Mathematics. — *Generalisations of* CARLEMAN's *Inequality*. By J. G. VAN DER CORPUT.

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In this note I assume that the numbers $a_1, a_2, \dots$ are $\geqq 0$, but not all zero, that $\beta_1, \beta_2, \dots$ are positive and I write $\beta_1 + \beta_2 + \dots + \beta_n = \sigma_n$.

Theorem 1:

$$\sum_{n=1}^{\infty} (a_1^{\beta_1} a_2^{\beta_2} \dots a_n^{\beta_n})^{\frac{1}{\sigma_n}} < \sum_{n=1}^{\infty} \left(\frac{\sigma_{n+1}\,\beta_n}{\sigma_n\,\beta_{n+1}}\right)^{\frac{\sigma_n}{\beta_n}} a_n, \quad . \quad . \quad . \quad . \quad (1)$$

provided that the second series converges.

If $c_1, c_2, \dots$ are positive, the theorem of the arithmetic and geometric means gives

$$S = \sum_{n=1}^{\infty} (a_1^{\beta_1} \dots a_n^{\beta_n})^{\frac{1}{\sigma_n}} = \sum_{n=1}^{\infty} (c_1^{\beta_1} \dots c_n^{\beta_n})^{-\frac{1}{\sigma_n}} \prod_{m=1}^{n} (c_m\, a_m)^{\frac{\beta_m}{\sigma_n}}$$

$$< \sum_{n=1}^{\infty} (c_1^{\beta_1} \dots c_n^{\beta_n})^{-\frac{1}{\sigma_n}} \sum_{m=1}^{n} \frac{\beta_m}{\sigma_n} c_m\, a_m$$

$$= \sum_{m=1}^{\infty} \beta_m\, c_m\, a_m \sum_{n=m}^{\infty} \frac{(c_1^{\beta_1} \dots c_n^{\beta_n})^{-\frac{1}{\sigma_n}}}{\sigma_n},$$

unless all $c_m\, a_m$ are equal. I choose

$$c_1^{\beta_1} \dots c_n^{\beta_n} = \left(\frac{\sigma_{n+1}}{\beta_{n+1}}\right)^{\sigma_n},$$

when

$$c_n = \left(\frac{\sigma_{n+1}}{\beta_{n+1}}\right)^{\frac{\sigma_n}{\beta_n}} \left(\frac{\beta_n}{\sigma_n}\right)^{\frac{\sigma_{n-1}}{\beta_n}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\sum_{n=m}^{\infty} \frac{(c_1^{\beta_1} \dots c_n^{\beta_n})^{-\frac{1}{\sigma_n}}}{\sigma_n} = \sum_{n=m}^{\infty} \frac{\beta_{n+1}}{\sigma_n\, \sigma_{n+1}} = \sum_{n=m}^{\infty} \left(\frac{1}{\sigma_n} - \frac{1}{\sigma_{n+1}}\right) \leqq \frac{1}{\sigma_m},$$

and then

$$S < \sum_{m=1}^{\infty} \frac{\beta_m\, c_m\, a_m}{\sigma_m} = \sum_{m=1}^{\infty} \left(\frac{\sigma_{m+1}\,\beta_m}{\sigma_m\,\beta_{m+1}}\right)^{\frac{\sigma_m}{\beta_m}} a_m,$$

unless $a_m = \frac{q}{c_m}$. If $a_m = \frac{q}{c_m}$ where $q > 0$, then the left side of the stated inequality is

$$q \sum_{n=1}^{\infty} (c_1^{\beta_1} \dots c_n^{\beta_n})^{-\frac{1}{\sigma_n}} = q \sum_{n=1}^{\infty} \frac{\beta_{n+1}}{\sigma_{n+1}}$$

and the right side

$$q \sum_{n=1}^{\infty} \frac{\beta_n}{\sigma_n} > q \sum_{n=1}^{\infty} \frac{\beta_{n+1}}{\sigma_{n+1}}.$$

This completes the proof of the theorem.

Theorem 2: *If* $k > -1$ *and*

$$\frac{\sigma_{n+1}}{\beta_{n+1}} - \frac{\sigma_n}{\beta_n} \leqq \frac{1}{k+1} \qquad (n = 1, 2, \dots), \quad . \quad . \quad (3)$$

then

$$\sum_{n=1}^{\infty} (a_1^{\beta_1} \dots a_n^{\beta_n})^{\frac{1}{\sigma_n}} < e^{\frac{1}{k+1}} \sum_{n=1}^{\infty} a_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

provided that the last series converges.

