with $R = 2.87 \times 10^6$, $f = 6.67 \times \times 10^{-8}$, $M = 2.00 \times 10^{33}$ c.g.s., we get

$$\varrho = \varrho_e \, e^{-10^{18}\frac{h^2}{r^3}} \quad \ldots\ldots\ldots \quad (13)$$

Whether we are allowed to speak of a disc depends upon the drop of density in axial direction, that is, upon what fraction is represented by the exponential factor. The latter increases with increasing r. Let us therefore substitute in (13) the distance of Neptune from the sun, or $r = 4.5 \times 10^{14}$ cm. We then obtain

$$\varrho = \varrho_e \, e^{-2220\left(\frac{h}{r}\right)^2} = \varrho_e \, . \, 10^{-1000\left(\frac{h}{r}\right)^2} \quad \ldots\ldots \quad (14)$$

The density represented by (14) drops to a value of only 10^{-10} times the density in the equatorial plane, at a distance as small as $h = 0.1\,r$. In other words, *if a solar nebula has ever existed, it has been a thin disc indeed at least as far as Neptune.* Ignoring (10) and applying (12) throughout is a procedure comparable with neglecting the totality of interstellar matter, when discussing the mass distribution in the solar system. Even when computing with the aid of (12) the total mass of the nebula

$$m = 4\pi \int_0^\infty r\,\varrho_e\,dr \int_0^\infty e^{-\frac{fM}{2RT_i}\frac{h^2}{r^3}}\,dh, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and its moment of momentum

$$\Theta = 4\pi \int_0^\infty r^3\,\omega\,\varrho_e\,dr \int_0^\infty e^{-\frac{fM}{2RT_i}\frac{h^2}{r^3}}\,dh \quad . \quad . \quad . \quad . \quad . \quad (16)$$

we do not commit any fundamental error. On the contrary, we obtain the advantage of skimming off the mass of the solar nebula proper from an infinite material interstellar background, which practically exists, but does not interest us in planetary questions.

Integrating (15) and (16), we get

$$m = (2\pi)^{3/2}\left(\frac{RT_i}{fM}\right)^{1/2}\int_0^\infty r^{5/2}\,\varrho_e\,dr \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and

$$\Theta = (2\pi)^{3/2}\left(\frac{RT_i}{fM}\right)^{1/2}\int_0^\infty r^{9/2}\,\omega\,\varrho_e\,dr \quad . \quad . \quad . \quad . \quad . \quad (18)$$

As the pressure gradient in our gaseous disc is slight,

$$r^{3/2}\,\omega = (fM)^{1/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

approximately and (18) becomes

$$\theta = (2\pi)^{3/2}\,(RT_i)^{1/2}\int_0^\infty r^3\,\varrho_e\,dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

A curious property of this relation is, that the solar mass M has dropped out. There is nothing trivial in this fact. It proves that on the basis of equilibrium alone no relation can be established between the solar mass and the moment of momentum of the planetary system. The puzzling problem, why the ratio of the planetary and solar masses does not in the least conform to the distribution of the moments of momentum, must find its solution along other lines, which I do not intend to pursue here.

Since the density distribution in the equatorial plane must be known, if we want to obtain a complete solution of our problem of the structure and motion of the disc, let us try to get, if possible, some general information about this distribution. *If* any relation exists between density and solar distance, it may be derived from the influence of viscocity. Viscosity acts on an infinitesimal ring, in the plane of symmetry, of radius r and density ϱ_e with a tangential force proportional to

$$\frac{d}{dr}\left(r^2 \varrho_e \frac{d\omega}{dr}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Since the disc must be looked upon as the product of a secular condensation process, it naturally tends to a configuration continually recreating its fundamental properties, as contraction progresses. As this seems best warranted by homologous contraction in the plane of symmetry, we arrive at the question, when viscosity generates homologous contraction. This occurs only if (21) is proportional to the momentum of the ringmatter. Consequently, we may anticipate the approximate validity of a relation of the form

$$\frac{d}{dr}\left(r^2 \varrho_e \frac{d\omega}{dr}\right) = \tfrac{3}{2}\, a \varrho_e\, \omega\, r \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

a factor $\frac{3}{2}$ being added to the constant of proportionality a, because it simplifies the following formulae. As (19) also holds approximately, we get

$$\frac{d}{dr}\left(\varrho_e\, r^{-1/2}\right) = -\, a\, \varrho_e\, r^{-1/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

or

$$\varrho_e = c\, r^{1/2}\, e^{-ar} \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

c being a constant. Let us label this case, for identification purposes, with the name of *approximate case*. It is characterized by the fact that the density in the equatorial plane of the disc does not decrease monotonously with increasing r, as we might have superficially expected, but increases first and only decreases asymptotically to zero after having passed through a maximum.

