Mathematics. — *Ricci-Calculus and Functional Analysis.* By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN).

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In this paper a short sketch is given of a general method for treating differential geometric problems in function-spaces. On a later occasion I hope to return upon the question in greater detail, and to give more references to existing literature. Here I might only mention a recent paper of KAWAGUCHI [1]), which contains a rather extensive list of older literature, and in particular to the papers of A. D. MICHAL, mentioned there. The main idea of our method consists in taking as contravariant functions (or vectorials) *absolutely additive set functions* instead of ordinary functions. This allows to unify and to generalize the different groups of functional transformations ("HILBERT"-, "FREDHOLM"- and "PICARD"-transformations) and to avoid the uninvariant δ-symbols, used by KAWAGUCHI.

§ 1. *Algebra in function-space.*

1. Let R be a separable topological space, $x, y, z, \ldots$ its elements ("points"), $X, Y, Z, \ldots$ its BOREL's subsets [2]); P the set of all real or complex numbers $\lambda, \mu, \nu, \ldots$; Λ, M, N the BOREL's subsets [2]) of P. We call *functions of the first kind* all *bounded measurable* (in the sense of BOREL) functions on R with values (called the *coordinates* or *components* of the function) in P. These functions are denoted by $f, g, h, \ldots$ The value which f takes in a point x of R is denoted by f_x; the $\limsup_{x \varepsilon R} |f_x|$ [3]) by $|f|$. A *function of the second kind* is an *absolutely additive set-function* F, which determines a real or complex number F^X with respect to each BOREL's subset X of R; the values F^X are called the *coordinates* or the *components* of F. We define $|F| = \lim\sup |F^{X_i}|$, where the X_i form any dissection of R into disjoint subsets. Further E^X denotes the "characteristic function of the first kind" of the set X, and

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1) A. KAWAGUCHI, Die Differentialgeometrie in den verschiedenen Funktionalräumen, I. Vektorialen und Tensorialen, Jn. Fac. of Sc. Hokkaido Imp. Univ. (1) 3, 43—106 (1935).

2) Or more generally a closed family of subsets, i. e. a family, which contains with each subset X also its complement $R - X$ and with each sequence of sets $X_1, X_2, \ldots$ also its intersection. BOREL's subsets are all sets, obtained from open sets by application of these two processes.

3) ε means "belongs to" or "is an element of".

E_x the "characteristic function of the second kind" of the point x[4]); both are defined by their coordinates

$$\mathsf{E}_x^X = \begin{cases} 1 \text{ if } x \varepsilon X \\ 0 \text{ if } x \varepsilon R - X \end{cases} \quad \ldots \ldots \ldots \quad (1)$$

2. If f and F are a function of the first and the second kind respectively, the limit $\lim \Sigma F^{X_i} f_{x_i}$ exists, where the X_i form a dissection of R, such that the variation of f_x on each X_i is $\leqq \varepsilon\ (\varepsilon > 0)$, and that x_i is an arbitrary point in X_i. Moreover this limit is independent of the choice of the X_i and the x_i. It is called the *transvection* of F and f, and denoted by $\int F^{dx} f_x$ [5]), or shortly by Ff.

We have $|Ff| \leqq |F|\,|f|$. In particular we note the relations

$$\left.\begin{aligned} \int \mathsf{E}_x^{dy} f_y &= f_x \\ \int F^{dy}\, \mathsf{E}_y^X &= F^X \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (2)$$

for any f or F.

3. A *functional* of the first (second) kind is a law, which determines a number $L[f]$ ($l[f]$) with respect to each function of the first (second) kind. After a well-known theorem of F. RIESZ, generalized by J. RADON, each *bounded linear*[5a]) functional of the first kind determines a function of the second kind, viz $L^X = L[\mathsf{E}^X]$, such that $L[f] = \int L^{dx} f_x$. An analogous theorem does not hold for functionals of the second kind.

