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**Mathematics.** — *Electromagnetism, independent of metrical geometry. 4. Momentum and energy; waves.* By D. VAN DANTZIG [1]). (Communicated by Prof. J. A. SCHOUTEN).

(Communicated at the meeting of November 24, 1934).

## § 1. *Introduction.*

A well-known difficulty [2]) in general relativity is: *which is the relation existing between the density* $\mathfrak{S}_i^{\cdot h}$ *of stress, momentum and energy, and the vector* $p_i$ *of momentum and energy?* This relation should be of a form somewhat like $cp_i = \int \mathfrak{S}_i^{\cdot 4}\, d\,\mathfrak{S}$ or $cp_i = \int \mathfrak{S}_i^{\cdot j}\, d\,\mathfrak{S}_j$, integrated over a three-dimensional space. Neither of these integrals, however, is invariant, for integration of a vector $\mathfrak{S}_i^{\cdot j}\, d\,\mathfrak{S}_j$ presumes addition of vectors in different points, a process which has no meaning, except in Euclidian spaces. Moreover, and apart from the special way in which $p_i$ is to be expressed by $\mathfrak{S}_i^{\cdot h}$, *a vector like* $p_i$ *cannot exist* [2a]) *at all in a consequent field-theory.* Indeed: a vector must always be applied to a certain point [3]),

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[1]) Cf. D. VAN DANTZIG, Electromagnetism, independent of metrical geometry, 1. The foundations, 2. Variational principles and further generalisation of the theory, 3. Mass and motion, these Proceedings, **37** (1934) 521—525; 526—531; 643—652, abbreviated as EM 1, 2, 3.

[2]) Cf. W. PAULI, A. S. EDDINGTON, l.c. EM 1, 523, note [3]).

[2a]) With "exist" we mean here: exist, independent of any choice of coordinates.

[3]) For the transformation-formulae $p_i \rightarrow p_{i'} = A^j_{i'}\, p_j$ are only well-defined if it is known, in which point the $A^j_{i'} = \partial_{i'}\, \xi^j$ are to be calculated.

whereas $p_i$ should be the total amount of momentum and energy, contained in a *finite volume* and therefore would belong to this *volume* instead of to a definite point. Hence *total* (= integrated) *momentum and energy of a field is not a well-defined notion.*[4])

Nevertheless light-waves exist, in general relativity also, and so does the wave-vector $\varkappa_i$. *But how then can* PLANCK'*s law* $E = h\nu$ *and* EINSTEIN'*s extension* $p_i = h\varkappa_i$ *have a meaning?* It is this problem, which will be solved in this paper.

The result is: *for a plane-polarized monochromatic wave*[5]) $\mathfrak{S}_i^{\cdot h}$ *degenerates into a product*

$$\mathfrak{S}_i^{\cdot h} \sim \varkappa_i \mathfrak{W}^h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$\mathfrak{W}^h = \mathfrak{H}^{hj}\varphi_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is the "action-current", used in EM 2 (8). Hence, instead of *first* integrating $\mathfrak{S}_i^{\cdot h}$ and *then* dividing by $\varkappa_i$ (which should give $hc$), we must *first* divide by $\varkappa_i$ (which is possible because of (1)) and *then* integrate, so that the law of PLANCK-EINSTEIN takes the invariant form $\int \mathfrak{W}^h d\mathfrak{S}_h = hc$. In the special case of *plane* waves, where the $\varkappa_i$ are *constant numbers*, both forms are equivalent.

Moreover, it will be shown (§ 4) that relations (1) and (2) hold not only for *light-waves*, i. e. waves without current, but also for *waves with current*[6]) (Cf. § 2), supposed that the latter is "vibrating" also, and that $\mathfrak{S}_i^{\cdot h}$ is replaced by a generalized quantity $\mathfrak{T}_i^{\cdot h}$, in which terms containing the current and the potentials are included (§ 3). Aside $\mathfrak{s}^i = 0$, i. e. light-waves (Cf. § 5), the other special cases $F_{ij} = 0$ and $\mathfrak{h} = \frac{1}{2} H_{ij}\mathfrak{H}^{ij} = 0$ are remarkable. In these latter cases $\mathfrak{W}^i$ is proportional with the electric current $\mathfrak{s}^i$, whereas in the former case $\mathfrak{W}^i$ is because of (1) $^\nu/_c$ times the Pointing-vector, together with the energy density. Moreover, $\mathfrak{W}^i$ satisfies the "equation of continuity" $\partial_i \mathfrak{W}^i = 0$ for *arbitrary* single waves as a consequence of the equation of motion (Cf. EM 3, § 4). Apparently $\mathfrak{W}^i$ can take entirely the place (but for a factor $^\nu/_c$) of the energy-current;

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[4]) Contrary to the *density* of momentum and energy $\mathfrak{S}_i^{\cdot 4}$ and also to the momentum and energy of a material *point*.

