Mathematics. — *Electromagnetism independent of metrical geometry. 2. Variational principles and further generalisation of the theory.* By D. VAN DANTZIG. (Communicated by Prof. H. KRAMERS).

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§ 1. *Variational principles.*

Instead of MAXWELL's equations (EM 1 (5) [1])) I will further use the equations

$$\text{I} \quad \mathfrak{s}^i = \partial_j \mathfrak{H}^{ij}, \quad \text{II} \quad F_{ij} = 2\,\partial_{[i}\,\varphi_{j]} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the potentials φ_i ($-\varphi_4 = \varphi$ = electric potential, $\bar{A} = (\varphi_1, \varphi_2, \varphi_3)$ = magnetic potential vector, so that $\bar{E} = -\operatorname{grad}\varphi - \frac{1}{c}\dot{\bar{A}}$, $\bar{B} = \operatorname{rot}\bar{A}$) have been introduced. The formal analogy between the two equations (1) is made evident, if the first of them is brought into the form, equivalent with it,

$$s_{ijk} = -3\,\partial_{[i}\,H_{jk]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

($H_{ij} = \frac{1}{2}\,\mathfrak{E}'_{ijkl}\,\mathfrak{H}^{kl}$). Hence, whereas F_{ij} and s_{ijk} are wholly determined, φ_i is only determined but for the gradient of a scalar and H_{ij} but for the rotation of a covariant vector (hence $\mathfrak{H}^{ij}$ is determined but for the divergence of a contravariant trivectordensity).

Now let $\mathfrak{W}$ be an arbitrary function (scalar density of weight 1) of the φ_i and the $\partial_{[i}\,\varphi_{j]}$, and let us *define* F_{ij} by (1, II), $\mathfrak{H}^{ij}$ and $\mathfrak{s}^i$ by

$$\mathfrak{H}^{ij} = \frac{\partial \mathfrak{W}}{\partial F_{ij}}, \qquad \mathfrak{s}^i = -\frac{\partial \mathfrak{W}}{\partial \varphi_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The variation of the integral

$$W = \int^4 \mathfrak{W}\, d\Sigma \text{ [2]}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

gives

$$\delta \mathfrak{W} = \frac{\partial \mathfrak{W}}{\partial \varphi_i}\,\delta\varphi_i + \frac{\partial \mathfrak{W}}{\partial(\partial_j \varphi_i)}\,\delta\,\partial_j\,\varphi_i = -\,\mathfrak{s}^i\,\delta\,\varphi_i + \mathfrak{H}^{ji}\,\partial_j\,\delta\,\varphi_i$$

because $\partial\mathfrak{W}/\partial(\partial_i\,\varphi_j) = -\,\partial\mathfrak{W}/\partial(\partial_j\,\varphi_i) = \partial\mathfrak{W}/\partial F_{ij}$ as only the *alternated* part of $\partial_i\,\varphi_j$ occurs in $\mathfrak{W}$. Hence

$$\delta\,\mathfrak{W} = \partial_j\,(\mathfrak{H}^{ji}\,\delta\,\varphi_i) + (-\,\mathfrak{s}^i - \partial_j\,\mathfrak{H}^{ji})\,\delta\,\varphi_i\,.$$

[1]) Cf. D. VAN DANTZIG, Electromagnetism independent of metrical geometry. 1. The foundations, these Proceedings **37**, (1934) 521—525, abbreviated as EM 1.

[2]) $\int^4$ is an abbreviation for $\int\!\!\int\!\!\int\!\!\int$; $d\Sigma$ is the fourdimensional volume-element (density of weight -1).

Hence $\delta W = 0$ leads to (1, I), whereas (1, II) has been used as a definition.

Another variational principle exists which leads to (1, II), if (1, I) is assumed. Indeed let now $\mathfrak{W}$ be an arbitrary function of the $\mathfrak{H}^{ij}$ and the $\mathfrak{s}^i$, the latter being defined by (1, I) (and *not* of the φ_i, F_{ij}) and let us define F_{ij}, φ_i by

$$F_{ij} = \frac{\partial \mathfrak{W}}{\partial \mathfrak{H}^{ij}} \qquad \varphi_i = -\frac{\partial \mathfrak{W}}{\partial \mathfrak{s}^i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Then we have because of

$$\frac{\partial \mathfrak{W}}{\partial (\partial_k \mathfrak{H}^{ji})} = 2 A^k_{[i} \frac{\partial \mathfrak{W}}{\partial \mathfrak{s}^{j]}} = -2 A^k_{[i} \varphi_{j]} \quad . \quad . \quad . \quad . \quad (6)$$