4. Because of RIESZ' theorem each bounded linear transformation of functions of the first kind $f \rightarrow g = \mathsf{P} f$ determines a set of components P_x^X, which for any fixed X (or x) are the coordinates of a function of the first (or second) kind, such that $g_x = \int \mathsf{P}_x^{dy} f_y$, with a finite value of $|\mathsf{P}| = \lim \sup \dfrac{\left| \int\int F^{dx}\, \mathsf{P}_x^{dy} f_y \right|}{|F|\,|f|}$. Evidently the identical transformation has because of (2) the components E_x^X, defined by (1). Moreover P_x^X determines also a linear transformation for functions of

[4]) Hence E^X is for each given X a function of the first kind with values E_x^X ; E_x is for each given x a function of the second kind with the same values E_x^X.

[5]) Of course we could just as well drop the integral-sign and adopt the summation-convention.

[5a]) "Linear" in the strong sense: also for infinite sums $\Sigma f_{\nu x} = f_x$ and non-uniform convergence $L[f] = \Sigma L[f_\nu]$.

the second kind, viz $F \rightarrow G = F\mathsf{P}$, with $G^X = \int F^{dy} \mathsf{P}_y^X$. The product $\mathsf{R} = \mathsf{QP}$ of two transformations has the components $\mathsf{R}_x^X = \int \mathsf{Q}_x^{dy} \mathsf{P}_y^X$. We note the inequalities $|\mathsf{P}f| \leqq |\mathsf{P}||f|$; $|F\mathsf{P}| \leqq |F||\mathsf{P}|$; $|\mathsf{QP}| \leqq |\mathsf{Q}||\mathsf{P}|$. Q is called a lefthanded or righthanded or unique inverse of P, if $\mathsf{QP} = \mathsf{E}$ or $\mathsf{PQ} = \mathsf{E}$ or $\mathsf{QP} = \mathsf{PQ} = \mathsf{E}$ respectively.

5. If R' is a second topological space (with points x' etc. and subsets X' etc.), we can consider also linear transformations of functions on R into functions on R'. We denote the components of such a transformation by $\mathsf{E}_x^{X'}$ and suppose it to have a unique inverse with the components $\mathsf{E}_{x'}^X$:

$$\int \mathsf{E}_x^{dy'} \mathsf{E}_{y'}^X = \mathsf{E}_x^X; \qquad \int \mathsf{E}_{x'}^{dy} \mathsf{E}_y^{X'} = \mathsf{E}_{x'}^{X'}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Then we can consider the coordinates f_x (or F^X) of any function f (or F) of the first (or second) kind and those of its transformed, viz

$$f_{x'} = \int \mathsf{E}_{x'}^{dy} f_y \qquad (\text{or } F^{X'} = \int F^{dy} \mathsf{E}_y^{X'}), \quad . \quad . \quad . \quad (4)$$

as different sets of components of one single object, which is called a *covariant* (or *contravariant* respectively) [6] *vectorial*. Evidently the transvection is an invariant:

$$\int F^{dx'} f_{x'} = \int F^{dx} f_x .$$

Now we can define in the usual way general affinorials and tensorials, e.g. G^{XY}, h_{xy}, P_x^X etc., Hermitean tensorials etc. A particular tensorial [7] is obtained as soon as a volume-measurement in R is given, viz

$$G^{X,Y} = G^{Y,X} = M^{X.Y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where M^X is the volume ("measure") of X in the sens of LEBESGUE, and $X.Y$ is the intersection of X and Y. It is to be noted, that neither a Hermitean nor an ordinary tensorial can have properties analogous to those of a tensor of highest rank: for all G^{XY} and h_{xy} $\int G^{Xdy} h_{yz} \neq \mathsf{E}_z^X$.

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[6]) Evidently the words covariant and contravariant as well as the upper and lower suffixes could have been interchanged; they have been defined such that the relation with ordinary differential geometry which is discussed in Art. 15 becomes as simple as possible. Cf. [11]).

[7]) It can also be considered as a Hermitean tensorial.

6. An ordinary integral equation of the second kind has the form $(\mathsf{E}+\mathsf{K})f=g$ with $\mathsf{K}_x^X=\int G^{Xdy}K_{yx},\quad G^{X,Y}=M^{X.Y}$.