[5]) Shortly: a *single* wave. The wave-front need *not* be plane.

[6]) We do not ask here, what is the exact physical meaning of such waves. It may be sufficient that they are possible solutions of MAXWELL's equations. For a particle in rest, i.e. $\mathfrak{s}^a \underset{*}{=} 0$, the waves are stationary, i.e. $\varkappa_4 \underset{*}{=} 0$.

also for arbitrary single waves. It seems that in the general case, where $\mathfrak{W}^i$ and $\mathfrak{S}^i$ are not proportional, intimate connections with the phenomenon of spin exist.

Finally it will be shown in § 8 that not only MAXWELL's equations, but also the equation of motion and the wave-equations can be brought into a very simple form, invariant under arbitrary transformations in *five* coordinates, but for the condition that the physical quantities are independent of the fifth coordinate. It cannot be decided yet, if this invariance has another than a merely formal meaning.

## § 2. *General single waves.*

A single (i. e. plane-polarized and monochromatic) wave is usually represented by equations of the form

$$\varphi_i = \overset{\circ}{\varphi}_i \sin \Phi, \text{ [7]} \qquad (3)$$

where the phase $\Phi$ is a linear form in the coordinates $\xi^i$ if the waves are plane, whereas the "amplitudes" $\overset{\circ}{\varphi}_i$ are constant numbers. Both conditions can only hold for special (e. g. CARTESian) coordinates. We will drop the first condition altogether, i. e. we allow $\Phi$ to be an *arbitrary* function of the coordinates. Then the wave-vector is

$$\varkappa_i = \partial_i \Phi, \qquad (4)$$

so that $\mathbf{k} \overset{*}{=} (\varkappa_1, \varkappa_2, \varkappa_3)$ [8] is the ordinary (three-dimensional) wave-vector, $\varkappa_4 \overset{*}{=} -\nu/c = \partial \Phi / \partial ct$. [9] The condition of constancy of the $\overset{\circ}{\varphi}_i$ cannot be omitted entirely, because (3) would become trivial (e. g. with $\Phi =$ constant). We replace it by

$$\partial_{[i} \overset{\circ}{\varphi}_{j]} = 0, \qquad (5)$$

which evidently is satisfied if the $\overset{\circ}{\varphi}_i$ are constant, but is more general and independent of the choice of coordinates. From MAXWELL's equation

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[7]) $h, \ldots, m = 1, 2, 3, 4$; $a, \ldots, g = 1, 2, 3$.

[8]) In order to compare easily our results with the usual ones, we split up some equations into their spacelike and timelike parts with respect to Cartesian coordinates. The symbol $\overset{*}{=}$ denotes that the equations are not generally invariant. In space the ordinary vector-symbolism will be used. Vectors will be denoted by fat letters. We must, however, keep in mind that **E**, **H**, **A**, **k** are really covariant vectors, i.e. pairs of *planes*, whereas **B**, **D**, **I**, **W** are covariant *bivectors*, i.e. tubes (or contravariant vector densities, i.e. ordinary vectors per volume).

[9]) We measure phases in *radians*, not in circonferences. Hence our $\nu$ is usually called $2\pi\nu$. We use also DIRAC's $h$, i. e. PLANCK's $h$, divided by $2\pi$.

$F_{ij} = 2\, \partial_{[i}\, \varphi_{j]}$ (Cf. EM 2 (1) II) and (3) follows $F_{ij} = 2\, (\partial_{[i}\, \overset{\circ}{\varphi}_{j]}\, sin\, \Phi +$ $+\, 2\, \varkappa_{[i}\, \overset{\circ}{\varphi}_{j]}\, cos\, \Phi$. Hence (5) expresses the condition that the bivector $F_{ij}$ is entirely a consequence of the *wave-character* of the field only.

In a part of space-time, where is nothing but light, we have $\mathfrak{s}^i = 0$. It seems however natural, because of the fargoing analogy of MAXWELL's equation $\mathfrak{s}^i = \partial_j\, \mathfrak{H}^{ij}$ (Cf. EM 2 (1) I) with II (see above), to consider also the more general case, where $\mathfrak{s}^i$ and $\mathfrak{H}^{ij}$ shall also be "waving" functions:

$$\mathfrak{H}^{ij} = \overset{\circ}{\mathfrak{H}}{}^{ij}\, sin\, \Phi \quad \ldots\ldots \quad (6)$$

where the condition analogous to (5) will be

$$\partial_j\, \overset{\circ}{\mathfrak{H}}{}^{ij} = 0 \quad \ldots\ldots \quad (7)$$