(which equation is a consequence of the fact that $\partial_k \mathfrak{H}^{ji}$ does only occur in $\mathfrak{W}$ in the combination $\partial_j \mathfrak{H}^{ij}$):

$$\delta \mathfrak{W} = \tfrac{1}{2} \frac{\partial \mathfrak{W}}{\partial \mathfrak{H}^{ij}} \delta \mathfrak{H}^{ij} + \tfrac{1}{2} \frac{\partial \mathfrak{W}}{\partial (\partial_k \mathfrak{H}^{ji})} \delta (\partial_k \mathfrak{H}^{ji}) =$$
$$= \tfrac{1}{2} F_{ij} \delta \mathfrak{H}^{ij} - A^k_{[i} \varphi_{j]} \partial_k \delta \mathfrak{H}^{ji} =$$
$$= \partial_k (\varphi_i \delta \mathfrak{H}^{ki}) + \tfrac{1}{2} (F_{ij} - 2 \partial_{[i} \varphi_{j]}) \delta \mathfrak{H}^{ij}$$

so that $\delta W = 0$ leads to (1, II).

Hence we have a full parallelism between the F_{ij} and the s_{ijk} and between the $\mathfrak{H}^{ij}$ and the $-\varphi_i$ [1] [2].

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[1] This parallelism may be brought into evidence by the use of a five-dimensional formalism, which greatly simplifies the calculations. This may be done in two ways. By the first method equations (1, I, II) take the forms

$$\text{I} \quad \partial_\mu \mathcal{H}^{\lambda\mu} = 0, \qquad \text{II} \quad \partial_{[\nu} \mathcal{F}_{\mu\lambda]} = 0;$$

by the second method they are taken together into one equation of the form

$$\mathcal{F}_{\nu\mu\lambda} = 3 \partial_{[\nu} \mathcal{H}_{\mu\lambda]},$$

($\varkappa, \lambda, \mu, \nu = 1, 2, 3, 4, 5$). I will, however, not yet make much use of these formalisms, because until now the fifth coordinate seems to have only a formal significance.

[2] Hence we may compare the electromagnetical quantities in two ways with the mechanical quantities

$q^\alpha \longleftrightarrow \varphi^i$	or	$q^\alpha \longleftrightarrow \mathfrak{H}^{ij}$
$\dot{q}^\alpha \longleftrightarrow F_{ji}$	"	$\dot{q}^\alpha \longleftrightarrow \mathfrak{s}^i$
$p_\alpha \longleftrightarrow \mathfrak{H}^{ji}$	"	$p_\alpha \longleftrightarrow \varphi_i$
$k_\alpha \longleftrightarrow \mathfrak{s}^i$	"	$k_\alpha \longleftrightarrow F_{ji}$
$(k_\alpha = \dot{p}_\alpha)$		

In the latter case the corresponding quantities are also physically analogous, $\mathfrak{s}^j$ being related to the velocity and F_{ij} to the forces. Hence we will prefer to work with the second variational principle; the physical meaning of the analogy between the $\mathfrak{H}^{ij}$ and the φ_i with the q^α and the p_α will be shown later.

In the important special case, when $\mathfrak{W}$ is (in the first case) *homogeneous of degree 2* (but not necessarily quadratic!) in the φ_i and F_{ij} together or (in the second case) homogeneous of degree 2 in the $\mathfrak{H}^{ij}$ and the $\mathfrak{S}^i$ together, we find from EULER's condition for homogeneous functions

$$2\,\mathfrak{W} = \tfrac{1}{2} F_{ij}\,\mathfrak{H}^{ij} - \varphi_i\,\mathfrak{S}^i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It is remarkable that in this case $\mathfrak{W}$ itself is a divergence:

$$2\,\mathfrak{W} = \partial_i\,\mathfrak{H}^{ij}\,\varphi_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

