has a sharp maximum. The fact that the angle between the two branches of the isopycnals becomes very much sharper means that the phenomenon of maximum density becomes much more striking at higher pressures. In connection with this it will be interesting to investigate the course of the isopycnals in the solid state [1]. Especially also the properties of the point where the lambda-curve meets the melting curve deserves attention. It is to be expected that these properties will have many points in common with those of the lambdapoint in the saturated vapour pressure curve [2].

At the lower temperatures the isopycnals of helium II become more and more horizontal. This is in harmony with what NERNST's heat theorem leads us to expect. It is a question whether the isopycnals at lower temperatures than those at which we experimented still have a flat maximum, so that the expansion coefficient changes its sign. The course of the isopycnals as far as measured does not make this probable.

[1]) In the experiments dealt with in this paper we did not go so far, as we preferred not to block the capillaries leading to and from the piezometer by solid helium.

[2]) Cf. W. H. KEESOM, Comm. Leiden Suppl. N^0. 71*e* § 1, and Suppl. N^0. 71*d* § 3.

Hydrodynamics. — *On the application of statistical mechanics to the theory of turbulent fluid motion.* VI. [1]) By J. M. BURGERS. (Mededeeling No. 26 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.)

(Communicated at the meeting of May 27, 1933).

7. *General remarks concerning the application of the formulae obtained.*

The results of the foregoing section may be summarized as follows: It has been shown that for a special choice of the function λ a set of functions $\chi_{\alpha k}$ can be obtained, which enable us to develop the stream function ψ of the relative motion into an expression of the type (26), while at the same time the exponent of the distribution function (12) is transformed into a homogeneous quadratic function of the coefficients A and B. If the choice of λ is taken for granted, and if provisionally it is assumed that the constant β occurring in the exponent be known, then it will be clear that — apart from difficulties connected with the numerical evaluation of integrals, etc. — it is possible to calculate the statistical mean values of quantities of the type $\overline{A_{\alpha k} A_{\alpha l}}$ etc. Further, having regard to equations (27) and (28), it must be possible to write down expressions for $-\overline{u'v'}$ and for $\overline{z}$ as functions of the coordinate η (or of y).

[1]) Part V has appeared in these Proceedings, **36**, p. 390, 1933.

Before starting with such calculations it will be useful to sketch in general terms the way in which it was intended to apply the results in order to work out the ideas indicated roughly in § 11 of Part III. It must be stated once more, however, that all considerations introduced have been tentative and provisional in character, and it will appear that several of the formulae arrived at do not seem very promising. The greatest difficulty in this respect is that the integral (30) apparently does not converge, as will be seen in § 10.

According to the programme drawn up in § 2 of Part IV, λ was meant to represent an unknown function, which ought to be determined in such a way, that the results to be obtained for $-\overline{u'v'}$ and for $\overline{U}$ should fulfill eq. (4). As the rigorous observation of this condition probably will lead to a functional equation of a rather hopeless type, it may be investigated whether an approximate fulfilment of eq. (4) can be obtained with a properly choosen function λ, the degree of approximation depending upon the number of adjustable parameters that might be introduced into this function. In this train of thought the function defined by eq. (36) had been taken, as it promised some simplification in the treatment of the equations. In this function the parameter b is undetermined, while also a certain freedom had been left with regard to the course of $d\lambda/dy$ in the neighbourhood of the walls.

Now although material has been prepared for the calculation of $-\overline{u'v'}$, no definite statement has yet been made concerning the problem presented by the determination of $\overline{U}$. According to the general idea put forth in § 11 of Part III, the distribution of the mean motion over a section of the channel also must come out as a result of the statistical formulae, in a similar manner as it was the case with the distribution of the relative motion. In fact the point of view accepted l.c. was that the statistical considerations should be applied to the *actual motion* (i.e. the sum of the relative motion and the mean motion), which actual motion is described by a stream function Ψ. Now the coordinates to be introduced into the generalized space in which the function Ψ is represented (in § 11 this space had been called the "ξ, η-space"), can be divided into two groups. One group is formed by the A's and B's used in describing the stream function ψ of the relative motion, that are all those coordinates which give rise to zero mean values of Ψ (taken with respect to x). The second group on the other hand must describe the distribution of the mean values $\overline{\Psi}$ as a function of y. This second group of coordinates can be considered as being "orthogonal" to the coordinates of the first group, in consequence of the circumstance that the exponent of the distribution function (10) could be divided into two parts, one part depending exclusively on the mean motion, the other part depending exclusively on the relative motion. It thus will be in line with our programme if we try to obtain the statistical mean value of U from an investigation of the distribution function:

$$e^{\frac{\beta}{R}\int dy\left\{\frac{\partial U}{\partial y}\frac{d\lambda}{dy}-\left(\frac{\partial U}{\partial y}\right)^2\right\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

This function can be considered as representing the quotient of the original distribution function (10) and the distribution function (12) of the relative motion. Hence we arrive at the conclusion that every function λ that might be choosen will determine a certain function $\overline{U}$.

In working out this idea we are confronted with a certain difficulty relating to the choice and the number of the coordinates that shall describe the course of the function $\overline{\Psi}$, especially as their number, or rather their "spacing", in some way must bear a relation to the "spacing" of the coordinates used to describe ψ. Moreover as the exponent of the function (57) is not homogeneous with respect to U, it does not give rise to a variational problem from which a system of normal functions can be deduced. However, we may try whether a simple Fourier expansion of the type:

$$\overline{\Psi} = \Sigma\, U_n \sin(2n+1)\pi y \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

may be used. The form of this expression ensures that $\partial\overline{\Psi}/\partial y$ shall be zero at the walls of the channel. In order that the value of $\overline{\Psi}$ itself shall be $\pm\frac{1}{2}$ at the walls a certain relation must exist between the coefficients U_n, which can be easily written down. — Now let us put:

$$\lambda = \Sigma\, \lambda_n \cos(2n+1)\pi y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

which expression makes $\lambda = 0$ at the walls. There will exist also a certain relation between the coefficients λ_n, on account of eq. (9) to which λ is subjected. — If (58) and (59) are introduced into (57) the statistical problem can be worked out. The result is:

$$\overline{U}_n = \frac{\lambda_n}{2\pi(2n+1)} + \frac{24}{\pi^4}\frac{(-1)^n}{(2n+1)^4}.$$

This leads to the following formula for $d\overline{U}/dy$:

$$\frac{d\overline{U}}{dy} = \frac{1}{2}\frac{d\lambda}{dy} - 6y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

This formula is substantially the same as the one obtained in § 4 of Part II, and, as has been mentioned there, apparently leads to values which in the central part of the channel are much too high. Hence in this respect no improvement has been obtained in comparison with the method of Parts I and II [1]), while — as will be seen below — in another respect we have drifted away even farther from experimental values, as the magnitude of $d\overline{U}/dy$ at the wall appears to surpass many times the estimate made formerly.

Notwithstanding this result, let us proceed to see how it stands about the application of eq. (4). This equation can be written:

$$2\,C - \frac{d}{dy}\,\overline{u'v'} + \frac{1}{R}\,\frac{d^2\overline{U}}{dy^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

Integrating with respect to y we obtain:

$$2\,C\,y - \overline{u'v'} + \frac{1}{R}\,\frac{d\overline{U}}{dy} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Now let us assume the course of $d\lambda/dy$ which is described by eq. (36) for the region $\delta < \eta < \frac{1}{2}$ (i.e. $-\frac{1}{2} + \delta < y < 0$), and by eq. (38) for the region in the immediate neighbourhood of the wall, while for positive values of y we simply change the sign of $d\lambda/dy$. If λ is found by an integration, then eq. (9) leads to a connection between b and δ, which approximately works out to: $\delta = 2b/R$. So there is only one adaptable parameter in λ. Consequently, as the pressure drop $2C$ is still unknown, we can make eq. (61) fit at two points at most, and then shall obtain relations that determine both b and C. Apparently it will be convenient to take as one of these points the point: $\eta = 0$ $(y = -\frac{1}{2})$, where we have $-\overline{u'v'} = 0$ (as will be seen if it is remembered that $\chi_{\alpha k}$ and $d\chi_{\alpha k}/d\eta$ are zero for $\eta = 0$). Eq. (60), when applied to the same point, gives us approximately: $1/R \,.\, d\overline{U}/dy = 1/8\,b$, and thus we obtain:

$$C \cong 1/8\,b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

The second point will be taken at a distance from the wall which is great compared with δ, but still can be considered as small in comparison with the half breadth of the channel. Then we may neglect the contribution of $d\overline{U}/dy$ into eq. (61). Further it will be seen in § 9 that in this region an approximate expression can be deduced for $-\overline{u'v'}$ which is independent of η. As the term $2\,Cy$ in (61) for $\eta \ll \frac{1}{2}$ still may be replaced by $-C$, we obtain:

$$C \cong -\overline{u'v'} \quad (\text{for } \tfrac{1}{2} \gg \eta \gg \delta). \quad . \quad . \quad . \quad . \quad . \quad (63)$$

The right hand side of this equation is a function of b and of β.

[1]) It thus appears that the expectation expressed in § 11 of Part III in connection with eq. (77) of that paper comes out negatively.

We thus have arrived at two equations between the three constants C, b, β. It remains to find a third equation, and it is evident that this equation must be furnished by the dissipation condition. This condition is expressed by eq. (7); written explicitly it takes the form (after division by 2):

$$C - \frac{1}{R}\int_0^{1/2} d\eta \overline{\left(\frac{\partial U}{\partial y}\right)^2} - \frac{1}{R}\int_0^{1/2} d\eta\, \overline{z} = 0 \quad . \quad . \quad . \quad . \quad (64)$$

The first integral perhaps may be replaced by [1]:

$$\frac{1}{R}\int_0^{1/2} d\eta \left(\frac{d\overline{U}}{dy}\right)^2 \cong \frac{1}{24\, b}.$$

If this is taken for granted, it remains to calculate $\int dy\, \overline{z}$. This integral can be expressed as a sum with respect to the indices k and α. As has been mentioned, however, this sum does not converge, if the domain of both k and α is stretched out to infinity. Hence eq. (64) can be used only if the "spectrum" of normal functions is cut off in some way. It is possible that a limit for the index k will be imposed by the finite breadth of the channel (see below, § 8), but with α the matter is much more serious.

We shall come back to this point in § 10, but first it will be indicated in which way expressions may be obtained for $-\overline{u'v'}$ and $\overline{z}$.

8. *Data concerning the normal functions* $\chi_{\alpha k}$.

The results obtained in § 6 lead us to a system of normal functions defined by the formula:

$$\chi_{\alpha k} = N_k\, \alpha\, \eta \int d\zeta\, e^{\alpha\eta\zeta}\, w_k(\zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

where N_k is a numerical constant, while w_k is the function denoted formerly by w_I and given by eq. (46a), for the case: $p = p_k = 4(2k+1)\sqrt{k^2+k}$, $r = 2k+1$, $s = 2\sqrt{k^2+k}$. In this case the hypergeometric function reduces to a polynomial of degree $2k-1$ in $1-t$. The factor N_k must be determined in such a way that the normalizing condition (23) shall be fulfilled with $d\lambda/dy$ given by (36) down to $\eta = 0$; it is independent of α.

In connection with the formula: $p = b\Lambda$ it follows that the characteristic values of Λ are given by:

$$\Lambda_k = p_k/b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

[1]) This point ought to be investigated with the aid of the distribution function (57). (See a remark in Part VII).

They are independent of a. The condition that all Λ_k shall be greater than unity (which condition was stated in § 4) requires $b < p_1$. As p_1 is about 17, this result, when taken in connection with eq. (62), leads to $C > 0{,}0073$, which is many times superior to experimental values.

