4. Die gegebene Differentialform ist für $\psi \equiv 0$ ein Quadrat. Es gelten also die bekannten Entwicklungen bei der quadratischen Differentialform. Weil der WEYLsche Tensor im binären Gebiet verschwindet, gibt es keine Differentialinvarianten.

5. Wenn $\theta_1 = 0$ und $\theta_2 = 0$, ist H ein Biquadrat. Setzen wir

$$-8\,H = (b_1\,dx_1 + b_2\,dx_2)^2, \quad \ldots \ldots \quad (63)$$

so ist

$$f = (a_1\,dx_1 + a_2\,dx_2)\,(b_1\,dx_1 + b_2\,dx_2)^3. \quad \ldots \quad (64)$$

Die Transformationsformeln für die relativen Vektoren a_i und b_i sind

$$\left.\begin{array}{l} \bar{a}_\alpha = \Delta^r\,\tau^{-1}\,a_\mu\,e^\mu_\alpha \\ \bar{b}_\alpha = \Delta^{-r-1}\,\tau\,b_\mu\,e^\mu_\alpha \end{array} \quad (r = -\tfrac{3}{2}) \right\}. \quad \ldots \quad (65)$$

Man könnte jetzt die konformen Differentialinvarianten von n relativen Vektoren im n-ären Gebiet

$${}_h\bar{a}_\alpha = \Delta^{r_h}\,\tau^{s_h}\,{}_h a_\mu\,e^\mu_\alpha \qquad (h = 1, \ldots, n) \quad \ldots \quad (66)$$

bestimmen und das Resultat auf unser Problem anwenden, also $n = 2$ setzen, aber grade für $n = 2$ gibt es keine Differentialinvarianten.

Mathematics. — *Casts of points, rays and planes.* By J. A. BARRAU. (Communicated by Prof. W. VAN DER WOUDE).

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§ 1. In $(n-1)$-dimensional space S_{n-1} a *cast* (Dutch: *worp*)

$$[A_1\,;\,A_2\,;\,A_3\,;\,\ldots\,;\,A_n\,;\,A_{n+1}\,;\,A_{n+2}]$$

is formed by $(n+2)$ points, no n of which belong to a S_{n-2}, and which are taken in a given order.

A cast is numerically defined by the set of homogeneous coordinates

$$\{a_1\,;\,a_2\,;\,a_3\,;\,\ldots\,;\,a_n\}$$

of the last point A_{n+2} with regard to a system where the first n points, in the given order, are fundamental points and the last point but one, A_{n+1}, is unit-point.

It is clear that casts are invariant under the projective group.

§ 2. In S_n a number of $(n+2)$ rays a through one point A, no n of which belong to a S_{n-1}, intersect any S_{n-1} not containing A, in

$(n+2)$ points, which form a cast. As all these casts are the same, we may take it as the cast of the $(n+2)$ rays:

$$[\alpha_1; \alpha_2; \ldots; \alpha_{n+2}] \equiv \{a_1; a_2; \ldots; a_n\}.$$

The locus of points P in S_n, from which $(n+2)$ given points (no $n+1$ of which lie in a S_{n-1}) are projected by rays forming a given cast, is a rational normal curve C_n of degree n.

We take the given points as fundamental points, the last one as unit-point. The rays projecting these points from an arbitrary point

$$\{x_1; x_2; \ldots; x_{n+1}\}$$

intersect the fundamental space

$$x_{n+1} = 0$$

resp. in the n fundamental points of this space, in

$$\{x_1; x_2; x_3; \ldots; x_n; 0\}$$

and in

$$\{x_1 - x_{n+1}; x_2 - x_{n+1}; \ldots; x_n - x_{n+1}; 0\},$$

thus forming the cast

$$\left\{\frac{x_1 - x_{n+1}}{x_1}; \frac{x_2 - x_{n+1}}{x_2}; \ldots; \frac{x_n - x_{n+1}}{x_n}\right\}.$$

This cast must accept the prescribed value

$$\{a_1; a_2; \ldots; a_n\},$$

hence

$$\frac{x_1 - x_{n+1}}{a_1 x_1} = \frac{x_2 - x_{n+1}}{a_2 x_2} = \ldots = \frac{x_n - x_{n+1}}{a_n x_n} = \lambda$$

or

$$(1 - a_1 \lambda) x_1 = (1 - a_2 \lambda) x_2 = \ldots = (1 - a_n \lambda) x_n = x_{n+1}.$$