If K_{xy} is symmetrical and completely continuous, the proper functions form a complete orthogonal system. Writing φ_x^n instead of $\varphi_n(x)$ and ψ_n^X instead of $\int\limits_X \overline{\varphi}_n(x)\,dx$ we have

$$\int \psi_n^{dx}\varphi_x^m=\delta_n^m, \qquad (6)$$

$(m,\ n=1,\ 2,\ 3,\ldots)$. Evidently we can consider also general systems φ_x^n, ψ_m^X with the property (6) and with $\psi_n^X\neq\int\limits_X\overline{\varphi}_x^n\,dx$; these are usually called (without much reason) "bi-orthogonal systems". The development of functions of either kind (if possible at all) is given by

$$f_x=\varphi_x^n f_n,\qquad f_n=\int\psi_n^{dx}f_x, \qquad (7)$$

$$F^X=F^n\,\psi_n^X,\quad F^n=\int F^{dx}\,\varphi_x^n. \qquad (8)$$

Evidently the coefficients of the development f_n, F^n can be considered as a new kind of components or coordinates of the vectorials f and F, just like the f_x and F^X. The only difference is (apart from the irrelevant fact that the f_n, F^n form a countable, the f_x, F^X an uncountable set) that the latter are not independent, whereas the former are. This however is not so very important; for functions of the second kind we cannot find (in general at least) any independent coordinates at all, so that we are forced here to work always with superabundant coordinates.

If the number of functions φ_x^n, ψ_n^X is finite, we can also form the sum

$$\varphi_x^n\,\psi_n^X=\mathsf{D}_x^X; \qquad (9)$$

if it is infinite however the series (9) is generally divergent.

Finally we remark that it would be more consequent to write D_x^n, D_n^X instead of φ_x^n, ψ_n^X, as all these quantities are different components of the same geometric object, viz the projection of all functions of either kind on a definite linear subset.

Instead of a sequence of functions D_n^X (or ψ_n^X) we can also consider an arbitrary set of functions of the second kind D_ξ^X, where ξ runs through any topological space Σ. If Ξ runs through the BOREL's subsets of this space, the functions of the first kind D_x^η (or φ_x^η) have to be replaced by D_x^Ξ, and the relations (6), (7), (8), (9) become:

$$\int D_\xi^{dx} D_x^\Xi = D_\xi^\Xi = \begin{cases} 1 \text{ if } \xi \varepsilon \Xi \\ 0 \text{ if } \xi \varepsilon \Sigma - \Xi. \end{cases} \quad \ldots \ldots \quad (6')$$

$$f_x = \int D_x^{d\xi} f_\xi, \qquad f_\xi = \int D_\xi^{dx} f_x, \quad \ldots \ldots \quad (7')$$

$$F^X = \int F^{d\xi} D_\xi^X, \qquad F^\Xi = \int F^{dx} D_x^\Xi \quad \ldots \ldots \quad (8')$$

$$\int D_x^{d\xi} D_\xi^X = \mathsf{D}_x^X. \quad \ldots \ldots \ldots \quad (9')$$

Here indeed non-trivial cases exist, where the integral (9′) is convergent.

7. If for any linear operator P (which we suppose to be bounded and to have finite components P_x^X) the proper-value-problem can be solved (e.g. if P is Hermitean with respect to a positive definite tensorial), it can be written in the form[8]):

$$\mathsf{P} = \int \lambda\, \mathsf{E}_\mathsf{P}^{d\lambda}, \qquad \text{i. e. } \mathsf{P}_x^X = \int \lambda\, \mathsf{E}_{\mathsf{P}\,x}^{d\lambda\,X}, \quad \ldots \ldots \quad (10)$$

where λ runs through the set P of all real numbers, and the components $\mathsf{E}_\mathsf{P}^\Lambda$ of the identity satisfy the conditions

$$\mathsf{E}_\mathsf{P}^\Lambda E_\mathsf{P}^M = \mathsf{E}_\mathsf{P}^{\Lambda . M}, \quad \ldots \ldots \ldots \quad (11)$$

$$\int_{\lambda=-\infty}^{\lambda=+\infty} \mathsf{E}_\mathsf{P}^{d\lambda} = \mathsf{E} \quad \ldots \ldots \ldots \quad (12)$$

