Our thanks are due to Mr. H. BRINKMAN for discussions on the subject, and to Mr. G. G. ZAALBERG for assistance in carrying out the experimental work. One of us (G. O. L.) is indebted to the Royal Commission for the Exhibition of 1851, whose award made his stay in Utrecht possible.

Hydrodynamics. — *On the application of statistical mechanics to the theory of turbulent fluid motion.* V. [1]) By J. M. BURGERS. (Mededeeling N°. 26 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft).

(Communicated at the meeting of April 29, 1933).

4. *Application of the normal functions obtained from equation* (19) *to the reduction of the exponent occurring in the distribution function* (12).

We proceed with the investigation of the statistical distribution of the relative motion and shall introduce the development of the stream function ψ of an arbitrary mode of relative motion (which satisfies the boundary conditions) according to the system of normal functions deduced from eq. (19).

It must be borne in mind that with the normalizing conditions assumed in § 3 the characteristic values of the parameter Λ all will be positive, provided that a is positive. It is convenient to adhere to the restriction of a to positive values, and to the type of functions obtained in § 3, but in the development of the stream function also terms will occur in which the sign of a is reversed (that is to say terms representing functions which are the symmetrical ones — with respect to the line $x=0$ — of those obtained before). As moreover both types of terms may have arbitrary phases with respect to x, we must expect that any stream function ψ will be built up from an aggregate of terms of the form:

$$\left.\begin{array}{l} a\,(\chi_I \cos ax + \chi_{II} \sin ax) + b\,(-\chi_I \sin ax + \chi_{II} \cos ax) + \\ + c\,(\chi_I \cos ax - \chi_{II} \sin ax) + d\,(\chi_I \sin ax + \chi_{II} \cos ax) \end{array}\right\} \quad . \quad . \quad . \quad (25)$$

It is convenient to introduce complex quantities, and so we assume that the stream function ψ_m of the mode of relative motion numbered m can be represented by the expression:

$$\psi_m = \tfrac{1}{2} \sum_{\alpha k} \left\{ e^{-i\alpha x} (A_{\alpha k}\, \chi_{\alpha k} + B_{\alpha k}\, \overline{\chi}_{\alpha k}) + e^{i\alpha x} (\overline{A}_{\alpha k}\, \overline{\chi}_{\alpha k} + \overline{B}_{\alpha k}\, \chi_{\alpha k}) \right\} \quad . \quad . \quad (26)$$

where $A_{\alpha k} = A^{I}_{\alpha k} + i\, A^{II}_{\alpha k}$, $\overline{A}_{\alpha k} = A^{I}_{\alpha k} - i\, A^{II}_{\alpha k}$, etc. Any mode of relative motion is now specified by the values of the A's and B's, and in calculating statistical mean values the summation with respect to the number m (i.e. the summation over the "ξ-space") can be replaced by an inte-

[1]) Part IV has appeared in these Proceedings, **36**, p. 276, 1933.

gration over all the A's and B's [1]). — In formula (26) the normal functions have been denoted by $\chi_{\alpha k}$, etc. in order to mark their dependence upon the parameter α; likewise we write $\Lambda_{\alpha k}$ for the characteristic numbers corresponding to a given value of α. The summation with respect to α properly speaking must be replaced by an integration; for simplicity of writing we provisionally keep to the notation used above.

From (26) we deduce:

$$\left.\begin{aligned} -\overline{u'v'} = \frac{i}{4} \sum_{\alpha kl} \alpha \, [(A_{\alpha k}\bar{A}_{\alpha l} - \bar{B}_{\alpha k} B_{\alpha l})(\chi'_{\alpha k}\bar{\chi}_{\alpha l} - \chi_{\alpha k}\bar{\chi}'_{\alpha l}) + \\ + A_{\alpha k}\bar{B}_{\alpha l}(\chi'_{\alpha k}\chi_{\alpha l} - \chi_{\alpha k}\chi'_{\alpha l}) - \bar{A}_{\alpha k} B_{\alpha l}(\bar{\chi}'_{\alpha k}\bar{\chi}_{\alpha l} - \bar{\chi}_{\alpha k}\bar{\chi}'_{\alpha l})] \end{aligned}\right\} \quad . \quad (27)$$

