aus der Gleichheit des Homotopietypen die Uebereinstimmung der *topologischen* Typen folgern. Beispielweise: Sind zwei *geschlossene Mannigfaltigkeiten* vom gleichen Homotopietypus immer homöomorph? Aus der positiven Beantwortung der letzten Frage würde sich sofort die *Richtigkeit der* POINCARÉ*schen Vermutung* [18]) *ergeben*, denn nach dem obigen hat eine $n$-dimensionale Mannigfaltigkeit deren Fundamentalgruppe und Homologiegruppen bis auf die letzte leer sind, den Homotopietypus der $n$-Sphäre.

[18]) Vgl. D II S. 523.

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**Mathematics.** — *Electromagnetism, independent of metrical geometry. 5. Quantum-theoretical commutability-relations for light-waves.* By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN).

(Communicated at the meeting of December 21, 1935).

§ 1. *Introduction.*

In some previous papers [1]) MAXWELL's equations were brought into the form:

$$I \quad \partial_j \mathfrak{H}^{ij} = \mathfrak{s}^i ; \qquad II \quad 2\, \partial_{[j}\, \varphi_{i]} = F_{ji} ; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whereas between the quantities $\mathfrak{s}^i$ (current and charge), $\varphi_i$ (potentials), $F_{ij}$ (fieldbivector, composed of **E** and **B**), $\mathfrak{H}^{ij}$ (composed of **H** and **D**) exist the "linking equations" (Cf. [7]))

$$I \quad \varphi_i = \int \gamma_{ik'}\, \mathfrak{s}^{k'}\, d\Sigma' ; \qquad II \quad \mathfrak{H}^{ij} = \tfrac{1}{2} \int \Gamma^{ijk'l'}\, F_{k'l'}\, d\Sigma' \quad . \quad (2)$$

(where $d\Sigma'$ is the four-dimensional volume-element), solved by

$$I \quad \mathfrak{s}^i = \int \Gamma^{ik'}\, \varphi_{k'}\, d\Sigma' ; \qquad II \quad F_{ij} = \int \tfrac{1}{2}\, \gamma_{ijk'l'}\, \mathfrak{H}^{k'l'}\, d\Sigma' \quad . \quad . \quad (3)$$

with

$$I \quad \Gamma^{ik'} = \partial_j\, \partial_{l'}\, \Gamma^{ijk'l'} ; \qquad II \quad \gamma_{ijk'l'} = 4\, \partial_{[i}\, \partial_{[k'}\, \gamma_{j]l']} \quad . \quad . \quad . \quad (4)$$

It is the purpose of this paper to show that also the fundamental equations of *quantum*-electrodynamics, viz. the commutability-relations for light-waves, can be brought into our scheme. These relations, which were studied by DIRAC [2]), JORDAN and PAULI [3]), HEISENBERG and PAULI [4]),

[1]) The preceding papers under the same main-title appeared in these proceedings, viz. I The foundations, **37** (1934) 521—525; II Variational principles and further generalisation of the theory, ibidem 526—531; III Mass and motion, ibidem 643—652; IV Momentum and energy; waves, ibidem 825—836. They are quoted here as I, II, III, IV.

[2]) P. A. M. DIRAC, Proc. Royal Soc. **114** (1927) 243, 710; **136** (1932) 433.

[3]) P. JORDAN und W. PAULI, Zur Quantenelektrodynamik ladungsfreier Felder, ZS. f. Ph. **47** (1928) 151—173.

[4]) W. HEISENBERG und W. PAULI, Zur Quantendynamik der Wellenfelder, ZS. f. Ph. **56** (1929) 1—61; **59** (1930) 168—190. Cf. also L. ROSENFELD, ibidem **76** (1932) 729—734.

BOHR and ROSENFELD[5]), FOCK[6]) and several other authors, take in our theory the simplest conceivable form, viz.[7])

$$[\varphi_i, \varphi_{j'}] = \frac{hc}{i}(\gamma_{ij'} - \gamma_{j'i}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

from which the relations for the field-bivector

$$[F_{ij}, F_{k'l'}] = \frac{hc}{i}(\gamma_{ijk'l'} - \gamma_{k'l'ij}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

follow immediately by means of (1, II) and (4, II).

They are deduced in § 2 by a method wich is much simpler than the usual ones given until now. § 3 contains the usual metrical specialisation of the linking quantities and the proof that equations (6) are really equivalent with the well-known commutability-relations.

