Die Teilkurve A der ungestörten Maxima (Figur 2) schliesst sich derjenigen der Minima genau an. Für die Schiefe der ungestörten Kurve findet man

$$\frac{M-m}{P} = 0.344.$$

Zum Schluss wurde die Differenzkurve $C = A - B$ gebildet, welche wieder symmetrisch verläuft. Das Minimum, zu $2^m.30$, fällt auf 2423106, also 6.5 Tage vor dem ungestörten Maximum; der Veränderliche erleidet beim Aufstieg eine Verfinsterung, welche ihn von 88 % seines Lichtes beraubt.

Leider habe ich nach Abschluss der Reduktionsarbeit erfahren, dass von 2427700 an sämtliche (31) Beobachtungen beim Eintragen in die Graphik um $+20^d$ verschoben sind. Dieser Fehler hat den Kurvenzug beim Maximum 7764 ein wenig geändert; überdies haben die Epochen vom Wendepunkt 7705 an negatieve Korrektionen bekommen, welche vom Maximum 7764 an den vollen Betrag von -20^d erreichten. Die paar letzten Epochen der Tabellen III und IV stimmen jetzt nicht genau mit der Figur 1 überein. Die mittlere Kurve brauchte nicht neu gebildet zu werden. Sämtliche Normalepochen bekamen aber Korrektionen von -1^d. Die Figur 2 ist noch mit den alten Epochen konstruiert; der Unterschied macht sich kaum bemerkbar.

Utrecht, Januar 1936.

Physics. — *Some remarks on the resolving power of the microscope measured with the "*GRAYSON*'s Rulings".* By P. H. VAN CITTERT. (Communicated by Prof. L. S. ORNSTEIN).

(Communicated at the meeting of January 25, 1936).

As is universally known ABBE has shown that a grating consisting of a great many very narrow parallel slits, separated by opaque bars, when illuminated parallel to the optical axis, is resolved by a microscope if the line-distance

$$\triangle \geqq \frac{\lambda}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $\lambda =$ the wave-length of the light used and $A =$ the sine of half the aperture 2α of the objective (if necessary multiplied by the refraction-index of the immersion liquid used.) The image is formed by the interference of the direct beam with at least both diffraction-beams of the first order.

With inclined illumination, however, the resolving power is twice as large, so that a grating with a line-distance of

$$\triangle \geqq \frac{\lambda}{2A} \quad \ldots \ldots \ldots \ldots \quad (2)$$

is still resolved. In this case the image is formed by the interference of the direct beam with only one of the diffraction-beams of the first order.

As now, when illuminating by wide-opened beams, so when using a condenser of sufficient aperture, under the different directions of illumination, the direction α is always present, it would be expected that in this case the limit of the resolving power is given by

$$\frac{2A\triangle}{\lambda} = 1.$$

In different textbooks it is indeed suggested that the resolving power of the microscope with a condenser is the same as in the case of inclined illumination, whereas in other textbooks, without any explanation, the resolving-power for the case of inclined illumination is given as the resolving power of the microscope. If, however, the ratio between the resolving powers with and without a condenser is measured by means of a grating-structure, the ratio 2 is never found, but always a smaller one.

As an example in table 1 the results of the measurement of the resolving

TABLE 1.

Without condenser.							
Obj.	N.A.	Number of resolved groups					
		Oc. 4	Oc. 5	Oc. 7	Oc. 10	Oc. 15	Theor.
5	?	—	—	1	1	1	?
8	0.2	1	1	1	2	2	1.8
10	0.3	1	2	2	3	3	2.7
20	0.4	3	3	4	4	4	3.6
40	0.65	5	6	6	6	6	5.9
60	0.9	7	7	8	8	8	8.2
90	0.9	7	8	8	8	8	8.2

power of a modern microscope by means of parallel light are tabulated. A GRAYSON's Test-plate, with 12 groups of parallel lines $\frac{1}{200}, \frac{1}{400}, \ldots \frac{1}{2400}$ mm apart respectively, was used. The indicated theoretical resolving power has been calculated for the wave-length of 5500 Å. The resolving powers observed by larger magnifications appear to be in good accordance with theory. Table 2, however, gives the values of the resolving powers measured

TABLE 2.