$$\left.\begin{aligned} \bar{z} = \frac{1}{2} \sum_{\alpha kl} [(A_{\alpha k}\bar{A}_{\alpha l} + \bar{B}_{\alpha k} B_{\alpha l})(\chi''_{\alpha k} - \alpha^2 \chi_{\alpha k})(\bar{\chi}''_{\alpha l} - \alpha^2 \bar{\chi}_{\alpha l}) + \\ + A_{\alpha k}\bar{B}_{\alpha l}(\chi''_{\alpha k} - \alpha^2 \chi_{\alpha k})(\chi''_{\alpha l} - \alpha^2 \chi_{\alpha l}) + \\ + \bar{A}_{\alpha k} B_{\alpha l}(\bar{\chi}''_{\alpha k} - \alpha^2 \bar{\chi}_{\alpha k})(\bar{\chi}''_{\alpha l} - \alpha^2 \bar{\chi}_{\alpha l})] \end{aligned}\right\} \quad . \; . \; . \quad (28)$$

from which the following expressions for the integrals are obtained:

$$-\int dy \, \overline{u'v'} \, \frac{d\lambda}{dy} = \frac{1}{2R} \sum_{\alpha k} \alpha^3 (A_{\alpha k}\bar{A}_{\alpha k} - B_{\alpha k}\bar{B}_{\alpha k}) \quad . \; . \; . \quad (29)$$

$$\int dy \, \bar{z} = \tfrac{1}{2} \sum_{\alpha k} \alpha^3 \Lambda_{\alpha k} (A_{\alpha k}\bar{A}_{\alpha k} + B_{\alpha k}\bar{B}_{\alpha k}) \quad . \; . \; . \; . \quad (30)$$

Here use is made of equations (21) — (24) and of two other orthogonality relations:

$$\int dy \frac{d\lambda}{dy} (\chi'_k \chi_l - \chi_k \chi'_l) = 0. \quad . \; . \; . \; . \; . \; . \; . \quad (31)$$

$$\int dy \, (\chi''_k - \alpha^2 \chi_k)(\chi''_l - \alpha^2 \chi_l) = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (32)$$

which latter (together with the equations obtained by changing every χ into its conjugate complex) are valid for all values of k and l, the case $k = l$ included.

Consequently the exponent occurring in the distribution function (12) assumes the form:

$$-\frac{\beta}{2R} \sum_{\alpha k} \alpha^3 \{ A_{\alpha k}\bar{A}_{\alpha k}(\Lambda_{\alpha k} - 1) + B_{\alpha k}\bar{B}_{\alpha k}(\Lambda_{\alpha k} + 1) \} \quad . \; . \quad (33)$$

The distribution function can have a meaning only if it remains finite

[1]) As the $A_{\alpha k}$ are complex quantities the integration with respect to $A_{\alpha k}$ in reality stands for an integration with respect to the two real variables $A^{I}_{\alpha k}$ and $A^{II}_{\alpha k}$. A similar remark applies to the $B_{\alpha k}$.

for all values (finite or infinite) of the variables. This requires that all characteristic numbers $\Lambda_{\alpha k}$ shall be greater than unity, and thus imposes a certain condition on the function λ. If the condition is fulfilled we can calculate the statistical mean value $\overline{f}$ of any quantity f depending on the A's and B's, by means of the formula:

$$\overline{f} = (\Sigma \, \overline{n}_m \, f_m) / (\Sigma \, \overline{n}_m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where — as mentioned before — the summation with respect to m can be replaced by an integration over all the A's and B's.

5. *Introduction of a special function* λ.

In the theories developed by PRANDTL and by VON KARMAN much attention is given to considerations of similarity, and one of the questions which have arisen in this connection is, whether in some region in the neighbourhood of the wall the relative motions for various distances from the wall may be considered as similar, the scale being proportional to the distance. It seems possible to introduce an analogous consideration into the problem we are treating here, and it may be asked whether there may exist a certain similarity amongst the normal functions $\chi_{\alpha k}$. As the scale in the direction of the coordinate x is determined by the parameter α, it may be asked if, for any given value of the number k, the functions $\chi_{\alpha k}$ might be functions of a single variable $\xi = \alpha\,(y + \frac{1}{2})$ (in the following lines we shall write η in stead of $y + \frac{1}{2}$ for the distance from the wall at $y = -\frac{1}{2}$). As the presence of the other wall at $y = +\frac{1}{2}$ disturbs the similarity, we shall provisionally assume that the other wall is situated at a very large distance, so that, if it may happen that we find functions $\chi_{\alpha k}$ which decrease sufficiently fastly for large values of ξ, the presence of this second wall may be without appreciable influence upon them.