§ 2. *The linking equations (General case).*

The parallelism we found in § 1 is lost if we introduce the "linking equation" *E M* 1 (9)

$$F_{ij} = \tfrac{1}{2}\,\eta_{ijkl}\,\mathfrak{H}^{kl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

as no such relation exists between the φ_i and the $\mathfrak{S}^i$. Equation (9), however, is extremely special and holds e.g. for special cases of *homogeneous* matter. Now it is well known that e.g. $\bar{B}$ and $\bar{H}$ may be defined by means of volume-integrals of the magnetisation-vector $\bar{B}-\bar{H}$ over the whole magnetic body (or even over the whole space), exclusive a small hole of a definite form at the point where the vectors are to be determined. Now the magnetisation cannot appear in our theory, $\bar{B}$ being a covariant *bivector*, whereas $\bar{H}$ is a covariant *vector*, so that their difference cannot be invariant. Hence it seems a rather natural assumption that, instead of these integral relations, we can express e.g. $\bar{B}$ by means of a volume-integral of $\bar{H}$, and just so $\bar{E}$ by a volume-integral of $\bar{D}$. Finally to get fourdimensional invariance we will take instead of volume-integrals fourdimensional-integrals. Hence we assume instead of (9)

$$F_{ij} = \int^{4} \tfrac{1}{2}\,\gamma_{ijk'l'}\,\mathfrak{H}^{k'l'}\,d\,\Sigma' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Here the coefficients $\gamma_{ijk'l'}$ are "two-point-functions", depending on the coordinates of two world-points: P (where F_{ij} is to be calculated) and P' (which runs through the whole space-time). With the first two indices $\gamma_{ijk'l'}$ transforms as a covariant bivector at P, with the latter two as a covariant bivector at P'. Hence with respect to the variable point P' the integrand of (10) is a *scalar*, so that the equation (10) is invariant under *arbitrary* transformations of coordinates. The integration with respect to the time has not only a formal, but also a physical meaning: it expresses that the F_{ij} (especially $\bar{B}$) may depend not only on the *present*, but also on *past* (or future) values of $\bar{H}$ in whole space, i.e. it allows to treat of phenomena of the kind of *remanent* magnetism [1]). Often we will suppose the $\gamma_{ijk'l'}$ to depend symmetrically on P and P';

$$\gamma_{ijk'l'}(P, P') = \gamma_{k'l'ij}(P', P). \qquad (11)$$

especially if we wish to deduce our equations from a variational principle. However, it will be necessary to drop this supposition, as soon as we will make difference between past and future, as it is necessary in the actual laws of physics.

Now it is known that the φ_i may be expressed as integrals of the $\mathfrak{s}^i$ over the three-dimensional light-cone. We assume these integrals to be a limiting case of fourdimensional integrals, and put

$$\varphi_i = \int^4 \gamma_{ij'} \mathfrak{s}^{j'} d\Sigma' \qquad (12)$$

where the integral is to be taken over the whole space-time. If we suppose the $\gamma_{ij'}$ to be symmetrical

$$\gamma_{ij'}(P, P') = \gamma_{j'i}(P', P) \qquad (13)$$

we must admit advanced as well as retardated potentials. We may also keep to the ordinary theory, by allowing the $\gamma_{ij'}$ and the $\gamma_{ijk'l'}$ to be symbolical functions of the kind of DIRAC's function $\delta(x)$ [2]). Indeed we get ordinary retardated potentials in empty space (of special relativity) by putting $\gamma_{ij'}(P, P') \overset{*}{=} g_{ij}\,\gamma(P P')$, where

$$\gamma(P, P') \overset{*}{=} -\frac{1}{4\pi r}\,\delta(ct - ct' - r) \;{}^{3)}\,{}^{4)} \qquad (14)$$

with respect to Galilean coordinates, where $r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$. Here $\gamma_{ij'}(P, P') \neq \gamma_{j'i}(P', P)$.

The action (4), (7) now becomes

$$W = \int^4 \mathfrak{W}\, d\Sigma = \int^4 d\Sigma \int^4 \mathfrak{w}(P, P')\, d\Sigma', \qquad (15)$$

$$2\,\mathfrak{w} = \tfrac{1}{4}\,\gamma_{ijk'l'}\,\mathfrak{H}^{ij}\,\mathfrak{H}^{k'l'} - \gamma_{ij'}\,\mathfrak{s}^i\,\mathfrak{s}^{j'}. \qquad (16)$$

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[1]) We might suppose that F_{ij} were to depend not only on the $\mathfrak{H}^{ij}$, but also on the $\mathfrak{s}^i$, past and present. This does make only a difference, however, in so far as boundary integrals are to be considered, which we will not do now. It is, however, not improbable that this will have to be done in the final version of the theory.