Substituting (24) in (17) and (18), we obtain

$$m = (2\pi)^{3/2} \left(\frac{RT_i}{fM}\right)^{1/2} c \int_0^\infty r^3\, e^{-ar}\, dr \,. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$\theta = (2\pi)^{3/2}\, (RT_i)^{1/2}\, c \int_0^\infty r^{3\,1/2}\, e^{-ar}\, dr \,. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

or

$$m = (2\pi)^{3/2} \left(\frac{RT_i}{fM}\right)^{1/2} c \,.\, 6a^{-4} \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$\theta = (2\pi)^{3/2}\, (RT_i)^{1/2}\, c \,.\, \tfrac{105}{16}\, \pi^{1/2}\, a^{-4\,1/2} \,. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Since m and θ are given quantities, we might determine both c and a if we knew R and T_i. However, we may determine at least a without knowing either R or T_i, for

$$\frac{\theta}{m}=\frac{105}{96}\left(\frac{fM\pi}{a}\right)^{1/2} \quad \text{. . . . . (29)}$$

The empirical value of this quotient, derived from the actual planetary system, is

$$1.02\times 10^{7}\,(fM)^{1/2} \quad \text{. . . . . (30)}$$

from which

$$a=3.63\times 10^{-14} \quad \text{. . . . . (31)}$$

follows.

A very suggestive consequence is, that the highest density in the disc occurred at a distance

$$r=\frac{1}{2a}=1.38\times 10^{13}\,cm \quad \text{. . . . . (32)}$$

from the sun. This corresponds to 0.93 or nearly 1 astronomical unit. This means that the highest density was found along a circle coinciding almost with the actual orbit of the earth. So the question arises, whether it is a result of pure chance that the earth actually possesses the highest density among all planets. Let us therefore compare the following sequence of planetary densities [1]) with the density distribution in our approximate case more thoroughly.

Mercury	3.58
Venus	5.02
Earth	5.53
Mars	4.09
Jupiter	1.34
Saturn	0.67
Uranus	1.47
Neptune	1.33

These values have been plotted against solar distance in figure 1 and joined by a smooth curve, which will be called the planetary density curve. The dotted curve represents (24) in arbitrary units. There is a striking parallelism between the two curves. It is true that the planetary density curve, after passing through a minimum, rises to a secondary maximum. But we know that (24) could not reveal anything more than the general trend of the density distribution and this it does with unexpected fidelity. Therefore, reversing the argumentation, it seems rational to assume that the actual density curve of the planets is very nearly *representative* of the original density distribution in the gaseous disc.

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[1]) C. A. VAN DEN BOSCH, Dissertation Utrecht, 1926.

Since the atomic weight is the only variable which enters into the density of solid bodies as well as into the density of gases, this relation may be due to a variability of the chemical composition of the disc with solar distance, but, for the moment, I prefer to avoid the resulting complications of the above theory. For, assuming a variability of mean atomic weight, we would be obliged to assume a variability of R, the gas constant. Moreover it would almost certainly imply a variability of the temperature too, which we supposed constant and perhaps even a variability of, say, the constant of gravitation, because some elements are and some are not notably affected by radiation pressure. Surely there are so many variables at hand that it would be an easy matter to suit them into a consistent scheme explaining a definite relation between the actual density of the planets and the original density of the gaseous disc. But I doubt, whether this relation would not remain too weakly founded, if it did not prove *remarkably successful,* when applied to the interpretation of two outstanding features of the actual system.

In the first place, (8) shows us that, where ϱ_e increases with r, the angular velocity is larger than the Keplerian velocity, whereas, where ϱ_e decreases with r, the angular velocity is smaller than the Keplerian velocity. Where the density attains a maximum or minimum the velocity is Keplerian. The cosmogonic consequences of this fact are evident. Each planet will, during its condensation, adjust its orbit to the moment of momentum, which its matter possessed, when it was still part of the disc. A planet born within a zone of outward increasing density will draw away from the sun, a planet born within a zone of outward decreasing density will draw nearer to the sun. But graphical interpolation of the planetary density curve of figure 1, revealed a first maximum at the proper distance of the earth, a