If we consider χ as a function of ξ, the differential equation (19), after division by α^4, takes the form:

$$\frac{d^4\chi}{d\xi^4} - 2\frac{d^2\chi}{d\xi^2} + \chi - i\,R\,\Lambda\left(\frac{1}{\alpha^2}\frac{d\lambda}{d\eta}\frac{d\chi}{d\xi} + \frac{1}{2\,\alpha^3}\frac{d^2\lambda}{d\eta^2}\chi\right) = 0 \quad . \quad . \quad (35)$$

It is easily seen that α and η will disappear as separate variables form this equation, if we assume that $d\lambda/d\eta$ is proportional to $1/\eta^2$. It is convenient to write:

$$\frac{d\lambda}{d\eta} = \frac{b}{R\eta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The constant b must be positive. In fact the expression can be applied in the neighbourhood of the wall $\eta = 0$ ($y = -\frac{1}{2}$) only, and it is evident that the statistical mean value of $-\overline{u'\,v'}$ must be positive here. From

(12) it will be seen that a positive statistical mean value of this quantity is to be expected only if $d\lambda/d\eta$ is positive.

Integrating (36) with respect to η, we obtain:

$$\lambda = constant - b / R\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

This expression, however, will violate the condition $\lambda = 0$ for $\eta = 0$ ($y = -\frac{1}{2}$) we have assumed in § 2. In order to amend this point we must assume that for very small values of η, say for $\eta < \delta$, the function $d\lambda/d\eta$ deviates from the course indicated by (36). This assumption, of course, is at variance with the similarity hypothesis; however, if we might suppose that the distance δ is sufficiently small, the similarity could be preserved approximately for values of η sufficiently surpassing δ. Such a case does not seem improbable; in fact an assumption of the same kind must be introduced by PRANDTL, as otherwise it would not be possible to understand how the constant a in eq. (**) of § 1 could take a definite value. — In order to fix the ideas we might assume f.i. that for $\eta < \delta$ the function $d\lambda/d\eta$ has the constant value:

$$\frac{d\lambda}{d\eta} = \frac{b}{R\delta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

For abbreviation we write $b\,\Lambda = p$. With ξ everywhere as the independent variable, and using primes, etc. to denote derivatives with respect to ξ, we now arrive at the following differential equations for the function χ:

(*a*) in the domain $\xi < \alpha\,\delta$:

$$\chi^{IV} - 2\,\chi'' + \chi - \frac{i\,p}{(\alpha\,\delta)^2}\,\chi' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (39a)$$

(*b*) in the domain $\xi > \alpha\,\delta$:

$$\chi^{IV} - 2\,\chi'' + \chi - i\,p\left(\frac{\chi'}{\xi^2} - \frac{\chi}{\xi^3}\right) = 0 \quad . \quad . \quad . \quad . \quad (39b)$$

Equation (39*a*) can be solved by means of functions of the type $e^{m\xi}$, m being one of the four roots of an equation of the fourth degree. Hence the general solution for χ in the domain $\xi < \alpha\,\delta$ is of the form:

$$\chi = \Sigma\, B_\nu\, e^{m_\nu \xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Equation (39*b*) can be reduced to a hypergeometric equation, which will be investigated in the next §.

6. *Investigation of equation* (39*b*).

In (39*b*) we write $\chi = \xi\,\chi_0$, and multiply by ξ; this gives:

$$\xi^2\,(\chi_0^{IV} - 2\,\chi_0'' + \chi_0) + 4\,\xi(\chi_0''' - \chi_0') - i\,p\,\chi_0 = 0 \quad . \quad . \quad . \quad . \quad (41)$$

This equation can be transformed by LAPLACE's method, if we put:

$$\chi_0 = \int d\xi \, e^{\xi\zeta} \, w(\zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

ζ being an auxiliary variable. Then w must satisfy the equation of the second order:

$$\frac{d^2 w}{d\zeta^2} + \frac{4\zeta}{\zeta^2-1}\frac{dw}{d\zeta} - \frac{ip\zeta}{(\zeta^2-1)^2} w = 0 \quad . \quad . \quad . \quad . \quad . \quad (43)$$

while besides it must be ensured that the value of the expression:

$$e^{\xi\zeta}(\zeta^2-1)^2\left(w\,\xi - \frac{dw}{d\zeta}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

shall be the same at both ends of the path of integration.