With condenser.						
Obj.	N.A.	Number of resolved groups.				
		Oc. 4	Oc. 5	Oc. 7	Oc. 10	Oc. 15
5	?	—	—	1	1	1
8	0.2	1	1	1	2	2
10	0.3	1	2	2	3	3
20	0.4	3	3	3	4	4
40	0.65	6	6	7	7	7
60	0.9	7	7	8	8	8
90	0.9	7	8	8	9	9

when using a condenser of large aperture. From this immediately appears that these are by no means twice as large as in the former case, but only about 1.2 times.

The cause of this discrepancy is at once clear, when it is considered that for the resolving of a grating-structure it is not only necessary for the observed image to show intensity-maxima and minima, but that is moreover necessary that the *visibility* V [1]) of the diffraction pattern should be large enough, to be observed with sufficient certainty. Now a visibility of about 10 % is necessary to observe a diffraction pattern clearly resolved whereas for the limit $\frac{2\triangle A}{\lambda} = 1$ the visibility of the diffraction pattern $= 0$, as in this case only the extreme beams can give a diffraction pattern: all other beams only cause a homogeneous illumination of the field of view. If the grating-constant becomes larger, the diffraction angles become smaller and so more beams contribute to the diffraction pattern: the visibility of the image increases. For the case treated by ABBE of an infinite grating with very narrow lines, observed with an objective with rectangular aperture, the visibility is plotted in fig. 2 against the line distance (—.— curve). In this case the visibility of 10 % is not reached for $\frac{2\triangle A}{\lambda} = 1$ but for $\frac{2\triangle A}{\lambda} = 1.05$.

In practice, however, it is much more unfavourable. In the first place the

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[1]) $V = \frac{I_1 - I_2}{I_1 + I_2}$, in which I_1 the intensity of a maximum and I_2 the intensity of a minimum.

aperture of the objective is not rectangular, but circular, through which it is the extreme beams forming the diffraction pattern that get less influence and in the second place the lines of the test-object are not infinitely narrow, but of finite width. This results in the diffraction beams of the first order being much less intensive than the direct beam through which the visibility of the diffraction pattern diminishes. Therefore in practice the theoretical value of the resolving power $\frac{2\triangle A}{\lambda} = 1$ cannot be expected to be found.

As the image formed when illuminating with wide-opened beams is approximately similar to that of a self-luminous object we will treat this latter case for the simplification of the calculations and compare this case with that of parallel illumination. In this case the illumination of the object is *completely non-coherent*, in the other case, however, *completely coherent*.

We will start from the well-known fact that an element $dX\,dY$ in the point $P(XY)$ of the object plane near the optical axis causes a vibration:

$$S \backsim \int\int d\xi\, d\eta \sin 2\pi \left[\frac{t}{T} - \frac{(x-X)\xi + (y-Y)\eta}{\lambda}\right] \quad . \quad . \quad (3)\,[1]$$

in that point Q of the image-plane which is conjugated to the point $p(xy)$ of the object-plane. In this ξ and η represent the sines of the angles which the projections of the line connecting the point P with the active element of the diffracting aperture on the X- and Y-plane make with the optical axis. The integrations must be extended over the whole diffracting aperture. If this aperture is circular, the integration gives the result:

$$S \backsim \frac{J_1\left(\frac{2\pi\varrho A}{\lambda}\right)}{\frac{2\pi\varrho A}{\lambda}}$$

in which $A =$ the numerical aperture of the objective and $\varrho =$ the distance pP. So the intensity-distribution is determined by the function

$$J = \frac{J_1^2(q)}{q^2}$$

which behaves like the diffraction-function $\frac{\sin^2 q}{q^2}$ of a right-angled aperture, with this difference, however, that the pattern is broader. The first zero-point is not $q = \pi$ but $q = 3.9$. If the object is a long self-luminous line

[1]) ABBE, Die Lehre von der Bildentstehung im Mikroskop, 1910, page 40, formula 29.

parallel to the Y axis the intensity-distribution is determined by the integral:

$$I_1 \backsim \int_{-Y_0}^{+Y_0} dY \frac{J_1^2 \left\{ \frac{2\pi A}{\lambda} \sqrt{(X-x)^2+(Y-y)^2} \right\}}{\frac{4\pi^2 A^2}{\lambda^2} \left\{ (X-x)^2+(Y-y)^2 \right\}}$$

in which Y_0 may be considered infinite compared with the extension of the diffraction pattern of a single point. Numerical integration gives a distribution I_1, which is constant in the Y-direction, and which in the X-direction behaves like a function, which is similar to the function $\frac{J_1^2(q)}{q^2}$, with this difference, however, that the minima are not $= 0$ and that they are nearer to the point $x = X$. The first minimum for instance is found at $q = 0.36$ instead of at $q = 0.39$.