If the polynomial for w_k is substituted into the integral (65), the integration of the separate terms leads to "confluent hypergeometric functions" (see WHITTAKER and WATSON's Modern Analysis, Chap. XVI). We mention the asymptotic expansion for very great values of $a\eta$:

$$\chi_{\alpha k} = constant.\ e^{-\alpha\eta} \left[(2\,a\eta)^{-a_1} - a_1\,(2\,a\eta)^{-a_1-1} \ldots\right] \quad . \quad . \quad . \quad (67)$$

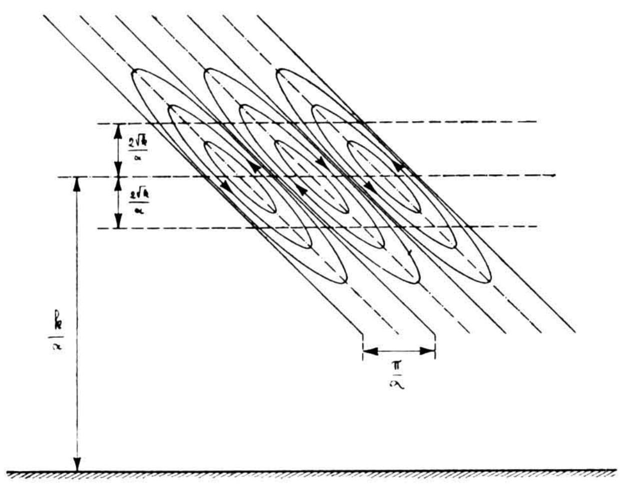
where: $-a_1 = (k+1) - i\sqrt{k^2+k}$. The first term of this series has its maximum for $a\eta = k+1$. Unfortunately for this value of $a\eta$ the asymptotic expansion becomes nearly useless. If notwithstanding this circumstance we consider this first term in order to have a very coarse image of the function $\chi_{\alpha k}$, it is found that for great values of k this term may be roughly approximated by the following expression:

$$\chi_{\alpha k} \simeq c\,e^{-\frac{(\alpha\eta')^2}{2k}}\,(\cos a\eta' - i \sin a\eta') \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

$a\eta'$ being written for $a\eta - k$, while c is a constant. This expression can be separated into its real and imaginary parts; if these parts are substituted into formula (14) the stream function of the mode of relative motion corresponding to $\chi_{\alpha k}$ is found to be:

$$\psi_{\alpha k} \simeq c\,e^{-\frac{(\alpha\eta')^2}{2k}} \cos\,(ax + a\eta + const.) \quad . \quad . \quad . \quad . \quad . \quad (69)$$

If the expression (68) is applied to the calculation of the integrals (23) and (24), it is found that the proper value of Λ does not come out: a value about half the true one is obtained. Still it may be supposed that some features indicated by formula (69) are not far from the truth,



viz. that the mode of motion corresponding to $\chi_{\alpha k}$ consists of a system of skew vortices, having their centers upon the line: $\eta = k/a$; the dimensions in the direction of x being $\pi/a = \pi\eta/k$, while the dimensions in the direction of η are of the order $2\sqrt{k}/a = 2\eta/\sqrt{k}$ (compare the accompanying sketch) [1]).

If a is increased while k is kept constant, the dimensions of the system are diminished proportionally. On the other hand, if, for a constant a, the value of k is increased, the system moves outwards. Clearly this cannot go on indefinitely, on account of the presence of the second wall at $\eta = 1$. Hence we are led to the supposition that k cannot increase beyond a certain amount of the order of a. An exact expression for this limit cannot be obtained from our calculations, as in the function λ that has been used no account is taken of the presence of this second wall. The point, however, is of importance in connection with what has been mentioned at the end of § 7. (An investigation of the character of the normal functions for a modified form of $d\lambda/dy$, which should change sign at $\eta = \frac{1}{2}$ $(y = 0)$, might be of interest).

9. *Calculation of* $-\overline{u'v'}$ *and* $\overline{z}$ *for values of* η *greatly surpassing* δ.

The values of $-\overline{u'v'}$ and $\overline{z}$ are obtained from eqs. (27) and (28) if for $A_{\alpha k}\, A_{\alpha l}$ etc. the statistical mean values are substituted. The way in which these statistical mean values can be calculated has been sketched in § 4 for the case that normal functions exactly corresponding to the course assumed for the function λ have been obtained. Now actually we have calculated a system of normal functions for a course of $d\lambda/dy$ described by eq. (36) right down to $\eta = 0$. If we adhere to this system of normal functions, and nevertheless at the same time wish to introduce the course of $d\lambda/dy$ which is determined by eqs. (36) and (38) taken conjointly, then it will be clear that the integrals (21), (23), (31) no longer will have the values indicated before. Without going into the details of the calculation we may suppose that certain corrections must be added to the right hand sides of these equations. Then the distribution function will not have the form given by eq. (33), but will contain certain additional terms, depending on these corrections.