The deduction is based upon the fact, taken over from the metrical theory, that the $\gamma_{ik'}$ and $\gamma_{ijk'l'}$ are essentially *unsymmetrical*. Indeed, if (2, I) determines the *retarded* potentials (denoted by $\overset{R}{\varphi}_i$), determined by any given charges $\mathfrak{s}^l$, the *advanced* potentials $\overset{A}{\varphi}_i$ are determined by

$$\overset{A}{\varphi}_i = \int \mathfrak{s}^{k'} \gamma_{k'i} \, d\Sigma' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now in the case of stationary light-waves, enclosed in a chamber, the walls of which are perfect mirrors, $\mathfrak{s}^{k'}$ is zero in the interior of the chamber; the integrals therefore are extended over the (unknown) charges in the walls, which interact with the waves. Because $\overset{R}{\varphi}_i$ and $\overset{A}{\varphi}_i$ satisfy in the metrical theory the equations $\Box \overset{R}{\varphi}_i = s_i$, $\Box \overset{A}{\varphi}_i = s_i$ the differences

$$\varphi_i = \overset{R}{\varphi}_i - \overset{A}{\varphi}_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

satisfy the equation $\Box \varphi_i = 0$ and may therefore be considered as the potentials of the light-waves.

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[5]) N. BOHR und L. ROSENFELD, Zur Frage der Messbarkeit der elektromagnetischen Feldgrössen, Kgl. Danske Vidensk. Selskap. Math. fys. Medd. **12**. 8 (1933).

[6]) V. FOCK, Zur Quantenelektrodynamik, Ph. ZS. der Sovjetunion **6** (1934) 425—469. Cf. also V. FOCK and B. PODOLSKY, ibidem **1** (1932) 801—817; P. A. M. DIRAC, V. FOCK and B. PODOLSKY, ibidem **2** (1932) 468—479.

[7]) As usual $[u, v]$ means $uv - vu$. The $h$ we use is PLANCK's $h$ divided by $2\pi$. It is to be noted that $\gamma_{ij'} = \gamma_{ij'}(P, P')$ is a two-point quantity; in $\gamma_{j'i}$ not only the indices but also the world-points are interchanged: $\gamma_{j'i}(P', P)$.

[8]) Prof. E. CARTAN kindly drew my attention to his paper, Sur les variétés à connexion affine et la relativité généralisée, Ann. Ec. Norm. **41** (1924) where he already in 1924 made the remark, (Chapter V, N⁰. 81) that MAXWELL's equations are independent of any kind of metric.

Though all equations of §§ 1, 2 are independent, not only of the choice of the system of coördinates, but also of *metrics*, I do not consider this independence as the most important *advantage* of the theory, but rather as a guiding and heuristic principle, though at the other hand the fact that also quantum-electro-dynamics appear to be independent of metric and the astonishingly simple and general form of relations (5) (6) might be considered as a rather strong argument for our point of view. The main advantage, to my view, however is the fact that equations (5) (6) and their deduction are *independent of the actual form of the* $\gamma_{ik'}$, which in the metrical theory is based on COULOMB's law. Hence a possible correction on COULOMB's law and therefore on the expressions (15) for $\gamma_{ik'}$ would *not* disturb equations (5), (6), supposed of course that the other fundaments of electromagnetism could be maintained [9]). A further advantage of the theory might be the fact that the *differential* equations (1) play only a rather unimportant and formal role in our theory, whereas the relevant part of it is contained in the *integral* equations (2), (3) [10]). Indeed, in a possible transition from the continuous field-theory to a discontinuum-theory of space-time, the integral-equations retain their meaning, as the integrals can easily be considered as mathematical abstractions and simplifications from sums over a very large but finite number of particles (just like in thermodynamics); the corresponding transition from differential-equations to equations with finite differences, however, is always rather artificial.

§ 2. *The commutability-relations.*

Just as an ordinary affinor $h_{ij}$ can be split up into the sum of products of vectors $h_{ij} = \overset{h}{u}_i \underset{h}{v}_j$, the two-point-quantity $\gamma_{ik'}$ can be split up in the following way:

$$\gamma_{ik'}(P, P') = \overset{r}{\varphi}_i(P)\, \underset{r}{\varphi}_{k'}(P') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The index $r$ will run neither through a finite nor through an infinite discrete series of symbols, but through a (fourdimensional) manifold, which need not be specified here. The transvection over $r$ means an integration (eventually combined with a summation). $P$ and $P'$ are two arbitrary world-points in space-time. As $\gamma_{ik'}$ is not symmetrical $\overset{r}{\varphi}_i(P)$ and $\underset{r}{\varphi}_i(P)$ will be different (except eventually for special values of $r$).