Hence such a generalized wave is characterized by

$$\left.\begin{array}{l} \varphi_i = \overset{\circ}{\varphi}_i\, sin\, \Phi\, ;\ \mathfrak{H}^{ij} = \overset{\circ}{\mathfrak{H}}{}^{ij}\, sin\, \Phi\, ; \\ F_{ij} = \overset{\circ}{F}_{ij}\, cos\, \Phi\, ;\ \mathfrak{s}^i = \overset{\circ}{\mathfrak{s}}{}^i\, cos\, \Phi\, ; \end{array}\right\} \quad \ldots\ldots \quad (8)$$

$$\boxed{F_{ij} \sim 2\, \varkappa_{[i}\, \varphi_{j]} \quad ; \quad \mathfrak{s}^i \sim \mathfrak{H}^{ij}\, \varkappa_j\ ^{10)}.} \quad \ldots\ldots \quad (9)$$

Splitting up (9) into space- and time-components, we get

$$\left.\begin{array}{l} \mathbf{B} \overset{*}{\sim} \mathbf{k} \times \mathbf{A} \quad ; \quad \mathbf{E} \overset{*}{\sim} -\, \mathbf{k}\, \varphi + {}^{\nu}/_{c}\, .\, \mathbf{A} \\ {}^{1}/_{c}\, \mathbf{I} \overset{*}{\sim} \mathbf{k} \times \mathbf{H} + {}^{\nu}/_{c}\, \mathbf{D} \quad ; \quad \varrho \overset{*}{\sim} \mathbf{k}\, .\, \mathbf{D} \end{array}\right\} \quad \ldots\ldots \quad (10)$$

Hence the direction of **B** is parallel with the intersection of **E** and the wave-front **k**. [11])

The equation of motion $F_{ij}\, \mathfrak{s}^j = 0$ (Cf. EM 3 (36)) becomes here (by exclusion of the trivial case $\varkappa_i = 0$, i. e. $\Phi =$ constant) by means of (9):

$$\varphi_j\, \mathfrak{H}^{jk}\, \varkappa_k = 0 \quad \ldots\ldots \quad (11)$$

[10]) We use the symbol $\sim$ to indicate that only the *amplitudes* of both members are equal. The phases in both members of (10) differ by $\pi/2$.

[11]) In the metrical theory the relations $\mathbf{E}\, .\, \mathbf{B} = 0$ and $\mathbf{k}\, .\, \mathbf{B} = 0$ would be interpreted by saying that **E** and **k** (i.e. the contravariant vectors orthogonal to the planes of **E** and of **k** respectively) were orthogonal to **B**. Scalar products are built with $+$-signs.

Now the "action-current" (2) becomes here

$$\mathfrak{W}^i = \overset{\circ}{\mathfrak{H}}{}^{ij} \overset{\circ}{\varphi}_j \sin^2 \Phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

or, if $\mathbf{W} \overset{*}{=} (\mathfrak{W}^1, \mathfrak{W}^2, \mathfrak{W}^3)$ and $\Omega \overset{*}{=} \mathfrak{W}^4$ are the space- and time-components of $\mathfrak{W}^i$:

$$\mathbf{W} \overset{*}{=} \mathbf{A} \times \mathbf{H} + \varphi \mathbf{D} \quad ; \quad \Omega \overset{*}{=} \mathbf{A} . \mathbf{D} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

From (12) follows by (5) and (7):

$$\partial_i \mathfrak{W}^i = 2\, \varkappa_i \overset{\circ}{\mathfrak{H}}{}^{ij} \overset{\circ}{\varphi}_j \sin \Phi \cos \Phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Hence the equation of motion (11) takes the form of a *conservation-law*:

$$\partial_i \mathfrak{W}^i = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

We shall see later (§ 5, 6) that (15) is the invariant form of the law of conservation of momentum and energy [12]). It expresses the fact *that the integral*

$$w = \int \mathfrak{W}^i \, d\,\mathfrak{S}_i \, , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

*extended over any space-like section through space-time, is constant.* This holds also if the section is taken through a *part* of space-time, bounded by a three-dimensional tube (e.g. a moving box), such that at its boundary

$$\mathfrak{W}^i \, d\,\mathfrak{S}_i = \mathfrak{H}^{ij} \, \varphi_j \, d\,\mathfrak{S}_i = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This condition will surely be satisfied if $\mathfrak{H}^{ij}\, d\mathfrak{S}_j = 0$. Then we have also $\mathfrak{s}^i \; d\,\mathfrak{S}_i = 0$, i. e. the walls of the box are *impenetrable for charges* [13]). Also (17) will be satisfied if $\varphi_{[i} \, d\,\mathfrak{S}_{j]} = 0$. Then we have also $\mathfrak{F}^{ij} \, d\,\mathfrak{S}_j = 0$, [14]) which means in the case of light-waves that the walls are *perfect mirrors* [15]). Generally we may interpret (17) as *impenetrability of the boundary against energy.*

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[12]) It is remarkable that also the equation of motion of *point*-mechanics (EM3 (14)) is equivalent with the conservation-law EM3 (18).

[13]) However, the condition $\mathfrak{s}^i \; d\,\mathfrak{S}_i = 0$ is *not* sufficient, except if $\mathfrak{W}^i$ and $\mathfrak{s}^i$ are proportional.