Under the additional condition that

$$\lim_{n \to \infty} \left(\frac{\sigma_{n+1}}{\beta_{n+1}} - \frac{\sigma_n}{\beta_n} \right) = \frac{1}{k+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and that σ_n *and* $\frac{\sigma_n}{\beta_n}$ *tend to infinity, the constant* $e^{\frac{1}{k+1}}$ *is the best possible.*

From (3) it follows

$$\frac{\sigma_{n+1}}{\sigma_n} \frac{\beta_n}{\beta_{n+1}} \leqq 1 + \frac{1}{k+1} \frac{\beta_n}{\sigma_n}, \text{ hence } \left(\frac{\sigma_{n+1}}{\sigma_n} \frac{\beta_n}{\beta_{n+1}} \right)^{\frac{\sigma_n}{\beta_n}} < e^{\frac{1}{k+1}},$$

so that theorem 1 includes inequality (4).

If I write $a_n = \frac{q}{c_n}$ $(n = 1, 2, \dots, N)$, where c_n is defined by (2), and $a_{N+1} = a_{N+1} = \dots = 0$, then the two sides of (4) reduce to

$$S_N = \sum_{n=1}^{N} \frac{\beta_{n+1}}{\sigma_{n+1}} \text{ and } S'_N = e^{\frac{1}{k+1}} \sum_{n=1}^{N} \left(\frac{\sigma_{n+1}}{\sigma_n} \frac{\beta_n}{\beta_{n+1}} \right)^{-\frac{\sigma_n}{\beta_n}} \frac{\beta_n}{\sigma_n}.$$

On account of $\sigma_n \to \infty$ the series $\beta_1 + \beta_2 + \dots$ diverges, and so does $\frac{\beta_1}{\sigma_1} + \frac{\beta_2}{\sigma_2} + \dots$; by (5) we obtain

$$\left(\frac{\sigma_{n+1}}{\sigma_n} \frac{\beta_n}{\beta_{n+1}} \right)^{\frac{\sigma_n}{\beta_n}} = \left\{ 1 + \left(\frac{\sigma_{n+1}}{\beta_{n+1}} - \frac{\sigma_n}{\beta_n} \right) \frac{\beta_n}{\sigma_n} \right\}^{\frac{\sigma_n}{\beta_n}} \to e^{\frac{1}{k+1}}$$

and therefore

$$\lim_{N\to\infty} \frac{S'_N}{S_N} = 1.$$

Hence it follows that the constant $e^{\frac{1}{k+1}}$ is the best possible.

Remark. The special case $\beta_n = 1$, $k = 0$ affords CARLEMAN's inequality.[1] The argument above is only a generalisation of PÓLYA's proof of CARLEMAN's inequality.

Inequality (4) holds, when $\beta_n = \frac{\Gamma(n+k)}{\Gamma(n)}$ or $\beta_n = \binom{n+k-1}{n-1}$ where $k > -1$, for then

$$\frac{\sigma_n}{\beta_n} = \frac{n+k}{k+1} \text{ and } \frac{\sigma_{n+1}}{\beta_{n+1}} - \frac{\sigma_n}{\beta_n} = \frac{1}{k+1}.$$

Lemma. *Suppose that* $k > -1$ *and write*

$$\gamma_n = \frac{k\,\beta_{n+1}}{\beta_{n+1} - \beta_n} - n \qquad (n = 1, 2, \ldots).$$

1. *If* $\beta_{n+1} < \beta_n$, $\gamma_{n+1} \geqq \gamma_n$ $(n = 1, 2, \ldots)$ *and inequality* (3) *holds for* $n = 1$, *then it is true for any positive integer* n.

2. *If* $\beta_{n+1} > \beta_n$, $\gamma_{n+1} \leqq \gamma_n$ *and inequality* (3) *holds for* $n = 1$, *then it is valid for any positive integer* n.

From $\gamma_{n+1} \geqq \gamma_n$ I deduce

$$\left.\begin{aligned} (k+1)\beta_{n+1} &\leqq \frac{k\beta_{n+1}\beta_{n+2}}{\beta_{n+2} - \beta_{n+1}} - \frac{k\beta_{n+1}\beta_{n+1}}{\beta_{n+1} - \beta_n} + k\,\beta_{n+1} \\ &= \frac{k\beta_{n+1}\beta_{n+2}}{\beta_{n+2} - \beta_{n+1}} - \frac{k\beta_n\beta_{n+1}}{\beta_{n+1} - \beta_n} \qquad (n = 1, 2, \ldots) \end{aligned}\right\} \quad \ldots \quad (6)$$

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[1]) T. CARLEMAN, Sur les fonctions quasi-analytiques, Conférences faites au cinquième congrès des mathématiciens scandinaves (Helsingfors, 1923), 181—196.

G. PÓLYA, Proof of an inequality, Proc. L. M. S. (2), **24** (1926), Records of Proc. LVII.

G. VALIRON, Lectures on the general theory of integral functions (Toulouse, 1923), 186.

TH. KALUZA and G. SZEGÖ, Ueber Reihen mit lauter positiven Gliedern, Journ. L.M.S. **2** (1927), 266—272.