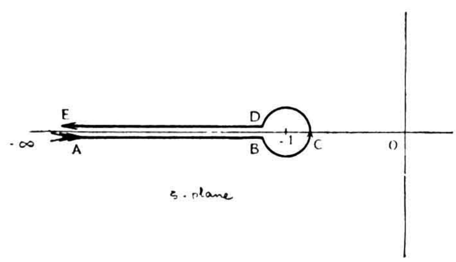
Equation (43) is a hypergeometric equation, having the singular points (all being regular):

$\zeta = -1$ with exponents: $\alpha_1 = \frac{1}{2}(-1-r+is)$, $\alpha_2 = \frac{1}{2}(-1+r-is)$
$\zeta = +1$ „ „ : $\beta_1 = \frac{1}{2}(-1-r-is)$, $\beta_2 = \frac{1}{2}(-1+r+is)$
$\zeta = \infty$ „ „ : $\gamma_1 = 3$, $\gamma_2 = 0$.

Here we have written $\sqrt{1+ip} = r + is$ etc.; p is a real and positive quantity, and we take r and s to be positive. — The function w thus can be defined by the scheme [1]:

$$w = P\left\{\begin{matrix} -1 & +1 & \infty & \\ \alpha_1 & \beta_1 & 3 & \zeta \\ \alpha_2 & \beta_2 & 0 & \end{matrix}\right\}$$

In order to obtain an integral for χ that does not become infinite for infinite values of ξ (which by nature is always real and positive), we must take the path of integration in such a way that the real part of ζ is always negative. This brings us to the path $ABCDE$; this



path at the same time ensures that the expression (44) shall vanish at both ends.

[1]) See f.i. E. T. WHITTAKER and G. N. WATSON, A course of modern analysis (Cambridge), § 10.7 and Chap. XIV.

In the vicinity of the point $\zeta = -1$ two linearly independent solutions of (43) are given by the following expressions, F being the symbol for the hypergeometric series:

$$w_1 = t^{\alpha_1} F(-1-r, -1+is, 1-r+is; t) \quad . \quad . \quad (45a)$$

$$w_2 = t^{\alpha_2} F(-1+r, -1-is, 1+r-is; t) \quad . \quad . \quad (45b)$$

Here t has been written for $(\zeta+1)/(\zeta-1)$; the series converge for all values of t satisfying $|t| \leqslant 1$, that is for all values of ζ having their real part negative or zero. For $|t| < 1$ the expression (45*a*) may be replaced by:

$$w_1 = t^{\alpha_1} (1-t)^3 F(2-r, 2+is, 1-r+is; t) \quad . \quad . \quad . \quad (45c)$$

It is also possible to write down two solutions, valid in the vicinity of $\zeta = \infty$; one of them is given by the series:

$$w_I = t^{\alpha_1} (1-t)^3 F(2-r, 2+is, 4; 1-t) \quad . \quad . \quad . \quad (46a)$$

while the other is of the form:

$$w_{II} = w_I \, lg\,(1-t) + c_0 + c_1(1-t) + c_2(1-t)^2 + \ldots \quad . \quad (46b)$$

These expressions are convergent for $|1-t| < 1$, the first one at any rate also for $|1-t| = 1$.

The functions w_1, w_2 are connected with the functions w_I, w_{II} by linear relations, which will be introduced subsequently. — In the special case when r is an integer $\geqslant 2$, the series defined by (45*a*) and (45*c*) and the one defined by (46*a*) break off, and reduce to polynomials. The function w_1 then is regular at $t = 1$ $(\zeta = \infty)$ and is equal to w_I multiplied by a constant factor.