[14]) Indices are always raised and lowered by means of the quadrivector-densities $\mathfrak{E}^{hijk}$, $\mathfrak{E}'_{hijk}$, introduced in EM1, p. 524. Cf. also my paper The fundamental equations of electromagnetism, independent of metrical geometry, Proceedings Cambridge Phil. Soc. **30** (1934) 421—427, cited as F E.

[15]) At the boundary both $\mathfrak{F}^{ij} \; d\,\mathfrak{S}_j$ and $\mathfrak{H}^{ij} \; d\,\mathfrak{S}_j$ must be the same for both media. In the walls of the box $\mathfrak{F}^{ij}$ must be zero. $\mathfrak{H}^{ij}$ must not, because of $\mathfrak{s}^i \neq 0$ in matter.

As $w/c$ has the dimensions of an action, the condition

$$\boxed{w = \int \mathfrak{W}^i \, d\mathfrak{S}_i = n\,h\,c^{16)}} \quad \ldots \ldots \quad (18)$$

is of *exactly* the same nature as the condition

$$s = \int \mathfrak{s}^i \, d\,\mathfrak{S}_i = n'\,e.\ ^{16)} \quad \ldots \ldots \ldots \quad (19)$$

Whereas (19) expresses the fact that the box contains a finite number of indivisible charges (electrons, etc.), (18) will express the fact that it contains a finite number of indivisible "atoms of action". Indeed, we shall see in § 4 that (18) is equivalent with the law of PLANCK-EINSTEIN.

§ 3. *Generalized stress, momentum and energy.*

The ordinary stress-momentum-energy-affinor-density is

$$\mathfrak{S}_i^{\cdot h} = + F_{ik}\,\mathfrak{H}^{hk} - \tfrac{1}{4} A_i^h\, F_{jk}\,\mathfrak{H}^{jk}\ ^{17)} \quad \ldots \ldots \quad (20)$$

Because of the analogy between MAXWELL's equations I and II it seems natural to extend it with terms $c_1\,\varphi_i\,\mathfrak{s}^h + c_2\,A_i^h\,\varphi_j\,\mathfrak{s}^j$. In order to obtain the right value for the energy-density (time-time-component) we must take $c_1 = 1$, $c_2 = -\tfrac{1}{2}$. Then instead of (20) we get:

$$\mathfrak{T}_i^{\cdot h} = F_{ik}\,\mathfrak{H}^{hk} + \varphi_i\,\mathfrak{s}^h - \tfrac{1}{2} A_i^h \left(\tfrac{1}{2} F_{jk}\,\mathfrak{H}^{jk} + \varphi_j\,\mathfrak{s}^j\right) \quad \ldots \quad (21)$$

In particular we have

$$-\mathfrak{T}_4^{\cdot 4} \stackrel{*}{=} \tfrac{1}{2}\left(\mathbf{E}\,.\,\mathbf{D} + \mathbf{B}\,.\,\mathbf{H} + {}^1/_c\,\mathbf{I}\,.\,\mathbf{A} + \varrho\,\varphi\right) \quad \ldots \quad (22)$$

Hence to the ordinary terms $-\mathfrak{S}_4^{\cdot 4}$, which belong to the energy contained in empty space come two new terms which constitute the part of the energy, contained in the charges, viz. the electrostatic energy-density $\tfrac{1}{2}\,\varrho\,\varphi$ and the density of the "self-potential" of the current $1/2\,c\,.\,\mathbf{I}\,.\,\mathbf{A}$. Also we have from (21):

$$\mathfrak{T}_a^{\cdot 4} \stackrel{*}{=} (\mathbf{D} \times \mathbf{B} + \varrho\,.\,\mathbf{A})_a \quad \ldots \ldots \quad (23)$$

Here ${}^1/_c\ \mathbf{D} \times \mathbf{B}$ and ${}^\varrho/_c\ \mathbf{A}$ are the well-known expressions for the density of the momentum of the field and of the charges respectively. In

$$-\mathfrak{T}_4^{\cdot a} \stackrel{*}{=} (\mathbf{E} \times \mathbf{H} + {}^\varphi/_c\,.\,\mathbf{I})^a \quad \ldots \ldots \ldots \quad (24)$$

16) $n$ and $n'$ are arbitrary integers.

17) $A_i^h$ is the unity-affinor; its components are $= 1$ for $i = h$, and $= 0$ for $i \neq h$.

the first term is POINTING's vector, whereas the second is $\varrho\,\varphi\,\mathbf{v}/c$[18]), i.e. the energy, transported by the charges. Finally, we see that the charges exert stresses, analogous to the MAXWELLian ones.