[2]) Properly spoken $\sigma(x)$ is not a point-function but $\sigma(x)\,dx$ is an interval function, viz. equal to 1 or 0 if the (finite) interval dx does or does not contain $x = 0$. This exact theory leads to STIELTJES-integrals. Cf. J. VON NEUMANN, Mathematische Grundlagen der Quantenmechanik, J. SPRINGER (1932).

[3]) $\overset{*}{=}$ means that the equation holds only with respect to special (e.g. orthogonal) coordinates.

[4]) The factor 4π occurs because we always use HEAVISIDE-LORENTZ-units.

Here $\mathfrak{w}$ is a two-point-density of weights (1, 1) [1]), whereas $\mathfrak{s}^i$ is *defined* by (1, I). If F_{ij} and φ_i are defined by (10), (12), and if we suppose that the conditions of symmetry (11), (13) are valid [2]), that all boundary-integrals and integrals resulting from eventual singularities of the $\gamma_{ij'}$, $\gamma_{ijk'l'}$ vanish, and that the integrations with respect to $d\Sigma$ and $d\Sigma'$ are interchangeable, it is easily seen that variation of W leads to (1, II).

Combining this with (12) we get

$$\left.\begin{aligned} F_{ij} &= 2\,\partial_{[i}\int^4 \gamma_{j]k}\,\mathfrak{s}^{k'}\,d\Sigma' = 2\int^4 (\partial_{[i}\,\gamma_{j]k'})\,\partial_{l'}\,\mathfrak{H}^{k'l'}\,d\Sigma' = \\ &= 2\int^4 \partial_{l'}\,(\mathfrak{H}^{k'l'}\,\partial_{[i}\,\gamma_{j]k'})\,d\Sigma' - 2\int^4 (\partial_{l'[i}\,\gamma_{j]k'})\,.\,\mathfrak{H}^{k'l'}\,d\Sigma'\ ^{3)} \end{aligned}\right\}\ . \quad (17)$$

If we suppose the first integral, which may be transformed into a boundary-integral, to vanish [4]), we find from (17) and (10) the identity

$$\int^4 (\tfrac{1}{2}\,\gamma_{ijk'l'} + 2\,\partial_{l'[i}\,\gamma_{j]k'})\,\mathfrak{H}^{k'l'}\,d\Sigma' = 0 \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Now $\gamma_{ij'}$ and $\gamma_{ijk'l'}$ characterise the properties of *matter*. Hence if we

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[1]) Its transformation-law is:

$$\mathfrak{w}\,(P, P') \rightarrow \mathfrak{w}\,(P, P')\,\triangle\,(P)^{-1}\,\triangle\,(P')^{-1}.$$

[2]) As only the symmetrical parts of $\gamma_{ij'}$ and $\gamma_{ijk'l'}$ occur in W, this supposition is necessary for a variational principle of this type.

[3]) $\partial_{l'i}$ is an abbreviation for $\partial_{l'}\,\partial_i$.

[4]) As the integrals in (17) are to be extended over the whole space-time, this condition seems acceptable as far as the infinity of *space* is concerned (if we accept a *closed* space, this condition disappears at all), but not for $t' \rightarrow \pm\infty$. As we have

$$\int^4 \partial_{l'}\,(\mathfrak{H}^{k'l'}\,\gamma_{jk'})\,d\Sigma' = \int^3 \mathfrak{H}^{k'l'}\,\gamma_{jk'}\,d\mathfrak{S}_{l'}$$

we need therefore only consider the integration over $d\mathfrak{S} = d\mathfrak{S}_4$, for $t' \rightarrow \pm\infty$, hence

$$\int^3 \partial_{[i}\,\mathfrak{H}^{k'l'}\,\gamma_{j]k'}\,d\,\mathfrak{S}_{l'} = \partial_{[i}\left[\int^3 \gamma_{j]k'}\,\mathfrak{H}^{k'4'}\,d\,\mathfrak{S}_{4'}\right]_{t'=-\infty}^{t'=+\infty} \stackrel{*}{=}$$

$$\stackrel{*}{=} \partial_{[i}\left[g_{j]a'}\int^3 \mathfrak{D}^{a'}\,.\,\frac{1}{4\pi r}\,\delta\,(ct - ct' - r)\,.\,r^2\,sin\,\varphi\,dr\,d\varphi\,d\vartheta\right]_{t'=-\infty}^{t'=+\infty} \stackrel{*}{=}$$