Then for any polynomial $\varphi(\mathsf{P})$ we have also $\varphi(\mathsf{P})_x^X = \int \varphi(\lambda)\, E_{\mathsf{P}\,x}^{d\lambda\,X}$, or, writing φ_P and φ_λ instead of φ (P) and $\varphi(\lambda)$:

$$\varphi_\mathsf{P} = \int \varphi_\lambda\, \mathsf{E}_\mathsf{P}^{d\lambda}, \qquad \text{i. e. } \varphi_{\mathsf{P}x}^{\;X} = \int \varphi_\lambda\, \mathsf{E}_{\mathsf{P}\,x}^{d\lambda\,X}. \quad \ldots \ldots \quad (13)$$

If φ is any measurable function of λ, (13) can be considered as the definition of φ_P. If we take in particular $\varphi = B^\Lambda$, where

$$B_\lambda^\Lambda = \begin{cases} 1 \text{ if } \lambda \varepsilon \Lambda, \\ 0 \text{ if } \lambda \varepsilon P - \Lambda, \end{cases} \quad \ldots \ldots \ldots \quad (14)$$

8) Cf. J. VON NEUMANN, Mathematische Grundlagen der Quantenmechanik, Berlin (1932) Ch. II, 6—9.

we obtain

$$\mathsf{B}_{\mathsf{P}}^{\Lambda} = \mathsf{E}_{\mathsf{P}}^{\Lambda}, \qquad \text{i. e. } \mathsf{E}_{\mathsf{P}x}^{\Lambda X} = (B_{\mathsf{P}}^{\Lambda})_x^X. \quad \ldots \ldots \quad (15)$$

Hence the $\mathsf{E}_{\mathsf{P}}^{\Lambda}$ are definite functions of P (viz characteristic functions). Evidently (13) is invariant if we perform on λ a transformation of the type considered in Art. 5. Then we obtain

$$\varphi_{\mathsf{P}} = \int \varphi_{\lambda'} E_{\mathsf{P}}^{d\lambda'} \quad \ldots \ldots \ldots \quad (16)$$

where λ' now runs through any topological space, which need not be a set of real numbers. Of course now the *form* of the functions φ_{P} and φ_{λ}, need no longer be the same, as it was in (13), at least for polynomials. At the other hand (16) can always be brought into the form (13) by means of LEBESQUE's definition of the integral, and is therefore not really more general than (13).

§ 2. *Analysis in function-space.*

8. Let $\Phi[f]$ be any functional of the first kind. The derivative of Φ with respect to a set X is defined [9]) as

$$\partial^X \Phi[f] = \left(\frac{\partial \Phi}{\partial f}\right)^X = \lim_{\varepsilon \to 0} \frac{\Phi[f + \varepsilon\, \mathsf{E}^X] - \Phi[f]}{\varepsilon} \quad \ldots \quad (17)$$

If this limit exists for some f and any X and is a bounded and continuous functional of f, it is (for a given f) a function of the second kind [10]). In that case the differential or "variation" $\delta\Phi$ of Φ under any variation δf, defined by $\delta\Phi = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \Phi[f + \varepsilon\delta f]$ is equal to

$$\delta\Phi = \int (\partial^{dx}\Phi)\, \delta f_x. \quad \ldots \ldots \ldots \quad (18)$$

For analytic functionals

$$\Phi[f] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int \Phi^{dx_1 \ldots dx_n} f_{x_1} \ldots f_{x_n}$$

with

$$\lim\, sup \frac{\left| \int \cdots \int \Phi^{dx_1 \ldots dx_n} \overset{1}{u}_{x_1} \ldots \overset{n}{u}_{x_n} \right|}{\left|\overset{1}{u}\right| \cdots \left|\overset{n}{u}\right|} \leqq M N^{-n}$$

9) Cf. D. VAN DANTZIG, La notion de dérivée d'une fonctionnelle, C. R. **201** (1935) 1008—1010, where the definition was given for functionals which need only be defined for all *continuous* functions.