Let us now consider the coherent case. In order to calculate the image of a line parallel to the Y axis, we must determine the integral

$$S \backsim \int_{-Y_0}^{+Y_0} dY \int\int d\xi \, d\eta \sin 2\pi \left\{ \frac{t}{T} - \frac{(x-X)\xi + (y-Y)\eta}{\lambda} \right\}.$$

Integration in regard to η gives

$$S \backsim \int_{-Y_0}^{+Y_0} dY \int_{-A}^{+A} d\xi \sin 2\pi \left\{ \frac{t}{T} - \frac{(x-X)\xi}{\lambda} \right\} \cdot \frac{\sin 2\pi \frac{(y-Y)\eta_0}{\lambda}}{2\pi \frac{(y-Y)}{\lambda}}$$

in which η_0 is a function of ξ. As Y_0 may be considered infinite, continued integration in regard to Y and ξ gives:

$$S \backsim \sin 2\pi \frac{t}{T} \cdot \frac{\sin \left(\frac{2\pi (x-X) A}{\lambda} \right)}{\frac{2\pi (x-X) A}{\lambda}} \quad \ldots \ldots \quad (4)$$

i.e. a vibration which is independent of the form of the aperture of the lens. So while in the non-coherent case the intensity of the diffraction-pattern is determined by the above-mentioned function $I_1(q)$, in the coherent case the amplitude is determined by the function $\frac{\sin q}{q}$, as would also be the case, if the aperture of the lens were not circular, but rectangular.

If we apply these results to the case of two, very narrow parallel lines $\triangle$ apart, we can immediately calculate the relation between the visibility of the diffraction-pattern and the line-distance. The result is plotted in fig. 1. In this figure AB represents the visibility curve for the non-coherent case and CD for the coherent one. It appears that in the first case the visibility 10 % is reached when $\frac{2\triangle A}{\lambda} = 1.1$ and in the latter case when $\frac{2\triangle A}{\lambda} = 1.4$ so that the ratio of the resolving powers is about 1.3. Moreover the

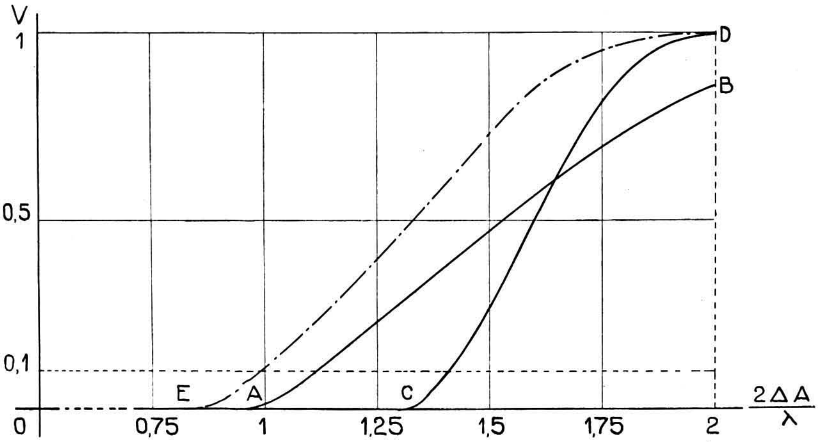

Fig. 1.

visibility curve for the non-coherent case is plotted for a rectangular aperture (ED), the coherent case of course gives the line CD again for a rectangular aperture. The ratio between the resolving powers is now about 1.4.