Hence the locus of P has the parameter-representation

$$\begin{aligned}
x_1 &= 1 \,.\, (1-a_2\lambda)(1-a_3\lambda) \ldots\ldots\ldots\ldots (1-a_n\lambda) \\
x_2 &= (1-a_1\lambda) \,.\, 1 \,.\, (1-a_3\lambda) \ldots\ldots\ldots\ldots (1-a_n\lambda) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_n &= (1-a_1\lambda)(1-a_2\lambda)(1-a_3\lambda) \ldots\ldots (1-a_{n-1}\lambda) \,.\, 1 \\
x_{n+1} &= (1-a_1\lambda)(1-a_2\lambda)(1-a_3\lambda) \ldots (1-a_{n-1}\lambda)(1-a_n\lambda),
\end{aligned}$$

it is a rational normal C_n. [1]

[1] For $n = 2$ a well-known property of conics; for $n = 3$, i. e. for the twisted cubic, STURM, *Die Lehre von den geometrischen Verwandtschaften* II, 1908, p. 176 and p. 320.

As all the C_n in S_n are projective, inversely these curves have the property that $(n+2)$ fixed points of the curve are projected from any point of the curve by rays forming a constant cast.

Another consequence is that in S_n the "fully twisted" C_n containing $(n+2)$ fixed points form a *linear* system of dimension $(n-1)$.

§ 3. In S_n a number of $(n+1)$ planes π through one line l, no $(n-1)$ of which belong to a S_{n-1}, intersect any S_{n-2} having no point in common with l, in $(n+1)$ points, which form a cast.

As all these casts are the same, it may be taken as the cast of the $(n+1)$ planes:

$$[\pi_1\,;\pi_2\,;\ldots;\pi_{n+1}] \equiv \{a_1\,;a_2\,;\ldots;a_{n-1}\}.$$

The system of lines p in S_n, from which $(n+1)$ given points (not belonging to a S_{n-1}) are projected by planes forming a given cast, is a V_n^{n-1} of dimension n and of degree $(n-1)$.

We take the $(n+1)$ given points $A_1, \ldots A_{n+1}$ as fundamental points and first find the lines p passing through an arbitrary point E, taking that as unit-point.

Lines through E intersect the fundamental S_{n-1}

$$x_{n+1} = 0$$

in points P; if E' is the point of intersection of EA_{n+1}, then EP will be a line p if, and only if, the cast

$$[PA_1;\, PA_2;\ldots;\, PA_n;\, PE']$$

is the prescribed one. Hence the locus of P is a normal C_{n-1} and the degree of V is (n—1).

The dimension of V is n, for the ∞^n points of S_n belong generally to ∞^1 lines p each, but each line contains ∞^1 points.

The two-dimensional p-cones belonging to the points of a line passing through a fundamental point, have the same directrix-curve in the fundamental S_{n-1} opposite to that fundamental point.

For $n=3$ the V_2^2 is the well-known tetrahedral complex.

For $n=4$ we have a system V_4^3 of ∞^4 lines.

The two-dimensional cone of lines through an arbitrary point is of degree 3, i. e. it is intersected in three points by a general plane.

The number of lines of the system in an arbitrary plane, which number we may call the "class" of the system, is equally 3.

For be the plane (without loss of generality)

$$u = ax + by + cz \;\;,\;\; v = x + y + z$$

and the line in it

$$\lambda x + \mu y + \nu z = 0,$$

then, calling U' and V' the projections, from that line, of U and V on the fundamental plane XYZ, we must have

$$[X;Y;Z;U';V'] = \{\alpha;\beta;\gamma\}.$$

the prescribed cast.

As

$$U' = \{\mu-\nu;\nu-\lambda;\lambda-\mu;0;0\}$$
$$V' = \{c\mu-b\nu;a\nu-c\lambda;b\lambda-a\mu;0;0\}$$

we must have

$$\left\| \begin{matrix} c\mu-b\nu & a\nu-c\lambda & b\lambda-a\mu \\ \alpha(\mu-\nu) & \beta(\nu-\lambda) & \gamma(\lambda-\mu) \end{matrix} \right\| = 0,$$

which gives three solutions for $\{\lambda;\mu;\nu\}$.

We see that the lines of the V_4^3, belonging to a general S_3, in that S_3 form a congruence of degree and class 3.