10) Contrary to VOLTERRA's definition, our definition does *not* depend on the geometric structure of R.

and $|f| < N$ the derivative (17) exists always and we have

$$\Phi^{x_1 \ldots x_n} = (\partial^{x_1} \ldots . \partial^{x_n} \Phi [f]_{f=0} .$$

9. If $\varphi [F]$ is any functional of the second kind we can define the derivative with respect to a point x by

$$\partial_x \varphi [F] = \left(\frac{\partial \varphi}{\partial F}\right)_x = \lim_{\varepsilon \to 0} \frac{\varphi [F + \varepsilon E_x] - \varphi [F]}{\varepsilon} \quad . \quad . \quad . \quad . \quad (19)$$

In this case, however, we cannot generally prove, but must *assume* explicitely, that the variation $\delta \varphi$, defined by $\delta \varphi = \lim\limits_{\varepsilon \to 0} \frac{d}{d \varepsilon} \varphi [F + \varepsilon \delta F]$ is equal to

$$\delta \varphi = \int \delta F^{dx} \partial_x \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

For analytic functionals of the second kind

$$\varphi [F] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \ldots . \int F^{dx_1} \ldots F^{dx_n} \varphi_{x_1 \ldots x_n}$$

with $lim\ sup\ |\varphi_{x_1 \ldots x_n}| \leqq M N^{-n}$ and $|F| < N$ the derivative (19) exists always and we have

$$\varphi_{x_1 \ldots x_n} = (\partial_{x_1} \ldots . \partial_{x_n} \varphi [F])_{F=0} .$$

In this case also the relation (18) holds for every δF.

10. In an analogous way we can define derivatives of more general functionals. As an example we consider a functional $\varphi [U]$, where U has the components $U^{XY}_{\cdot \cdot z}$. If $E_{xy}^{\cdot \cdot Z}$ is the mixed affinorial with the components $(E_{xy}^{\cdot \cdot Z})^{XY}_{\cdot \cdot z} = \mathsf{E}^X_x \mathsf{E}^Y_y \mathsf{E}^Z_z$, we define

$$\partial_{xy}^{\cdot \cdot Z} \varphi [U] = \lim_{\varepsilon \to 0} \frac{\varphi [U + \varepsilon E_{xy}^{\cdot \cdot Z}] - \varphi [U]}{\varepsilon} \quad . \quad . \quad . \quad . \quad (21)$$

For functionals of linear operators P with components P^X_x, we would obtain in the same way

$$\partial^X_{\cdot x} \varphi [\mathsf{P}] = \lim_{\varepsilon \to 0} \frac{\varphi [\mathsf{P} + \varepsilon E^X_{\cdot x}] - \varphi [\mathsf{P}]}{\varepsilon} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

11. In the last-mentioned case we can define another kind of derivative, which is simply the spur of (22), viz

$$\varphi' [\mathsf{P}] = \frac{d\varphi}{d\mathsf{P}} = \lim_{\varepsilon \to 0} \frac{\varphi [\mathsf{P} + \varepsilon \mathsf{E}] - \varphi [\mathsf{P}]}{\varepsilon} = \int \partial^{dx}_{\cdot x} \varphi [\mathsf{P}] . \quad . \quad . \quad (23)$$

Of course it can be used also if φ itself is (apart from being a functional of P) a function of either kind. If φ itself is a linear operator with components φ_x^X, φ'^X_x becomes the derivative $\left(\frac{d\varphi}{d\mathsf{P}}\right)_x^X$ which is usually considered [10a]) in operational calculus. Writing again φ_P instead of $\varphi\,[\mathsf{P}]$ and assuming φ_P to possess a development of the form (13) we find simply

$$\varphi'_\mathsf{P} = \int \frac{d\varphi}{d\lambda}\, \mathsf{E}_\mathsf{P}^{d\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

12. In an analogous way we could define an operational derivative of ordinary functions $\varphi(\lambda)$ of a real variable, viz

$$\partial_x^X \varphi = \lim_{\varepsilon \to 0} \frac{\varphi(\lambda + \varepsilon E_x^X) - \varphi(\lambda)}{\varepsilon}$$

This derivative, however, is not very important, as it is equal to $\frac{d\varphi}{d\lambda}\,\mathsf{E}_x^X$.