Because of the identity[19])

$$F_{ik}\,\mathfrak{H}^{hk} + H_{ik}\,\mathfrak{F}^{hk} = \tfrac{1}{2}\,A_i^h\,F_{jk}\,\mathfrak{H}^{jk} = \tfrac{1}{2}\,A_i^h\,H_{jk}\,\mathfrak{F}^{jk}, \quad . \quad . \quad . \quad (25)$$

holding for two *arbitrary* bivectors $F_{ij}, H_{ij}$[20]) we can write $\mathfrak{T}_i^{\cdot h}$ also in the form

$$\mathfrak{T}_i^{\cdot h} = -\,H_{ik}\,\mathfrak{F}^{hk} + \varphi_i\,\mathfrak{s}^h + \tfrac{1}{2}\,A_i^h\,(\tfrac{1}{2}\,H_{jk}\,\mathfrak{F}^{jk} - \varphi_j\,\mathfrak{s}^j). \quad . \quad . \quad (26)$$

where the coefficient of $A_i^h$ is equal to $\mathfrak{W}$.[21])

§ 4. *The law of* PLANCK-EINSTEIN.

Now let us consider a single wave as defined in § 2. Substitution of (8), (9) in (26) gives at once by (12):

$$\boxed{\mathfrak{T}_i^{\cdot h} \sim \varkappa_i\,\mathfrak{W}^h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Hence in the special case when the wave is *plane* (with respect to certain, e. g. CARTESian, coordinates), i. e. if the $\varkappa_i$ are constants, we can write the condition (18) in the form

$$c\,p_i \overset{*}{=} \int \mathfrak{T}_i^{\cdot h}\,d\mathfrak{S}_h \overset{*}{=} \varkappa_i \int \mathfrak{W}^h\,d\mathfrak{S}_h = n\,h\,\varkappa_i \quad . \quad . \quad . \quad (28)$$

which is the law of PLANCK-EINSTEIN. However, it is to be noted that $\mathfrak{W}^i = \overset{\circ}{\mathfrak{W}}{}^i \sin^2\Phi$, whereas $\mathfrak{T}_i^{\cdot h} = \overset{\circ}{\mathfrak{T}}{}_i^{\cdot h} \sin\Phi\cos\Phi$. Hence the general condition (18) may only be brought into the special form (28), if the phase-factor in $\mathfrak{T}_i^{\cdot h}$ is changed from $\sin\Phi\cos\Phi$ into $\sin^2\Phi$. This now is exactly what is done in the usual theory, where only the part $\mathfrak{S}_i^{\cdot h}$ of $\mathfrak{T}_i^{\cdot h}$ is considered and where the $\mathfrak{H}^{ij}$ are replaced by $\mathfrak{g}^{ik}\,\mathfrak{g}^{jl}\,F_{kl}$ which have with $\mathfrak{H}^{ij}$ a difference in phase of $\frac{1}{2}\pi$.

Hence we have proved that *the ordinary quantum-condition (28) is a special form of our condition (18)* which is of a much more general nature and, moreover, invariant under arbitrary transformations of coordinates.

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18) $\mathbf{v} = {}^1/_\varrho\,\mathbf{I}$ is the velocity of the charges.

19) Cf. 14). It is easily verified by means of the identities, given in FE, p. 425, footnote.

20) A special case of (25) is:

$$H_{ik}\,\mathfrak{H}^{hk} = \tfrac{1}{4}\,A_i^h\,H_{jk}\,\mathfrak{H}^{jk}$$

21) Cf. EM2 (7). In (21) the corresponding coefficient was equal to $-\,\mathfrak{H}$, Cf. EM3 (40).

§ 5. *The case* $\mathfrak{s}^i = 0$ *(light).*

From equations (9), (11) follow some identities which will often be used:

$$\left.\begin{array}{l} \mathfrak{F}^{ij}\varkappa_j = 0,\ \mathfrak{F}^{ij}\varphi_j = 0,\ \mathfrak{s}^i\varkappa_i = 0,\ \mathfrak{s}^i\varphi_i = 0,\ F_{ij}\mathfrak{H}^{ij} = 0, \\ \mathfrak{W}^i\varkappa_i = 0,\ \mathfrak{W}^i\varphi_i = 0,\ F_{ij}\mathfrak{W}^i = 0,\ F_{ij}\mathfrak{s}^i = 0,\ F_{ij}\mathfrak{F}^{hj} = 0. \end{array}\right\} \quad . \quad (29)$$

Now let us consider a light-wave, determined by

$$\mathfrak{s}^i \sim \mathfrak{H}^{ij}\varkappa_j = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