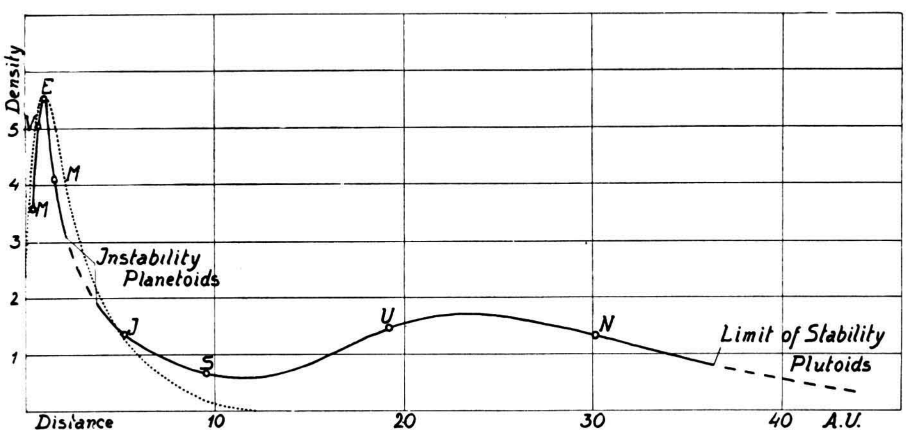

Fig. 1

minimum between Saturn and Uranus, a second maximum between Uranus and Neptune. Thus, *postulating that the distances from the sun where the planets were born, strictly obey some law,* we should find that Mercury, Venus and Uranus revolve too far away from the sun, Mars, Jupiter, Saturn and Neptune too near, whereas the distance Sun-Earth, in which the other distances are expressed as a unit, remains constant.

Let us express the law of the planetary distances by a simple exponential function, $r = r_0 p^n$ ($n = 0, 1, 2$, etc.), which is BODE's law, omitting the irrelevant additative constant. Then, nearest agreement is obtained with the ratio $p = 1.79$, as may be shown by the following table, which contains the distances of the planets from the sun in astronomical units.

	M	V	E	M	J	S	U	N
observed	0.39	0.72	1	1.52	5.20	9.55	19.22	30.11
computed	0.31	0.56	1	1.79	5.73	10.24	18.33	32.77
quotient	1.26	1.29	1	0.85	0.91	0.93	1.05	0.92

The last row contains the quotients of the observed and computed values. It shows us that as a matter of fact Mercury, Venus and Uranus are placed too far away, whereas Mars, Jupiter, Saturn and Neptune are placed too near.

On the one hand, this confirms our supposition that an exponential law has ruled the distances of the planets from the sun. I even dare say, in contradiction to suggestions from several sides, that *an exponential law is much more strictly obeyed, than has ever been believed.* It requires careful consideration, because it surely touches the root of the problem of the evolution of the solar system and may lead us to the solution of the fundamental question, as to which agency once transformed KANT's disc into the rings of LAPLACE [1]). On the other hand, we could hardly conceive of any stronger evidence of a gaseous disc having constituted the embryo of the planetary system.

The second question, which arises, is the question of stability. What is the condition that a gaseous equilibrium configuration, represented by

$$\frac{fM}{r^2} + \frac{RT_i}{\varrho_e}\frac{d\varrho_e}{dr} = \omega^2 r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

be stable? This is of course too complicated a problem to be solved in its integral form. My impression is that the most general criterion is the following. The gradient of angular velocity in radial direction should be less steep than the gradient, which follows from the formula

$$\omega r^2 = constant,$$

expressing the preservation of moment of momentum. For, in this case only, every mass element, which, after some arbitrary radial displacement,

[1]) Versuch einer Entwicklungsgeschichte der Planeten, Ergänzungsheft zu Gerlands Beiträge zur Geophysik, **17**, 1927; On the electrostatic field of the sun due to its corpuscular rays, Proceedings Amsterdam, **33**, 1930, p. 614; On the electrostatic field of the sun as a factor in the evolution of the planets, Proceedings Amsterdam **33**, 1930, p. 719.

takes up a new circular orbit, will move with such a velocity that it tends to return to its former position.

Multiplying (8) by r^3, we get

$$f M r + R T_i \frac{r^3}{\varrho_e} \frac{d\varrho_e}{dr} = \omega^2 r^4. \quad \ldots \ldots \quad (33)$$

Differentiating (33) with respect to r, the condition of stability

$$f M + R T_i \frac{d}{dr} \left(\frac{r^3}{\varrho_e} \frac{d\varrho_e}{dr} \right) > 0 \quad \ldots \ldots \quad (34)$$

immediately follows. The stability of the solar disc depends upon the radial density distribution in the equatorial plane. If the equation

$$f M + R T_i \frac{d}{dr} \left(\frac{r^3}{\varrho_e} \frac{d\varrho_e}{dr} \right) = 0 \quad \ldots \ldots \quad (35)$$

has real positive roots, *they represent the radii of the boundaries of zones of stability and instability*. Within the former zones the motion of the gas is laminar, within the latter zones it is turbulent.