However, if we restrict ourselves to values of η that are great compared with δ, then, judging by what has been found in the foregoing §, it may be assumed that those functions $\chi_{\alpha k}$ which materially contribute in the value of $-\overline{u'v'}$ or of $\overline{z}$, will be very small in the narrow region where $d\lambda/dy$ is given by eq. (38) (it will be remembered that the width δ of this region, according to what has been deduced in § 7, is of the

[1]) It is probable that a more accurate calculation will show that the slope of the vortices is not equal to 45° as would follow from eq. (69).

order b/R). Hence for these functions we may neglect the corrections in the integrals (21) etc. and keep to the expression (33) for the distribution function. From this expression the following results are obtained for the statistical mean values:

$$\left.\begin{array}{c}\overline{A_{\alpha k}\bar{A}_{\alpha k}}=\dfrac{2R}{\beta\,\alpha^3(\Lambda_k-1)},\ \overline{B_{\alpha k}\bar{B}_{\alpha k}}=\dfrac{2R}{\beta\,\alpha^3(\Lambda_k+1)}\\ \overline{A_{\alpha k}\bar{A}_{\alpha l}}=\overline{B_{\alpha k}\bar{B}_{\alpha l}}=\overline{A_{\alpha k}\bar{B}_{\alpha l}}=0\end{array}\right\} \quad . \quad . \quad . \quad . \quad (70)$$

Substituting these into (27) we have:

$$-\overline{u'v'}\simeq\frac{R}{\beta}\sum_{\alpha k}\frac{i\,(\chi'_{\alpha k}\bar{\chi}_{\alpha k}-\chi_{\alpha k}\bar{\chi}'_{\alpha k})}{\alpha^2(\Lambda_k^2-1)} \quad . \quad . \quad . \quad . \quad . \quad (71)$$

In order to carry out the summation with respect to α in (71), we avail ourselves of the remark made in § 4 that this summation properly speaking ought to be replaced by an integration. The transition from a sum to an integral can be effected if we assume that the (constant) interval between the values of α in the sum is given by a certain number θ. Then it is allowed to write: $\sum_\alpha(\ldots.)\simeq 1/\theta\,.\int d\alpha\,(\ldots.)$ [1]. Further by making use of eq. (65) it can be shown that the expression $i\,(\chi'_{\alpha k}\bar{\chi}_{\alpha k}-\chi_{\alpha k}\bar{\chi}'_{\alpha k})$ is of the form: $\alpha/b\,.\,f_k(\alpha\eta)$. Hence eq. (71) may be written:

$$-\overline{u'v'}\simeq\frac{R}{b\beta\theta}\sum_k\frac{1}{\Lambda_k^2-1}\int\frac{d\alpha}{\alpha}f_k(\alpha\eta).$$

Now the integral $\int_0^\infty\frac{d\alpha}{\alpha}f_k(\alpha\eta)$ — which appears to be convergent — yields a number, say σ_k, which is independent of η. Thus we find (with $\Lambda_k=p_k/b$):

$$-\overline{u'v'}\simeq\frac{Rb}{\beta\theta}\sum_k\frac{\sigma_k}{p_k^2-b^2}\quad(\text{for }\tfrac{1}{2}\gg\eta\gg\delta)\ . \quad . \quad . \quad . \quad (72)$$

The numbers σ_k have not been calculated, as this required the evaluation of certain rather complicated integrals; these integrals, however, can be expressed in the form of a series, and thus a numerical evaluation,

[1]) The introduction of θ does not bring a new unknown constant into the problem, as it will appear that in all further equations only the combination $\beta\theta$ occurs.

though laborious, will be possible. It is probable that the sum with respect to k is convergent. If b should prove to be nearly equal to p_1 (it must be $< p_1$), the first term of the sum will be far more important than the rest.