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9) Cf. e.g. BORN's new theory which is a *metrical* theory, different from the classical one, but also reposing on the supposition of *local* dependence of $F_{ij}$ on $\mathfrak{H}^{ij}$. Cf. M. BORN and L. INFELD, Foundations of the new field theory, Proc. Royal Soc. A **144** (1934) 425—451.

10) The importance of integral-equations in quantum-dynamics was first pointed out by K. LANCZOS, Ueber eine feldmässige Darstellung der neuen Quantenmechanik, ZS. f. Ph. **35** (1926) 812—830.

Now if any field $\mathfrak{S}^i$ is given, it determines by (2, I) the retarded potentials

$$\overset{R}{\varphi_i} = \underset{r}{c}\,\overset{r}{\varphi_i} \quad , \quad \underset{r}{c} = \int \mathfrak{S}^{k'}\,\underset{r}{\varphi_{k'}}\, d\Sigma' \, . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

and the advanced potentials

$$\overset{A}{\varphi_i} = \underset{r}{\varphi_i}\,\overset{r}{c} \quad , \quad \overset{r}{c} = \int \overset{r}{\varphi_{k'}}\,\mathfrak{S}^{k'}\, d\Sigma' \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Passing to quantum-electrodynamics, we suppose the $\underset{r}{c}$ and the $\overset{r}{c}$ to be symbols for operators (e.g. matrices). It is natural to assume that the $\overset{r}{\varphi_i}$ and $\underset{r}{\varphi_i}$ which are characteristic functions of the nucleus $\gamma_{ik'}$ of the integral equation (2, I), are commutable between each other, as well as with the $\underset{r}{c}$ and $\overset{r}{c}$, and that the constants $\underset{r}{c}$, $\overset{r}{c}$ satisfy the relations

$$\left[\overset{r}{c}, \underset{s}{c}\right] = \frac{hc}{i}\,\overset{r}{\underset{s}{\delta}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

where $\overset{r}{\underset{s}{\delta}}$ is the general KRONECKER-DIRAC-symbol [11]), whereas the $\overset{r}{c}$ between each other, and the $\underset{r}{c}$ between each other are commutable.

Hence any two retarded potentials as well as any two advanced potentials are commutable; only a retarded and the *corresponding* advanced potential are non-commutable. This seems a very natural result, if we interpret our formulae in the following way: *The integration over r is a mathematical idealization of a summation over a very large but finite number of elementary fields, belonging to charges situated in the walls of the box or outside. To each "jump" belongs an ingoing and an outgoing field* $\underset{r}{\varphi_i}$ *and* $\overset{r}{\varphi_i}$ *respectively. The field in the box is a linear* [12]) *superposition of all these elementary fields, determined by (10), (11). The only cause of inaccuracy in the measurements of field-components lies in the fact that an in- and an outgoing field, belonging to one single "jump", are not commeasurable. This is plausible, as a measurement of the ingoing field would disturb the jump and therefore the outgoing field. Finally we see from (9) that the quantity* $\gamma_{ik'}$ *which lies at the bottom of metric is determined by the elementary fields, though not as a mean value (as was predicted in I) but as a sum.*

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[11]) P. JORDAN, Ueber eine neue Begründung der Quantenmechanik, I. ZS. f. Ph. **40** (1926), 809—839, II ibidem **44** (1927), 1—25.

[12]) At least approximately, i.e. at a sufficiently large distance from the walls of the box.

If now the stationary waves are defined by (8) we find at once by substitution of (10, 11, 12):

$$\left[\varphi_i\,,\,\varphi_{k'}\right]=\left[\underset{r}{c}\,\overset{r}{\dot{\varphi}}_i-\underset{r}{\varphi_i}\,\overset{r}{\dot{c}}\,,\,\underset{s}{c}\,\overset{s}{\dot{\varphi}}_{k'}-\underset{s}{\varphi_{k'}}\,\overset{s}{\dot{c}}\right]=-\left[\underset{r}{\varphi_i}\,\overset{r}{\dot{c}}\,,\,\underset{s}{c}\,\overset{s}{\dot{\varphi}}_{k'}\right]-$$

$$-\left[\underset{r}{c}\,\overset{r}{\dot{\varphi}}_i\,,\,\underset{s}{\varphi_{k'}}\,\overset{s}{\dot{c}}\right]=-\left[\overset{r}{\dot{c}}\,,\,\underset{s}{c}\right]\left(\underset{r}{\varphi_i}\,\overset{s}{\dot{\varphi}}_{k'}-\underset{r}{\varphi_{k'}}\,\overset{s}{\dot{\varphi}}_i\right)=$$

$$=-\frac{hc}{i}\left(\underset{r}{\varphi_i}\,\overset{r}{\dot{\varphi}}_{k'}-\underset{r}{\varphi_{k'}}\,\overset{r}{\dot{\varphi}}_i\right)=\frac{hc}{i}\left(\gamma_{ik'}-\gamma_{k'i}\right),$$

which is equation (5).