Boundary conditions. — If we take w in the form $A_1 w_1 + A_2 w_2$, we obtain:

$$\chi = \xi \int d\zeta \, e^{\xi\zeta} (A_1 w_1 + A_2 w_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

There now are altogether six constants in our solution ($B_1, B_2, B_3, B_4, A_1, A_2$), which must be determined in such a way that $\chi = \chi' = 0$ for $\xi = 0$, while $\chi, \chi', \chi'', \chi'''$ must be continuous at $\xi = \alpha\delta$. Consequently there are also six homogeneous equations of the first degree for the six constants, and solutions different from zero can be obtained only if the determinant of the system vanishes. We shall not, however, try to develop an expression of this determinant for arbitrary values of δ, as this would require the evaluation of complicated integrals etc., but will turn at once to the case that δ becomes vanishingly small.

In that case the conditions $\chi = \chi' = 0$ for $\xi = 0$ can be applied at once to the expression (47), and lead to the equations: [1])

$$\lim_{\xi=0} \int d\zeta \, e^{\xi\zeta} \, \zeta (A_1 \, w_1 + A_2 \, w_2) = finite \quad . \quad . \quad . \quad . \quad (48)$$

$$\lim_{\xi=0} \int d\zeta \, e^{\xi\zeta} (A_1 \, w_1 + A_2 \, w_2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

In the general case, r not being an integer, w_1 and w_2 assume constant values for $\zeta = -\infty$. If these values are denoted by a_1, a_2 respectively for the case $arg\,(\zeta + 1) = -\pi$, $arg\, t = 0$, which are the values that can be assigned to the arguments along the part A B of the path of integration, then for the case $arg\,(\zeta + 1) = + \pi$, $arg\, t = 2\pi$ (which takes place along the part $D\,E$ of the path of integration) the limiting values become: $e^{2\pi i \alpha_1} a_1$, $e^{2\pi i \alpha_2} a_2$. Hence it will be seen that the condition (48) can be fulfilled only if:

$$A_1 \, a_1 \, (1 - e^{2\pi i \alpha_1}) + A_2 \, a_2 \, (1 - e^{2\pi i \alpha_2}) = 0 \quad . \quad . \quad . \quad . \quad (50)$$

It will appear from the results obtained below that this is also sufficient.

Investigation of the condition (49). — We begin with the parts of the integral relating to $A\,B$ and $D\,E$, which can be combined into the expression:

$$\lim_{\xi=0} \int_A^B d\zeta \, e^{\xi\zeta} \left[A_1 \, w_1 \, (1 - e^{2\pi i \alpha_1}) + A_2 \, w_2 \, (1 - e^{2\pi i \alpha_2})\right] \quad . \quad . \quad (51)$$

to be taken, as indicated, along $A\,B$.

We now put:

$$w_1 = a \, w_I + b \, w_{II}, \quad w_2 = c \, w_I + d \, w_{II} \quad . \quad . \quad . \quad . \quad (52)$$

the arguments of $\zeta + 1$ and of t in the points of $A\,B$ being as defined above. Making $\zeta = -\infty$ $(t = 1)$, and having regard to the expressions

[1]) Objections perhaps might be raised against the procedure of applying the conditions $\chi(0) = \chi'(0) = 0$ to the expression (47), as the point $\xi = 0$ is a singular point of eq. (39*b*). The same results, however, can be obtained in the following way: A finite value of σ is taken, and the equations expressing the continuity of $\chi, \chi', \chi'', \chi'''$ at $\xi = \alpha\sigma$ are written out in full. Then the exponential functions occurring in (40) are developed according to powers of σ; certain combinations of terms obtained in this way cancel in consequence of the relations $\Sigma B_\nu = \Sigma m_\nu B_\nu = 0$, which must be fulfilled in order that (40) satisfies the conditions at $\xi = 0$. Then a comparison is made of the terms of the lowest orders in σ on both sides of the equations. If now σ is made to decrease to zero, it is found that independently of the values of the B_ν the system of equations leads to certain relations between A_1 and A_2, viz. to eq. (50), which is equivalent with (48), and to eq. (49).

It may be remarked that also in the case of a finite value of σ the system of normal functions obtained for the case $\sigma \rightarrow 0$ can be used for the reduction of the integrals occurring in the distribution function, though the formulae will differ slightly from those deduced in § 4. We hope to come back to this point in a future paper.