$$\stackrel{*}{=} \partial_{[i}\left[g_{j]a'}\int \overline{\mathfrak{D}}^{a'}(r)\,\delta\,(ct - ct' - r)\,r\,dr\right]_{t'=-\infty}^{t'=+\infty} \stackrel{*}{=} \partial_{[i}\left[g_{j]a'}\,\overline{\mathfrak{D}}^{a'}(r)\,.\,r\right]_{r=\infty},$$

where $\overline{\mathfrak{D}}^{a'}(r)$ is the mean value of $\mathfrak{D}^{a'}$ over the surface of a sphere with radius r, hence het supposition that the boundary integrals for $t' \rightarrow \pm\infty$ vanish is also acceptable.

suppose these properties to be sufficiently general, so that they allow *every* possible electromagnetical field, (18) must hold for arbitrary values of $\mathfrak{H}^{k'l'}$, i. e. the integrand must vanish. Hence we find:

$$\left.\begin{aligned} \gamma_{ijk'l'} &= 4\,\partial_{[i[k'}\,\gamma_{j]l']} = {}^{1)} \\ &= \partial_{ik'}\,\gamma_{jl'} - \partial_{jk'}\,\gamma_{il'} + \partial_{jl'}\,\gamma_{ik'} - \partial_{jk'}\,\gamma_{il'} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19)$$

We may consider equations (10), (12) as integral equations of the first kind with the kernels $\gamma_{ijk'l'}$ and $\gamma_{ik'}$ respectively. If we call the resolving kernels $\Gamma^{k'l'm''n''}$ and $\Gamma^{k'm''}$ (allowing these, however, also to be "symbolic" functions, i. e. operators) so that

$$\left.\begin{aligned} &\int^{4} \gamma_{ik'}\,\Gamma^{k'm''}\,d\Sigma' = \int^{4} \Gamma^{m''k'}\,\gamma_{k'i}\,d\Sigma' = \triangle_i^{m''} \\ &\tfrac{1}{2}\int^{4} \gamma_{ijk'l'}\,\Gamma^{k'l'm''n''}\,d\Sigma' = \tfrac{1}{2}\int^{4} \Gamma^{m''n''k'l'}\,\gamma_{k'l'ij}\,d\Sigma' = \triangle_{i\ j}^{m''n''} \end{aligned}\right\} \quad . \quad (20)$$

where

$$\triangle_i^{m''} \stackrel{*}{=} A_i^m\,\delta(P,P'') \stackrel{*}{=} A_i^m\,\delta(x-x'')\,\delta(y-y'')\,\delta(z-z'')\,.\,\delta(ct-ct'') \quad (21)$$

$$\triangle_{i\ j}^{m''n''} = 2\,A_i^{[m''}\,A_j^{n'']}\,\delta(P,P') \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

the solution of (10), (12) may be written

$$\mathfrak{H}^{ij} = \tfrac{1}{2}\int^{4} \Gamma^{ijk'l'}\,F_{k'l'}\,d\Sigma' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$\mathfrak{s}^{i} = \int^{4} \Gamma^{ik'}\,\varphi_{k'}\,d\Sigma' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

With the identity (19) corresponds the identity

$$\Gamma^{ik'} = \partial_{jl'}\,\Gamma^{ijk'l'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and the action-density (16) becomes

$$2\,\mathfrak{w} = \tfrac{1}{4}\,\Gamma^{ijk'l'}\,F_{ij}\,F_{k'l'} - \Gamma^{ik'}\,\varphi_i\,\varphi_{k'} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Again by variation of W under assumption of the *second* set of (1) and of the symmetry

$$\Gamma^{ijk'l'}(P,P') = \Gamma^{k'l'ij}(P',P), \quad \Gamma^{ik'}(P,P') = \Gamma^{k'i}(P',P) \quad . \quad . \quad (27)$$

we get the *first* set of (1).

Hence we see that also if we use the *general* linking equations (10), (12), we can obtain MAXWELL's equations *in two ways* from a variational principle, viz. by varying either the φ_i or the $\mathfrak{H}^{ij}$ in (15) with (16) or (26).

1) In the symbols [[...]] the first opening bracket belongs to the first closing bracket, etc.