Let us now consider as an object a grating with comparatively many lines. In fig. 2 the visibility of the observed images are plotted for the non-coherent case against the line-distance for different values of the width b of the transparant slits. It is obvious that the visibility decreases as the value of b approaches that of $\triangle$. If the slits are broad compared with the opaque bars the resolving of the grating is very difficult. In the

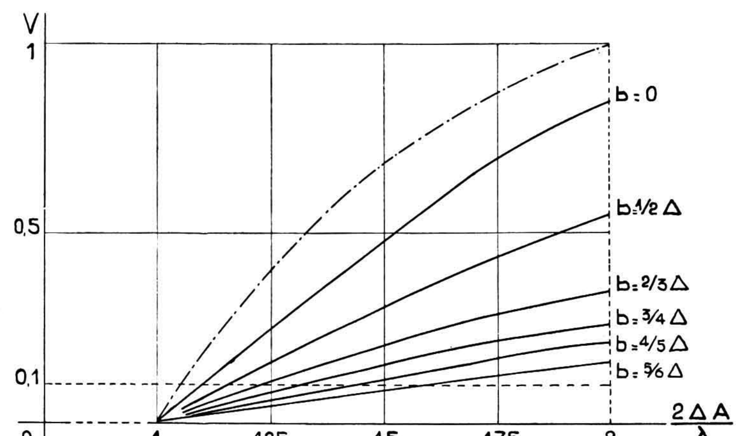

Fig. 2.

coherent case, however, the grating is not at all resolved if $\frac{2\triangle A}{\lambda} < 2$, but it is, if $\frac{2\triangle A}{\lambda} > 2$ and that always with sufficient visibility for the values of $\frac{b}{\triangle}$ considered above. For the image of an infinite grating is then determined by

$$I \backsim \left\{ 1 \pm 2 \frac{\sin \pi \frac{b}{\triangle}}{\pi \frac{b}{\triangle}} \cos 2\pi \frac{x}{\triangle} \right\}^2 \quad \ldots \ldots \quad (5)^{1)}$$

Now two cases are possible:

1°. $\frac{\sin \pi \frac{b}{\triangle}}{\pi \frac{b}{\triangle}} > \frac{1}{2}$, that is $b < 0.6\,\triangle$. Between the principal maxima faint secondary maxima are observed. Between these maxima the intensity $= 0$; so the visibility is 100 %. This is the case for $b = 0$ and $b = \frac{1}{2}\triangle$.

2°. $b > 0.6\,\triangle$. The secondary maxima have disappeared, the visibility is determined by

$$V = \frac{4r}{1 + 4r^2}$$

in which

$$r = \frac{\sin \pi \frac{b}{\triangle}}{\pi \frac{b}{\triangle}}.$$

This gives for $b = {}^2/_3\,\triangle$	$V = 98\,\%$
$b = {}^3/_4\,\triangle$	$V = 88\,\%$
$b = {}^4/_5\,\triangle$	$V = 76\,\%$
$b = {}^5/_6\,\triangle$	$V = 67\,\%$

So by coherent illumination the visibility will suddenly reach a more than sufficient visibility to make the resolving of the grating structure possible if $\frac{2\triangle A}{\lambda} = 2$. From fig. 2 it is obvious that the ratio of the resolving powers is always less than 2, e.g. if $b = \frac{4}{5}\,\triangle$ this ratio is only 1.37.

Let us now consider GRAYSON's Testplate. It consists of a number of groups of 10 or more lines. For the wider groups the ratio $\frac{b}{\triangle}$ can be

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1) ABBE, l.c. p. 104, f 95. The factor J_0^2 in this formula is constant for an infinite grating.

estimated with fairly great accuracy, because a microscope with great resolving power resolves these groups almost completely. The results show that for these groups $b = \frac{4}{5} \triangle$. For the narrower groups, however, this estimation is impossible and the ratio $\frac{b}{\triangle}$ is unknown.

In the non-coherent case the visibility curves for a grating of about 10 lines appear to be almost the same as those for an infinite one. In the coherent case formula (5) can be applied for the central part of the group [1]) provided that the whole diffraction beam of the first order passes through the objective. Now this diffraction beam is determined for a grating with p lines by

$$\frac{p-1}{p} < \frac{\triangle \xi}{\lambda} < \frac{p+1}{p}$$

and so the resolving will begin if $\frac{2 \triangle A}{\lambda} = \frac{2(p-1)}{p}$ and will reach its maximum if $\frac{2 \triangle A}{\lambda} = \frac{2(p+1)}{p}$ in other words for a group of for instance 10 lines the visibilities will not reach the above mentioned values discontinuously if $\frac{2 \triangle A}{\lambda} = 2$, but will rather rapidly increase continuously from $\frac{2 \triangle A}{\lambda} = 1.8$ till $\frac{2 \triangle A}{\lambda} = 2.2$. In first approximation the structure will yet be resolved if $\frac{2 \triangle A}{\lambda} = 2$. So, if also for the narrower groups $b = \frac{4}{5} \triangle$, a ratio between the resolving powers of about 1.3 can be expected, which is in good accordance with the experimental result 1.2.