In particular we have by (1):

$$\mathfrak{W}^i \overset{*}{\sim} -{}^c/_v\, \mathfrak{S}_4^{\cdot i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

i. e. the action-current is proportional with the combination of the POINTING-vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, and the energy-density $\mathfrak{T} = -\mathfrak{S}_4^{\cdot 4}$. Hence in this case (15) is equivalent with the law of conservation of momentum and energy. By splitting up (30) and (31) we find with the aid of (9) and (11):

$$\left.\begin{array}{c} -{}^v/_c\,\mathbf{D} \overset{*}{\sim} \mathbf{k} \times \mathbf{H}, \quad \mathbf{k} \cdot \mathbf{D} \overset{*}{\sim} 0, \quad \mathbf{W} \overset{*}{\sim} -{}^c/_v\,\mathbf{S} = {}^c/_v\,\mathbf{E} \times \mathbf{H}, \\ \Omega \overset{*}{\sim} {}^c/_v\,\mathfrak{T} = -{}^v/_c\,\mathbf{k} \cdot (\mathbf{H} \times \mathbf{A}). \end{array}\right\} . \quad (32)$$

With respect to the geometrical representation, mentioned in [8]) we remark that the initial plane of **k** is the wave-front, the initial plane of **E** is the polarization-plane, the intersection of the initial planes of **E** and **H** is the light-ray **W**. The vibrating electric and magnetic vector which are usually considered are **B** and **D** (and not **B** and **E**, or **H** and **D**); their directions are contained in **k**. In the special case of empty space the three planes of **k**, **E** and **H** form an orthogonal system.

§ 6. *The cases* $F_{ij} = 0$ *and* $\mathfrak{h} = 0$.

A second special case is given by

$$F_{ij} = 2\,\varkappa_{[i}\,\varphi_{j]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Then a scalar $\lambda$ exists, such that

$$\varphi_i = \lambda\,\varkappa_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Then from (9) and (11) follows:

$$\mathfrak{W}^h = \lambda\,\mathfrak{s}^h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

i. e. the action-current is proportional with the electric current. As the latter in this case is proportional with the energy-current, (15) again is equivalent with the law of conservation of energy and momentum. Substitution of (35) in (18), (19) shows (taking the case $n = n' = 1$) that the mean value $\bar{\lambda}$ of $\lambda$ is

$$\bar{\lambda} = hc/e. \quad \ldots\ldots\ldots \quad (36)$$

Hence we see that

$$1/hc \,.\, \mathfrak{W}^h \sim 1/e \,.\, \mathfrak{s}^h \quad \ldots\ldots\ldots \quad (37)$$

*may be considered as the probability-current.* Note, however, that both sides of (37) differ by a phase-factor $\sin^2 \Phi / \cos \Phi$, the period of vibration of $\mathfrak{W}^h$ being twice that of $\mathfrak{s}^h$.

If, according to EM3 (33), $p_i = {}^e/_c \,.\, \overline{\varphi}_i$ is interpreted as the momentum and energy of the particle in the sense of *point*-mechanics, we find from (34), (36), supposing $\varkappa_i$ to be approximately constant:

$$p_i = h \,\bar{\varkappa}_i \text{ [22]} \quad \ldots\ldots\ldots \quad (38)$$

From (9), (11) follows by the aid of (25), (29):

$$2\, \mathfrak{s}^{[h}\, \mathfrak{W}^{i]} = \mathfrak{H}^{hj}\, \mathfrak{H}^{ik}\, F_{jk} = - \mathfrak{H}^{hj}\, \mathfrak{F}^{ik}\, H_{jk} = \mathfrak{h}\, \mathfrak{F}^{hi} \quad \ldots \quad (39)$$

where

$$\mathfrak{h} = -\tfrac{1}{4} H_{ij}\, \mathfrak{H}^{ij} \overset{*}{=} \mathbf{H} \,.\, \mathbf{D} \quad \ldots\ldots\ldots \quad (40)$$

Hence an equation of the form (35) holds not only in the case $F_{ij} = 0$, but also in the case

$$\mathfrak{h} = 0 \quad \ldots\ldots\ldots \quad (41)$$

(and in no other one). From (41) follows easily (Cf. [20]) and (9))

$$H_{ij}\, \mathfrak{s}^j = 0 \quad \ldots\ldots\ldots \quad (42)$$

Hence a contravariant vector $u^i$ exists, such that

$$\mathfrak{H}^{ij} = 2\, u^{[i}\, \mathfrak{s}^{j]} \quad \ldots\ldots\ldots \quad (43)$$

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[22]) For $F_{ij} \neq 0$ the $p_i$ of point-mechanics will *not* be proportional with $\varkappa_i$. This seems to contradict (28). However, it is to be noted that a *single* wave will *not* have the character of a material point, even if it bears charge, so that it is not to be expected that it will have analogous properties. The consistency of (28) and (38) shows that for $F_{ij} = 0$ this may be the case.

As we may without loss of generality assume $u^4 \overset{*}{=} 0$, (43) gives by splitting up:

$$\mathbf{H} \overset{*}{=} {}^1/_c\, \mathbf{u} \times \mathbf{l}\,; \qquad \mathbf{D} \overset{*}{=} -\varrho\, \mathbf{u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

i.e. $-\mathbf{D}$ and $\mathbf{H}$ have the character of an electric and magnetic moment [23]). Of course the conclusions (36), (37) drawn from (35) remain also valid in the case $\mathfrak{h} = 0$.