(46*a*), (46*b*), we find: $a_1 = b\,c_0$, $a_2 = d\,c_0$. Hence if the relations (52) are substituted into the integral (51), it is found that the terms depending on w_{II} cancel in consequence of (50). The integral thus contains w_I only, and as w_I is of the order ζ^{-3} for $\zeta \to \infty$, it is convergent also when ξ is replaced by zero. The integral takes the form:

$$A_1\,(a - c\,a_1 / a_2)\;(1 - e^{2\pi i \alpha_1}) \int_A^B d\zeta\; w_I\,.$$

By comparing the values of the various solutions for $t \to 0$ and also for $t = 1$, we obtain: $a - c\,a_1 / a_2 = 1 / F_1$, where F_1 has been written for $F(2 - r, 2 + i\,s, 4;\; 1)$. We further introduce the expression (46*a*) for w_I into the integral and put: $\tau = 1 - t = -2 / (\zeta - 1)$, so that $d\zeta = 2\,d\tau / \tau^2$. In this way the integral to be evaluated becomes:

$$\frac{2\,A_1}{F_1}\,(1 - e^{2\pi i \alpha_1}) \int_0^{1-\sigma} d\tau\,(1 - \tau)^{\alpha_1}\,\tau\;\; F(2 - r, 2 + i s, 4;\; \tau)\;\;.\;\;. \qquad (53)$$

σ being the value of t at the point B. Replacing the factor τ before the function F by $1 - (1 - \tau)$, it is required to calculate [1]):

$$\int_0^{1-\sigma} d\tau\,(1 - \tau)^{\alpha_1}\,F - \int_0^{1-\sigma} d\tau\,(1 - \tau)^{\alpha_1 + 1}\,F.$$

We begin with the first integral. To abbreviate we write: $F = \Sigma\, f_h\, \tau^h$. Then we make use of the equation [2]):

$$I_h \equiv \int_0^{1-\sigma} d\tau\;(1 - \tau)^{\alpha_1}\,\tau^h = \frac{\Gamma(\alpha_1 + 1)\,\Gamma(h + 1)}{\Gamma(\alpha_1 + h + 2)} - \frac{1}{1 - e^{2\pi i \alpha_1}} \int_K d\tau\;(1 - \tau)^{\alpha_1}\,\tau^h,$$

where the circuit denoted by K is defined by the formula: $\tau = 1 - \sigma\,e^{i\theta}$, θ moving from 0 to 2π. We expand $(1 - \sigma\,e^{i\theta})^h$ according to powers of σ by means of the binomial theorem; in this expansion it is sufficient to retain such terms σ^n only, as leave the real part of $\alpha_1 + n = n - \frac{1}{2}(r + 1) + \frac{1}{2}\,i\,s$ negative, as the other terms may be made arbitrarily small by taking σ sufficiently near to zero. In this way we find:

$$I_h = \frac{\Gamma(\alpha_1 + 1)\,\Gamma(h + 1)}{\Gamma(\alpha_1 + h + 2)} - \sum_{n=0}^{n=m} \frac{(-1)^n\;h!\,\sigma^{\alpha_1 + n + 1}}{(\alpha_1 + n + 1)\,n!\,(h - n)!},$$

where m is the greatest integer contained in $\frac{1}{2}(r + 1)$.

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[1]) For the evaluation of the integral (53) I am indebted to the very valuable help of Dr. S. C. VAN VEEN at Dordrecht, and it is a pleasure for me to express my gratitude towards him also at this place.

[2]) This equation is obtained by a process similar to that used by WHITTAKER and WATSON, l.c., § 12.43.

The integral to be calculated now is given by the sum [1]: $\sum_h f_h I_h$. Taking the first term of I_h, it appears that the sum:

$$F^\star = \sum_h f_h \frac{\Gamma(\alpha_1+1)\,\Gamma(h+1)}{\Gamma(\alpha_1+h+2)}$$

can be transformed in such a way, that it takes the form of a hypergeometric series in which the independent variable has the value unity, together with some additional terms; the series can be summed by a known formula, so that $F^\star$ can be reduced to a relatively simple expression. — On the other hand it can be shown that for any value of $n < r$:

$$\frac{(-1)^n\,\sigma^{\alpha_1+n+1}}{(\alpha_1+n+1)\,n!}\sum_h f_h \frac{h!}{(h-n)!} = F_1\, g_n \frac{\sigma^{\alpha_1+n+1}}{\alpha_1+n+1},$$

where the g_n are the coefficients of the hypergeometric series

$$F(2-r,\, 2+is,\, 1-r+is;\, x).$$

A similar process can be carried out with the second integral, and in this way it is found that the original integral occurring in (53) can be transformed into an expression of the form:

$$\int_0^{1-\sigma} d\tau\,(1-\tau)^{\alpha_1}\,\tau F = (F^\star - F^{\star\star}) - F_1 \sum_n g_n \left(\frac{\sigma^{\alpha_1+n+1}}{\alpha_1+n+1} - \frac{\sigma^{\alpha_1+n+2}}{\alpha_1+n+2}\right). \quad (54)$$