K. KNOPP, Ueber Reihen mit positiven Gliedern, Journ. L. M. S. (3) (1928), 205—211.

A. OSTROWSKI, Ueber quasi-analytische Funktionen und Bestimmtheit asymptotischer Entwicklungen, Acta Math. **53** (1929), 181—266.

G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, Inequalities (Cambridge University Press, 1934), 249.

If $\beta_{n+1} < \beta_n$ and inequality (3) holds for $n=1$, then $\beta_2 \geqq (k+1)\beta_1$, hence

$$(k+1)\beta_1 \leqq \frac{k\beta_1\beta_2}{\beta_2 - \beta_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By addition of (6) and (7) I obtain

$$(k+1)\sigma_n \leqq \frac{k\beta_n\beta_{n+1}}{\beta_{n+1}-\beta_n},$$

hence

$$\frac{\sigma_{n+1}}{\beta_{n+1}} - \frac{\sigma_n}{\beta_n} = 1 + \sigma_n\left(\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n}\right) \geqq 1 - \frac{k}{k+1} = \frac{1}{k+1}.$$

If $\beta_{n+1} > \beta_n$, $\gamma_{n+1} \leqq \gamma_n$ and (3) holds for $n=1$, then I find similarly

$$(k+1)\sigma_n \geqq \frac{k\beta_n\beta_{n+1}}{\beta_{n+1}-\beta_n},$$

consequently

$$\frac{\sigma_{n+1}}{\beta_{n+1}} - \frac{\sigma_n}{\beta_n} = 1 - \sigma_n\left(\frac{1}{\beta_n} - \frac{1}{\beta_{n+1}}\right) \leqq 1 - \frac{k}{k+1} = \frac{1}{k+1}.$$

Theorem 3: *If $k \geqq 1$ or $-1 < k \leqq 0$, and furthermore $\beta_n = n^k$, then (4) is valid and $e^{\frac{1}{k+1}}$ is the best possible constant.*

The special cases $k=0$ and $k=1$ follow immediately from theorem 2. For $n=1$ inequality (3) reduces to the valid relation $\frac{1}{2^k} \leqq \frac{1}{k+1}$.

According to the lemma it is therefore sufficient to verify that

$$\psi(v) = \frac{k\,(v+\frac{1}{2})^k}{(v+\frac{1}{2})^k - (v-\frac{1}{2})^k} - v \qquad\qquad (v > \tfrac{1}{2})$$

is monotonic increasing if $-1 < k < 0$ and decreasing if $k > 1$. We have

$$\psi'(v) = \frac{k^2\,(v+\frac{1}{2})^{k-1}\,(v-\frac{1}{2})^{k-1}}{\{(v+\frac{1}{2})^k - (v-\frac{1}{2})^k\}^2} - 1.$$

If $x \geqq 0$ and $0 \leqq p \leqq 1$ or $p > 1$, the inequality

$$e^{px} - e^{-px} \leqq \text{ or } \geqq p\,(e^x - e^{-x})$$

affords

$$\left(\frac{v+\frac{1}{2}}{v-\frac{1}{2}}\right)^{\frac{1}{2}p} - \left(\frac{v+\frac{1}{2}}{v-\frac{1}{2}}\right)^{-\frac{1}{2}p} \leqq \text{ or } \geqq \frac{p}{\sqrt{v^2-\frac{1}{4}}}.$$

Therefore if $p = k > 1$

$$(v+\tfrac{1}{2})^k - (v-\tfrac{1}{2})^k \geqq k\,(v^2-\tfrac{1}{4})^{\frac{1}{2}(k-1)},$$

hence $\psi'(v) \leqq 0$. If $0 < p = -k < 1$, we obtain

$$(v+\tfrac{1}{2})^{-k} - (v-\tfrac{1}{2})^{-k} \leqq -k\,(v^2-\tfrac{1}{4})^{\frac{-k-1}{2}}.$$

Multiplication by $(v+\frac{1}{2})^k\,(v-\frac{1}{2})^k$ affords

$$(v-\tfrac{1}{2})^k - (v+\tfrac{1}{2})^k \leqq -k\,(v^2-\tfrac{1}{4})^{\frac{k-1}{2}},$$

accordingly $\psi'(v) \geqq 0$. This completes the proof of the theorem.