§ 7. *The general case.*

Now let us suppose $\mathfrak{s}^i \neq 0$, $F_{ij} \neq 0$, $\mathfrak{h} \neq 0$. Putting for some arbitrary value of $\lambda$

$$z^i = -\,{}^1/_{\mathfrak{h}} \,.\, (\mathfrak{W}^i - \lambda\, \mathfrak{s}^i)\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

we get from (39):

$$\mathfrak{F}^{hi} = 2\, z^{[h}\, \mathfrak{s}^{i]} \;{}^{24)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Without loss of generality we may assume $z^4 \overset{*}{=} 0$. By extending the integrals (18), (19) over a space $t =$ constant we then find $\int \mathfrak{h}\, z^i\, d\mathfrak{S}_i = 0$ hence $\bar{\lambda} = hc/e$, i.e. (36). Splitting up (46) we get

$$\mathbf{B} \overset{*}{=} \varrho\, \mathbf{z}, \qquad \mathbf{E} \overset{*}{=} {}^1/_c\, \mathbf{z} \times \mathbf{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Hence $\mathbf{B}$ has the character of an electric, $\mathbf{E}$ that of a magnetic moment (not inversely!) [23]). The vector $\mathbf{z}$ does not bear a phase-factor. From (39) follows also

$$\mathfrak{F}^{hi} = 2\, z^{[h}\, \mathfrak{W}^{i]}/\lambda \overset{*}{=} 2\, y^{[h}\, \mathfrak{T}_4^{i]} \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where $y^h \overset{*}{=} -\, {}^c/_{\nu\lambda} \,.\, z^h$, so that $\mathbf{B}$ and $\mathbf{E}$ have also the character of a static moment and of a moment of momentum respectively [23]).

Evidently the physical meaning of these waves, except in the cases $\mathfrak{s}^i = 0$ and $F_{ij} = 0$ is still very mysterious and rather doubtful.

§ 8. *Five-dimensional formalism.*

If we write

$$\left.\begin{array}{ll} \mathcal{F}_{ijk} = -\, s_{ijk}, & \mathcal{F}_{ij5} = F_{ij} \\ \mathcal{H}_{ij} = H_{ij}, & \mathcal{H}_{i5} = \varphi_i, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

---

[23]) I renounce here trying to interpret these relations and to connect them with the spin of electrons, as such an interpretation would be rather haphazard in this stadium.

[24]) An equation of the form (46) exists *always* as a consequence of the equation of motion $F_{ij}\, \mathfrak{s}^j = 0$. For $\mathfrak{h} = 0$, however, $z^i$ cannot be expressed in the form (45).

MAXWELL's equations EM2 (1 I, II) may be subsumed into one set, viz.

$$3\, \partial_{[\mu}\, \mathcal{H}_{\lambda\varkappa]} = \mathcal{F}_{\mu\lambda\varkappa} \quad \ldots \ldots \ldots \quad (50)$$

$\iota, \varkappa, \lambda, \mu, \ldots, = 1, 2, 3, 4, 5$), if, moreover, we put

$$\partial_5\, H_{ij} = 0 \quad \ldots \ldots \ldots \quad (51)$$

On the other hand the relation (50) between a bivector and a trivector in five-dimensional space, together with condition (51) leads to a set of equations of the MAXWELL-form. [25]) Raising and lowering indices with the five-vector-densities $\mathfrak{E}^{\iota\varkappa\lambda\mu\nu}$, $\mathfrak{E}'_{\iota\varkappa\lambda\mu\nu}$, defined by $\mathfrak{E}^{12345} = 1$, $\mathfrak{E}'_{12345} = 1$, we have

$$\left.\begin{array}{ll} \mathfrak{F}^{ij} = \mathfrak{F}^{ij}, & \mathfrak{F}^{i5} = -\, \mathfrak{s}^{i} \\ \mathfrak{H}^{ijk} = -\, \varphi^{ijk}, & \mathfrak{H}^{ij5} = \mathfrak{H}^{ij} \end{array}\right\} \quad \ldots \ldots \quad (52)$$

The equation of motion takes the very simple form

$$\mathfrak{F}^{[\varkappa\lambda}\, \mathfrak{F}^{\mu\nu]} = 0 \quad \ldots \ldots \ldots \quad (53)$$

or also $\mathfrak{F}^{\varkappa\lambda}\, \mathcal{F}_{\varkappa\lambda\mu} = 0$ (which is equivalent with (53)). Indeed (53) splits up into $F_{ij}\, \mathfrak{s}^{j} = 0$ which is the equation of motion, and $F_{[ij}\, F_{kl]} = 0$, which is a consequence of it.