As the coordinates $\xi^h, \xi^{h'}$ of the world-points $P, P'$ are independent variables, we find by differentiation with respect to $\xi^j$ and alternation:

$$[F_{ij}, \varphi_{k'}]=\frac{hc}{i}\left(2\,\partial_{[i}\,\gamma_{j]k'}-2\,\partial_{[i}\,\gamma_{|k'|j]}\right) \quad . \quad . \quad . \quad . \quad (13)$$

and by repeating this process with respect to $\xi^{l'}$:

$$[F_{ij}, F_{k'l'}]=\frac{hc}{i}\left(4\,\partial_{[i}\,\partial_{[k'}\,\gamma_{j]l']}-4\,\partial_{[k'}\,\partial_{[i}\,\gamma_{l']j]}\right) \quad . \quad . \quad . \quad (14)$$

i.e. equation (6).

§ 3. *Metrical specialisation.*

Taking

$$\gamma_{ij'} \stackrel{*}{=} -\frac{1}{4\pi r}\,\delta\,(ct-ct'-r)\,g_{ij}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where $\delta\,(x)$ is DIRAC's function, and splitting up (5) into space and time we get:

$$[A_x, A_{x'}] \stackrel{*}{=} [A_y, A_{y'}] \stackrel{*}{=} [A_z, A_{z'}] \stackrel{*}{=} -[\varphi, \varphi'] \stackrel{*}{=} \frac{hc}{i}\,.\,\triangle\,(P, P'), \quad (16)$$

all other combinations being commutable. Here

$$\triangle\,(P, P') \stackrel{*}{=} \frac{1}{4\pi r}\left\{\delta\,(ct-ct'-r)-\delta\,(ct'-ct-r)\right\},$$

is the so-called relativistic delta-function. In the same way we get from (6) (or also from (16) by differentiation):

$$\left.\begin{aligned}
[E_x, E_{x'}] &\overset{*}{=} -\frac{hc}{i}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x'} - \frac{1}{c^2}\frac{\partial}{\partial t}\frac{\partial}{\partial t'}\right)\triangle(P, P') \\
[E_x, E_{y'}] &\overset{*}{=} -\frac{hc}{i}\frac{\partial}{\partial x}\frac{\partial}{\partial y'}\triangle(P, P') \\
[B_x, B_{x'}] &\overset{*}{=} +\frac{hc}{i}\left(\frac{\partial}{\partial y}\frac{\partial}{\partial y'} + \frac{\partial}{\partial z}\frac{\partial}{\partial z'}\right)\triangle(P, P') \\
[B_x, B_{y'}] &\overset{*}{=} -\frac{hc}{i}\frac{\partial}{\partial x}\frac{\partial}{\partial y'}\triangle(P, P') \\
[E_x, B_{y'}] &\overset{*}{=} -\frac{hc}{i}\frac{1}{c}\frac{\partial}{\partial t}\frac{\partial}{\partial z'}\triangle(P, P') \\
[E_x, B_{x'}] &\overset{*}{=} 0.
\end{aligned}\right\} \quad . \quad . \quad (17)$$

All other commutability-relations are obtained by permuting the coordinates and the two world-points.

The relations (17) are entirely equivalent with those which are usually given, with the only exception that usually $[B_x, B_{x'}]$ is taken to be equal to $[E_x, E_{x'}]$. Now it is easily seen that indeed the expressions for these commutators are equal, as we have

$$\left(-\frac{\partial^2}{\partial x\,\partial x'} - \frac{\partial^2}{\partial y\,\partial y'} - \frac{\partial^2}{\partial z\,\partial z'} + \frac{1}{c^2}\frac{\partial^2}{\partial t\,\partial t'}\right)\triangle(P, P') \overset{*}{=} 0 \quad . \quad (18)$$

Hence we see that from the *metrical* standpoint the relations (17) are equivalent with the usual ones. From our point of view, however, (17) or rather (6) must be considered as the preferable form which only in the special case of COULOMB-potentials leads to $[E_x, E_{x'}] = [B_x, B_{x'}]$, etc.