By means of some further transformations the difference $(F^\star - F^{\star\star})$ can be brought into the following form:

$$F^\star - F^{\star\star} = -\frac{24}{p^2}\left\{\frac{\cos\frac{\pi}{2}(r+is)}{\cos\frac{\pi}{2}(r-is)} + 1\right\} \quad . \quad . \quad . \quad . \quad (55)$$

It remains to consider the part of the integral (49) relating to the circuit BCD. Here we may put at once $\xi = 0$. Further, as the real part of α_2 is positive, the contribution of w_2 into the integral can be made as small as we please by diminishing σ; hence it is sufficient to consider: $A_1 \int d\zeta\, w_1$. For the function w_1 the expression (45c) will be taken; upon integrating by terms it is found that the same series appears as occurred in (54).

Having regard to the constant factors before the integrals, it is easily seen that upon adding together the various terms, these series cancel in

[1]) The series $F = \Sigma f_h \tau^h$ is uniformly convergent in the domain $0 \leq \tau \leq 1 - \sigma$ and so term by term integration is allowed.

the final result, and thus equation (49) takes the form [1]):

$$-\frac{48\,A_1}{p^2\,F_1}\,(1-e^{2\pi i\alpha_1})\left\{\frac{\cos\frac{\pi}{2}(r+is)}{\cos\frac{\pi}{2}(r-is)}+1\right\}=0\quad.\quad.\quad.\quad(56)$$

Characteristic values of the parameter p. — The characteristic values of p are determined by the condition that the coefficient of A_1 in (56) shall vanish. This condition reduces to the equation [2]):

$$\cos\frac{\pi}{2}(r+is)+\cos\frac{\pi}{2}(r-is)=0,$$

having the roots: $r=1,\ 3,\ 5,\ldots.$ or, in general: $r=2k+1$, k being an integer. As: $1+ip=(r+is)^2$, it is found that the values of p are given by the expression:

$$p=4\,(2k+1)\sqrt{k^2+k}.$$

The case $k=0$ $(r=1)$ must be excluded, as p must be greater than zero.

(To be continued).

[1]) In connection with the expression obtained for I_h and with the calculation of $\Sigma\, f_h\, I_h$ it may be remarked that it has not been proved that the whole sum of the neglected terms vanishes for $\sigma\rightarrow 0$ in such a way that the summation with respect to h can be executed absolutely safely. However, as Dr. VAN VEEN has pointed out to me, any difficulties arising from this circumstance can be obviated, by considering first the integrals for the case that r is a positive number included between the limits ε_1 and $1-\varepsilon_2$ (ε_1 and ε_2 being arbitrary positive numbers $<{}^1/_2$). In that case the real parts of both α_1 and α_2 are >-1, and the integrals of both w_1 and w_2 along the circuit BCD can be discarded, and also the integral along the circuit K occurring in the expression for I_h, so that the additional terms in the result for I_h are got rid of. By means of the theory of analytic continuation it then can be shown that the final result obtained for the integral (49) remains valid for all cases provided $r>-2$.

[2]) The factor $1/F_1$ has the value: $\Gamma(2+r)\,\Gamma(2-is)\,/\,6\,\Gamma(r-is)$.

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Instrumente: *S* und *R*. Die Beobachtungen wurden alle auf *R* reduziert: die Reduktion *R*—*S* beträgt —$0^{m}.19$. Spektrum M3e (*Harv. Ann.* **79**, 164).

Der Stern ist von Anfang April bis Anfang Juni nicht beobachtbar: die letzte Beobachtung im Frühjahr erhielt ich am 3. April, die erste Sommerbeobachtung am 5. Juni. Es konnten mehrere Minima und Maxima ent-