The action-density is

$$\mathfrak{W} = \tfrac{1}{4}\, \mathcal{H}_{\lambda\mu}\, \mathfrak{F}^{\lambda\mu} = \mathfrak{W} \quad \ldots \ldots \ldots \quad (54)$$

The generalized affinor-density $\mathfrak{T}_{i}^{\cdot h}$ of stress, momentum and energy is the space-time part of

$$\mathcal{T}_{\lambda}^{\cdot\varkappa} = -\, \mathcal{H}_{\lambda\mu}\, \mathfrak{F}^{\varkappa\mu} + \mathcal{A}_{\lambda}^{\varkappa}\, \mathfrak{W} \quad \ldots \ldots \quad (55)$$

The components are

$$\left.\begin{array}{ll} \mathcal{T}_{i}^{\cdot h} = \mathfrak{T}_{i}^{\cdot h}, & \mathcal{T}_{5}^{\cdot h} = \mathfrak{F}^{hj}\, \varphi_{j}, \\ \mathcal{T}_{i}^{\cdot 5} = -\, H_{ij}\, \mathfrak{s}^{j}, & \mathcal{T}_{5}^{\cdot 5} = +\, \tfrac{1}{4}\, F_{ij}\, \mathfrak{H}^{ij} + \tfrac{1}{2}\, \varphi_{i}\, \mathfrak{s}^{i} = -\, \mathfrak{H}. \end{array}\right\} \quad \ldots \quad (56)$$

[25]) Another five-dimensional formalism which is possible is obtained by putting f. i.

$$\mathcal{F}_{ij} = F_{ij} \quad , \quad \mathcal{F}_{i5} = -\, \varphi_{i} \quad , \quad \partial_5\, F_{ij} = F_{ij}$$

$$\mathcal{H}_{ijk} = s_{ijk} \quad , \quad \mathcal{H}_{ij5} = H_{ij} \quad , \quad \partial_5\, s_{ijk} = -\, s_{ijk}.$$

Then MAXWELL's equations become

$$\partial_{[\nu}\, \mathcal{F}_{\mu\lambda]} = 0 \quad , \quad \partial_{[\nu}\, \mathcal{H}_{\mu\lambda\varkappa]} = 0.$$

In the case of waves, determined by

$$\left.\begin{array}{ll} \mathcal{H}_{\lambda\mu} = \overset{\circ}{\mathcal{H}}_{\lambda\mu} \sin \Phi, & \mathcal{F}_{\varkappa\lambda\mu} = \overset{\circ}{\mathcal{F}}_{\varkappa\lambda\mu} \cos \Phi \\ \partial_{[\nu} \overset{\circ}{\mathcal{H}}_{\mu\lambda]} = 0, & \partial_{[\nu} \overset{\circ}{\mathcal{F}}_{\mu\lambda\varkappa]} = 0 \end{array}\right\} \quad . \; . \; . \; . \quad (57$$

it becomes always

$$\mathcal{T}_{\lambda}^{\cdot\varkappa} = k_{\lambda} \mathfrak{W}^{\varkappa} \cot \Phi, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (58)$$

with

$$\left.\begin{array}{c} \mathfrak{W}^{\varkappa} = \tfrac{1}{4} \mathfrak{H}^{\varkappa\lambda\mu} \mathcal{H}_{\lambda\mu} \\ \mathfrak{W}^{i} = \mathfrak{W}^{i}, \quad \mathfrak{W}^{5} = \tfrac{1}{4} \mathfrak{H}^{ij} H_{ij} = -2\mathfrak{h} \end{array}\right\} \quad . \; . \; . \; . \; . \; . \quad (59)$$

$$k_{\lambda} = \partial_{\lambda} \Phi \qquad k_{i} = \varkappa_{i} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (60)$$

The theory treated in § 2—7 belongs to the special case $k_5 = 0$, in accordance with (51).

Though the physical meaning of the fifth coordinate is not yet quite clear [26]), the formal simplification to which it leads is striking.

[26]) Though of course it may be interpreted in one of the well-known ways. Cf. the references, given in EM1, p. 522, note [2]).

If the radius of the spheres is made 30 cm then the values of $f(\gamma)$ for various values of r are

| r | | $f(\gamma)$ |
|---|---|---|
| 0 | cm | 2.00000 |
| 0.1 | „ | 1.99476 |
| 0.5 | „ | 1.97384 |
| 1 | „ | 1.94764 |

It is easy to make the radius r of the pendular body smaller than 0.5 cm, but even when it has this value, neglecting r causes a difference of only

$$\frac{2.00000-1.97384}{2} \times 100 = 1.3\%.$$

If water is used at a temperature of 17° centigrade, then the force becomes

$$P = 30\,\pi\, f(\gamma) \times 0.07434 = 7.02\, f(\gamma) \text{ grammes.}$$

So for various values of the radius r the force becomes

| r | | P | |
|---|---|---|---|
| 0 | cm | 14.014 | grammes |
| 0.1 | „ | 14.003 | „ |
| 0.5 | „ | 13.856 | „ |
| 1 | „ | 13.672 | „ |