The case $k>1$ may be treated also in the following way. If $f(u)=u^k$, then

$$\frac{f''(u)}{f'(u)} = \frac{k-1}{u} \qquad (u>0)$$

is monotonic decreasing; in the intervals $v > \frac{1}{2}$ and $0 \leqq w \leqq \frac{1}{2}$ we have

$$\log f'(v+w) + \log f'(v-w) - \log f'(v+\tfrac{1}{2}) - \log f'(v-\tfrac{1}{2})$$

$$= \int_w^{\frac{1}{2}} \left(\frac{f''(v-u)}{f'(v-u)} - \frac{f''(v+u)}{f'(v+u)} \right) du \geqq 0,$$

hence

$$\tfrac{1}{2}\left(f'(v+w) + f'(v-w)\right) \geqq \sqrt{f'(v+w)\,f'(v-w)} \geqq \sqrt{f'(v+\tfrac{1}{2})\,f'(v-\tfrac{1}{2})}.$$

By integration we obtain

$$f(v+\tfrac{1}{2}) - f(v-\tfrac{1}{2}) \geqq \sqrt{f'(v+\tfrac{1}{2})\,f'(v-\tfrac{1}{2})},$$

hence $\psi'(v) \leqq 0$.

The case $k = -1$ leads us to the following two theorems.

Theorem 4. *If p is arbitrary, and $\beta_n = \frac{1}{n}$, there are positive numbers $a_1, a_2, \ldots$ such that*

$$\sum_{n=1}^{\infty} (a_1^{\beta_1} \ldots a_n^{\beta_n})^{\frac{1}{\sigma_n}} > p \sum_{n=1}^{\infty} a_n. \quad \ldots\ldots \quad (8)$$

If $a_n = \frac{q}{c_n}$ $(n = 1, 2, \ldots, N)$, where c_n is defined by (2), and $a_{N+1} = a_{N+2} = \ldots = 0$, the two sides of (8) are

$$S_N = \sum_{n=1}^{N} \frac{\beta_{n+1}}{\sigma_{n+1}} \text{ and } S'_N = p \sum_{n=1}^{N} \left(\frac{\sigma_{n+1}\,\beta_n}{\sigma_n\,\beta_{n+1}} \right)^{-\frac{\sigma_n}{\beta_n}} \frac{\beta_n}{\sigma_n}.$$

We have

$$\frac{\sigma_n}{\beta_n} \log \left(\frac{\sigma_{n+1}\,\beta_n}{\sigma_n\,\beta_{n+1}}\right) = \frac{\sigma_n}{\beta_n} \log \left(1 + \frac{\beta_n}{\sigma_n} + \frac{\beta_n - \beta_{n+1}}{\beta_{n+1}}\right) \to \infty$$

by

$$\frac{\sigma_n}{\beta_n}\,\frac{\beta_n - \beta_{n+1}}{\beta_{n+1}} = \sigma_n\, n(n+1)\left(\frac{1}{n} - \frac{1}{n+1}\right) = \sigma_n \to \infty,$$

and consequently $\frac{S'_N}{S_N} \to 0$, if N tends to infinity.

Theorem 5: *If* $\beta_n = \frac{1}{n}$ *and* C *is* EULER's *constant, then*

$$\sum_{n=1}^{\infty} \left(a_1^{\beta_1} \ldots a_n^{\beta_n}\right)^{\frac{1}{\sigma_n}} < e^{1+C} \sum_{n=1}^{\infty} (n+1)\, a_n, \quad \ldots \quad (9)$$

provided that the second series converges; the constant e^{1+C} *is the best possible.*

On account of

$$(\sigma_n - \log (n+1)) - (\sigma_{n-1} - \log n) = \int_n^{n+1} du \left(\frac{1}{n} - \frac{1}{u}\right) > 0$$

$\sigma_n - \log (n+1)$ is increasing; $\sigma_n - \log(n+1) \to C$ implies therefore $\sigma_n - \log (n+1) < C$, hence

$$\frac{\sigma_n}{\beta_n} \log \left(\frac{\sigma_{n+1}\,\beta_n}{\sigma_n\,\beta_{n+1}}\right) = n\,\sigma_n \log \left(1 + \frac{\sigma_n + 1}{n\,\sigma_n}\right)$$

$$< \sigma_n + 1 < 1 + C + \log (n+1),$$

and (9) follows from (1).

If I take $a_n = \frac{q}{c_n}$ $(n = 1, 2, \ldots, N)$, where c_n is defined by (2), and $a_{N+1} = a_{N+2} = \ldots = 0$, then the two sides of (9) are

$$S_N = \sum_{n=1}^{N} \frac{\beta_{n+1}}{\sigma_{n+1}} \text{ and } S'_N = e^{1+C} \sum_{n=1}^{N} (n+1) \left(\frac{\sigma_{n+1}\,\beta_n}{\sigma_n\,\beta_{n+1}}\right)^{-\frac{\sigma_n}{\beta_n}} \frac{\beta_n}{\sigma_n}$$

and $\frac{S'_N}{S_N} \to 1$; consequently e^{1+C} is the best possible constant.