13. We can introduce now *functional transformations* of different kinds. Let us consider as an example those which correspond with the case treated in Art. 8. Therefore let $\varphi_{x'}[f]$ be a functional of the first kind, and at the same time a function of the first kind on some set R'. Moreover, let it possess a continuous derivative

$$\mathsf{E}_{x'}^X[f] = \partial^X \varphi_{x'}[f], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and let the transformation $f_x \rightarrow f_{x'} = \varphi_{x'}[f]$ possess a unique inverse one. Then $\mathsf{E}_{x'}^X$ and $\mathsf{E}_x^{X'}$ satisfy (3) if the "same" function f (with coordinates f_x or $f_{x'}$) is substituted. Then we can extend the definitions of vectorials, tensorials and affinorials to such quantities which are functionals of f, all transformations being performed by means of $\mathsf{E}_x^{X'}[f]$ and $\mathsf{E}_{x'}^X[f]$.

14. Also linear connections can be introduced, which in this case belong to *contra*variant derivation. Indeed, if $\Gamma_z^{XY} = \Gamma_z^{XY}[f]$ are defined with respect to each system of coordinates $f^x, f^{x'}$, etc., such that the law of transformation is

$$\Gamma_{z'}^{X'Y'} = \iiint \mathsf{E}_x^{X'}\, \mathsf{E}_y^{Y'}\, \Gamma_z^{dx,dy}\, \mathsf{E}_{z'}^{dz} + \int \mathsf{E}_{z'}^{dz}\, \partial^{X'}\, \mathsf{E}_z^{Y'}, \quad . \quad . \quad . \quad (26)$$

the contravariant derivatives of e. g. vectorials v_x and V^X, viz.

$$\left.\begin{aligned} \nabla^Y v_x &= \partial^Y v_x + \int \Gamma_x^{Y\,dz}\, v_z, \\ \nabla^Y V^X &= \partial^Y V^X - \int \Gamma_z^{YX}\, V^{dz}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

[10a]) In particular if φ is a *function* (not merely a functional) of P.

are evidently affinorials. For the affinorials of torsion and of curvature we find in the usual way

$$S^{ZY}_{\cdot\cdot x} = 2\Gamma^{[ZY]}_{x}, \qquad (28)$$

$$R^{ZYX}_{\cdot\cdot\cdot w} = 2\,\partial^{[Z}\Gamma^{Y]X}_{w} + 2\int \Gamma^{[Z|du|}_{w}\Gamma^{Y]X}_{u}, \qquad (29)$$

and

$$\left.\begin{aligned} 2\,\nabla^{[Z}\nabla^{Y]}\,v_w &= \int R^{ZYdx}_{\cdot\cdot\cdot w}\,v_x - 2\int (\nabla^{dx}\,v_w)\,S^{ZY}_{\cdot\cdot x}, \\ 2\,\nabla^{[Z}\nabla^{Y]}\,V^X &= \int R^{ZYX}_{\cdot\cdot\cdot w}\,V^{dw} - 2\int (\nabla^{dw}\,V^X)\,S^{ZY}_{\cdot\cdot w} \end{aligned}\right\} . \qquad (30)$$

Hence the whole theory of linear connections, of parallel displacement, etc. can be extended, except of course the theory of Riemannian and conformal connections, as no tensorial G^{XY} or g_{xy} has a reciprocal one. To generalize these theories also we must consider the case in ordinary differential geometry, where tensors of lower rank are given. If we have e. g. two tensorials G^{XY}, h_{xy}, such that

$$\int G^{Xdy}\,h_{yz} = \mathsf{D}^X_z$$

is idempotent: $\mathsf{D}^2 = \mathsf{D}$, then we can define a Riemannian connection for such functions only, which are invariant under the transformation D: $\mathsf{D}f = f$ or $F\mathsf{D} = \mathsf{D}$.