Summing up we can conclude that by observation with parallel light GRAYSON's Testplate gives a resolving power that nearly completely agrees with the theoretical value $\frac{2 \triangle A}{\lambda} = 2$. When using a condenser, however, it is impossible to measure the double value of this resolving power, but a resolving power that is not only determined by the aperture of the objective but also by the nature of the object is measured. In order to compare the resolving power of a microscope with the theoretical value the use of parallel illumination is advisable [2]).

ABBE objected to the use of a testplate, because owing to him only the

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[1]) The factor J_0^2 has its maximum there. The transparant spaces between the different groups have very little influence on the visibility in the centre of a group.

[2]) Therefore this is the reason why in my book "Descriptive Catalogue of the Collection of Microscopes in charge of the Utrecht University Museum" only measurements with parallel illumination are recorded. If the measurements had been done with a condenser, all resolving powers would have been multiplied by a meaningless factor 1.2. Moreover it would not have been possible then to compare the measurements with GRAYSON's Testplate with those done with NOBERT's Testplate.

outer zones of the objective contributed to the formation of the image. However, when using a condenser the resolving power measured with GRAYSON's Testplate is $\frac{1.2}{2} = \frac{3}{5}$ times as small as the theoretical resolving power, the resulting refraction-angles will also be $\frac{3}{5}$ times as small. And as the co-operating diffractionbeams have moreover a finite aperture on account of the small number of grating slits, the whole objective will practically contribute to the formation of the image. So when using a condenser, ABBE's objection against measuring with a Testplate does not hold good.

Physics. — *Ueber das antiferromagnetische Austauschproblem bei tiefen Temperaturen.* Von L. HULTHÈN. (Communicated by Prof. H. A. KRAMERS).

(Communicated at the meeting of January 25, 1936).

Zusammenfassung. Das Benehmen eines Antiferromagnetikums bei tiefen Temperaturen wird mit Hilfe einer von KRAMERS und HELLER [1]) angegebenen Methode untersucht. Es ergibt sich, dass die Entropie (ohne Magnetfeld) wie T^3 geht, und die Suszeptibilität wie const. (1-const. T^2).

§ 1. *Problemstellung und Uebersicht.*

Bei tiefen Temperaturen hat die energetische Wechselwirkung der magnetischen Atome grossen Einfluss auf das Benehmen magnetischer Kristalle. Ist das sog. Austauschintegral, das die Wechselwirkung zwischen Nachbaratomen beschreibt, negativ, so kann nach HEISENBERG [2]) und BLOCH [3]) Ferromagnetismus auftreten; dies entspricht einer Wechselwirkungsenergie, die dem Cosinus des Winkels zwischen den Richtungen der betreffenden Elementarmagnete proportional ist und zwar mit einem negativen Wert des Koeffizienten. Den Fall, wo eine solche Wechselwirkung vorhanden ist, aber mit positivem Koeffizienten, wollen wir als *antiferromagnetisch* bezeichnen; er liefert ein mögliches Modell für magnetische Kristalle, die selbst bei den tiefsten Temperaturen paramagnetisch bleiben [4]).

In dieser Arbeit wird versucht, die Eigenschaften eines solchen Modells

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1) G. HELLER und H. A. KRAMERS, Proc. Royal Acad. Amsterdam, **37**, 378 (1934), im folgenden als HELLER—KRAMERS zitiert.

2) W. HEISENBERG, Zs. f. Phys. **49**, 619 (1928).

3) F. BLOCH, Zs f. Phys. **61**, 206 (1930).

4) Nach den Untersuchungen von BECQUEREL, DE HAAS und VAN DEN HANDEL (Physica **1**, 383 (1934)), VAN VLECK und HEBB (Phys. Rev. **46**, 17 (1934)) tritt eine solche Wechselwirkung beim CeF_3-Kristall auf. Allerdings wird bei wasserhaltigen Kristallen die direkte magnetische Wechselwirkung die wichtigste Rolle spielen. Vgl. H. A. KRAMERS, Physica **1**, 182 (1934).