The result that the value of $-\overline{u'v'}$ is independent of η applies only to a domain of values of η satisfying the conditions: $\frac{1}{2} \gg \eta \gg \delta$. In this domain the shearing force acting in layers parallel to the x-axis is still substantially equal to the value at the wall itself, which value in our system of non dimensional variables is given by the constant C. —

A similar calculation carried out starting from eq. (28) leads to the result that $\overline{z}$ must be proportional to η^{-2}, though the nature of the factor has not been investigated.

10. *The expression for* $\int_0^{1/2} d\eta\, \overline{z}$.

As we have found that $\overline{z}$ is proportional to η^{-2} for $\eta \gg \delta$, it may be expected that in this integral the upper limit of η may be replaced by ∞ without great error.

The values of the integrals (22), (24), (32) are not affected by the course assumed for the function λ. Hence eq. (30) must remain true also if $d\lambda / dy$ is determined by eqs. (36) and (38) taken conjointly.

If in eq. (30) we substitute the approximate values of $\overline{A_{\alpha k} \bar{A}_{\alpha k}}$ etc. as given in (70), we find:

$$\int_0^\infty d\eta\, \overline{z} \simeq \frac{2R}{\beta} \sum_{\alpha k} \frac{\Lambda_k^2}{\Lambda_k^2 - 1} \cong \frac{2R}{\beta\theta} \int d\alpha \sum_k \frac{\Lambda_k^2}{\Lambda_k^2 - 1} \quad . \; . \; . \quad (73)$$

This expression is *divergent* — both with respect to k and with respect to α. The summation with respect to k probably may be limited, as has been indicated, the number of terms being of the order α. But at the present moment I do not see from where a limit for α might arise.

It is true that the approximations (72) are not valid for modes of relative motion which wholly, or at least for the greater part, fall into the region defined by $\eta < \delta$. But as far as I can see an exact calculation will not give values for $\overline{A_{\alpha k} \bar{A}_{\alpha k}}$ etc. which decrease sufficiently fastly in order to make the integration with respect to α convergent. This may be inferred already from the general formulae of § 4. Suppose that the exact system of normal functions has been determined corresponding to the precise course of the function $d\lambda / dy$, with all modifications in this function that might have been considered appropriate. In that case formulae (29), (30) and (33) are exact and we should obtain:

$$\tfrac{1}{2}\,a^3 \Lambda_{\alpha k} (\overline{A_{\alpha k} \bar{A}_{\alpha k}} + \overline{B_{\alpha k} \bar{B}_{\alpha k}}) = \frac{2R\,\Lambda^2_{\alpha k}}{\beta\,(\Lambda^2_{\alpha k} - 1)} \quad . \; . \; . \; . \quad (74)$$

It thus would appear that the dissipation for every separate mode of relative motion approaches to the constant limiting value $R\beta$ (it may be supposed that also in the general case the characteristic values $\Lambda_{\alpha k}$ soon will become great in comparison with unity) [1]). Hence the total dissipation threatens to become infinite, unless a limit can be found to the system of modes of motion. Although the discussion of the properties of the system of normal functions determined by eq. (19) with an arbitrary form of $d\lambda/dy$ will be necessary to settle such a point, I cannot find any indication of such a limit.

Perhaps there may be a limit to a of the order of magnitude of $1/\delta$ (i.e. of R). If such a limit is assumed artificially, then an estimate of the integral for the dissipation could be made, and the third equation between the constants C, b, β (or rather, $\beta\,\theta$) might be written down.

I must leave the problem at this point, the main object of the foregoing lines having been to give a somewhat more detailed view of the ideas which in a rather crude form had been indicated already formerly. A few additional remarks will be made in a concluding paper, in which it will be tried moreover to apply the statistical method to an imaginary mechanical problem, of a simpler character than that presented by the motion of a fluid, as this perhaps may afford an easier basis for a criticism of the method.

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[1]) The expression (74) actually gives the dissipation for two modes of relative motion, one of which is of the nature indicated by eq. (69), while the other is obtained by changing x into $-x$ in this expression. In those cases where $\Lambda_{\alpha k}$ does not differ much from unity, the former of these is much stronger than the second one; if $\Lambda_{\alpha k}$ is very great, the intensities of the two become nearly equal.