Evidently analogous definitions can be given for functional transformations of the second kind $F^{X'} = \Phi^{X'}[F]$, for transformations of operators $\mathsf{P}^{X'}_{x'} = \varphi^{X'}_{x'}[\mathsf{P}]$, etc. where the derivatives, defined in Art. 9 and 10 are used.

15. The theory which is represented here shows a twofold relation with ordinary differential geometry. First, the latter is a special case of the former. Indeed, if we take for R a *finite* set, the functions of the first and second kind can be identified with co- and contravariant vectors or reciprocally.

A second and more interesting relation is obtained, if we take R to be an ordinary differentiable manifold of n dimensions (an X_n). A scalarfield p in R becomes now a function p_x of the first kind. A covariant vectorfield w_i must be written w_{xi}. If x^i are the coordinates of the point x and $y^i = x^i + dx^i$ those of a neighbouring point, w_{xi} determines the differential form $w_{xi}\,dx^i$, which is, but for quantities of the second order, equal to $\frac{1}{2}(w_{xi} + w_{yi})\,dx^i$ or to $\frac{1}{2}(w_{xi} + w_{yi})(y^i - x^i)$. Denoting the latter quantity by w_{xy}, we see that w_{xy} is an alternating two-point-function. In the same way, if $w_{i_1..i_k} = w_{x i_1..i_k}$ is a k-vectorfield, and if $x^i_r = x^i_0 + d_r x^i$ $(r = 1, .., k)$, are k neighboring points of $x^i_0 = x^i$, $w_{i_1..i_k}$ determines the differential form

$$w_{x i_1 \dots i_k} d_1 x^{i_1} \dots d_k x^{i_k} \sim \frac{1}{k+1} (w_{x i_1 \dots i_k} + \sum_{r=1}^{k} w_{x_r i_1 \dots i_k}) d_1 x^{i_1} .. d_k x^{i_k} \sim$$

$$\sim \frac{1}{k+1} \sum_{r=0}^{k} w_{x_r i_1 \dots i_k} \cdot \sum_{r=0}^{k} (-1)^r x_0^{i_1} \dots x_{r-1}^{i_r} x_{r+1}^{i_r+1} \dots x_k^{i_k} .$$

Hence a covariant k-vector corresponds with an alternating $(k+1)$-point-function, or rather with a class of such functions, which differ only by quantities of the $(k+1)^{\text{th}}$ order of smallness, if the mutual distances of the $k+1$ points are small of the first order. If in particular the k-vector is the exterior derivative of a $(k—1)$-vector:

$$w_{x i_1 \dots i_k} = k!\, \partial_{[i_1} v_{i_2 \dots i_k]}, \quad \dots \dots \dots \quad (31)$$

we find that the relation between $w_{x x_1 .. x_k}$ and $v_{x_1 .. x_k}$, corresponding with (31), is

$$w_{x x_1 \dots x_k} = (k+1)!\, 1_{[x} v_{x_1 \dots x_k]}, \quad \dots \dots \dots \quad (32)$$

where 1_x is the function (corresponding with the scalar 1), which takes the value 1 in each point of R. Hence we see that the operation of derivation in X_n corresponds with the purely *algebraical* operation (32) in the corresponding alternating function-space [11]). Analogous relations exist between $(k+1)$-fold alternating set-functions $F^{X_0 \dots X_k}$ and contravarient k-vector-densities of weight 1 $\mathfrak{F}^{i_1 \dots i_k}$; with $\partial_{i_1} \mathfrak{F}^{i_1 \dots i_k}$ corresponds here $\int 1_x F^{dx X_1 \dots X_k}$.

The analogy considered here is of importance for the abstract theory of differentiation and integration [12]) and for the foundations of topology.

[11]) This is the reason why we have written the point-functions with *lower* suffixes. Cf [6]).

[12]) Cf. J. W. ALEXANDER, On the chains of a complex and their duals; On the ring of a compact metric space , Proc. Nat. Ac. Sc. **21** 509—511; 511—512 (1935).