For a first orientation in this matter of stability and instability, let us return to our approximate case (24). Substituting (24) in (8), we get

$$fM + r\left(\tfrac{1}{2} - ar\right) R T_i = \omega^2 r^3 \quad \ldots \ldots \quad (36)$$

whereas the limit of stability follows from (35)

$$fM + r\left(1 - 3\,ar\right) R T_i = 0 \quad \ldots \ldots \quad (37)$$

(37) has always one real positive root. The radius

$$r = \frac{1 + \sqrt{1 + 12\,a\,(fM)\,(RT_i)^{-1}}}{6\,a} \quad \ldots \ldots \quad (38)$$

divides the disc into an inner stable zone and an outer instable zone. We could localize the boundary, if we knew R and T_i. This being not the case, let us draw a conclusion, which is independant of these data.

Denoting the value of the angular velocity at the boundary by ω_b, the Keplerian value at the same distance by ω_k, we find by elimination of RT_i from (36) and (37)

$$\frac{\omega_b^2}{\omega_k^2} = \frac{2\,ar - \frac{1}{2}}{3\,ar - 1} \quad \ldots \ldots \ldots \quad (39)$$

Suppose a planet to be born at the boundary. With the generation of a planet from the disc it automatically adjusts its actual distance r_a to the value ω_b, whereas we shall call r_t the theoretical distance corresponding with the Keplervalue ω_k. We then get evidently

$$\frac{r_a}{r_t} = \frac{2\,ar - \frac{1}{2}}{3\,ar - 1} \quad \ldots \ldots \ldots \quad (40)$$

Suppose, instability sets in on the descending branch of the density curve. Then $2ar > 1$ and the quotient (40) is limited between 1 and $^2/_3$. *We then* *may observe our planet down to two thirds of its theoretical distance from the sun at most, but if so, irregularities in the behaviour of the planets beyond this hypothetical one are to be expected.*

Fortunately, an example of this approximate case immediately presents itself. The great bulk of planetoids between Mars and Jupiter, on the first descending branch of the density curve, sets in at about 2.0 A.U., that is at 0.63 of their theoretical distance 3.20 A.U. This strongly suggests that the riddle of their origin can be solved in a remarkably simple manner. *The planetoids were born within a zone of instability and turbulence in the gaseous disc.* This fact explains their not being united in one body, as well as their puzzling dispersity and excentricity.

The inner radius of this zone of instability must have been smaller than 3.20 A.U. If the equatorial density had rigorously followed the function (24), with its one maximum, the disc would have been unstable from the inner limit up to infinity. The planetary density curve, however, rises a second time beyond Saturn. Consequently an outer limit to the zone of instability might have been anticipated. The outer radius has evidently exceeded 3.20 A.U., but it must have been smaller than 5.73 A.U. the theoretical distance of Jupiter.

As instability occurred on the first descending branch of the density curve, we may expect it to occur once more on the second descending branch of the density curve, beyond the density maximum between Uranus and Neptune, *and it evidently did occur*. For, does not the recent discovery of the excentric Pluto make it extremely probable that astronomers have got hold of the first member of a new family of planetoids, which might be well called plutoids?

The mean distance of Pluto from the sun is 39.9 A.U., that is 0.68 of its theoretical distance 58.7 A.U., again strikingly near to the theoretical limit of two thirds. It is dubious, whether or not the density of the planets increases a third time in outward direction. It is therefore dubious, whether the second zone of instability, where the plutoids originated, extends to infinite distance, or that there is a chance for some more regular planets to pursue still larger orbits (See figure 1).

I think it rather improbable that big transneptunian planets should exist. It is far more probable that we have to look at the comets as messengers descending to the sun from the outmost instable and turbulent portions of the solar nebula, where it merged into the interstellar medium.

Concluding this paper I should like to emphasize that we have found such convincing evidence of a gaseous disc having constituted the original state of our planetary system that I feel obliged to express doubt, whether the Tidal Theory of the origin of this system will remain any longer a serious competitor besides some other theory, which follows the line of thought of DESCARTES, KANT and LAPLACE.

The Hague, March 1932.