§ 4. By the $(n+3)!$ permutations of $(n+3)$ points in S_n their casts take, for $n>1$, in general $(n+3)!$ values:

$$\{a\} = F(a_1, a_2, \ldots, a_{n+1}).$$

These $(n+3)!$ transformations form a group $G_{(n+3)!}$; the functions F are of five types:

1. A permutation only affecting the first $(n+1)$ points causes the corresponding permutation of the a.

2. By the permutation of the last two points every a_i is changed into its inverse:

$$[A_1;A_2;\ldots;A_{n+1};A_{n+3};A_{n+2}] = \left\{\frac{1}{a_1};\frac{1}{a_2};\ldots;\frac{1}{a_{n+1}}\right\}.$$

3. Transposition of an $A_i\,(i<n+1)$ with the last A but one, e. g. $(A_1\,A_{n+2})$, has the effect

$$\{a\} = \{a_1;a_1-a_2;\ldots;a_1-a_{n+1}\}.$$

4. Transposition of a A_i (e. g. A_1) with the last A gives

$$\{a\} = \left\{1;\frac{a_2}{a_2-a_1};\frac{a_3}{a_3-a_1};\ldots;\frac{a_{n+1}}{a_{n+1}-a_1}\right\}.$$

5. Combination of 3 and 4, $i\neq j$, e. g. $i=1$, $j=2$, has the effect:

$$\{a\} = \left\{\frac{a_1}{a_2};1;\frac{a_1-a_3}{a_2-a_3};\ldots;\frac{a_1-a_{n+1}}{a_2-a_{n+1}}\right\}.$$

All these transformations are rational; they are also birational, as the inverse transformation belongs to the inverse permutation.

The degree of the transformations is 1 for the types 1 and 3, $(n+2)!$ in number; it is n for the others, hence:

For $n > 1$ the transformations of the casts of $(n+3)$ points in S_n form a group $G_{(n+3)!}$ of $(n+2)!$ collineations and $(n+2).(n+2)!$ birational transformations of degree n.

§ 5. Taking the $\{a\}$ as points in S_n, we see that the $G_{(n+3)!}$ arranges these points in sets of $(n+3)!$, thus creating an involution of degree $(n+3)!$

Singular points of this involution are those from which the indeterminate set

$$\{a\}_0 \equiv \{0; 0; 0; \ldots; 0\}$$

can be derived; they satisfy

$$\text{either } a_i = 0, \quad \text{or} \quad a_i = a_j.$$

Coinciding points are those, invariant for a subgroup (other than mere identity) of the $G_{(n+3)!}$

For $n = 1$ all the points are quadruple points and the group G reduces to the well-known G_6 of anharmonic ratios.

We shall give the coinciding points for the case $n = 2$, thus describing in the plane point-sets analogous with harmonic and equi-anharmonic point-sets in the line. Let A, B, C be the (real) fundamental points, D the (real) unit-point and

$$E \equiv \{x; y; z\},$$

which makes

$$[A; B; C; D; E] = \{x; y; z\}.$$

There are 3 lines and 12 conics of *double points*, viz.

1. $x = y + z$ $(AD)(BC)$ 4. $x^2 = yz$ $(BC)(DE)$

2. $y = x + z$ $(BD)(AC)$ 5. $y^2 = xz$ $(AC)(DE)$

3. $z = x + y$ $(CD)(AB)$ 6. $z^2 = xy$ $(AB)(DE)$

7. $yz = xy + xz$ $(AE)(BC)$ 8. $xz = xy + yz$ $(BE)(AC)$ 9. $xy = xz + yz$ $(CE)(AB)$

10. $yz - 2xy + x^2 = 0$ $(BE)(CD)$ 11. $yz - 2xz + x^2 = 0$ $(CE)(BD)$

12. $xz - 2yz + y^2 = 0$ $(AD)(CE)$ 13. $xz - 2xy + y^2 = 0$ $(AE)(CD)$

14. $xy - 2xz + z^2 = 0$ $(AE)(BD)$ 15. $xy - 2yz + z^2 = 0$ $(AD)(BE)$

The substitution for which the cast $[A; B; C; D; E]$ is invariant is given with each number.

There are 30 *quadruple points*, which lie in pairs on these lines and conics, viz.

1.	$\{1+i;\ 1;\ i\}$ and $\{1+i;\ i;\ 1\}$	$(CABD)$
2.	$\{i;\ 1+i;\ 1\}$ and $\{1;\ 1+i;\ i\}$	$(ABCD)$
3.	$\{1;\ i;\ 1+i\}$ and $\{i;\ 1;\ 1+i\}$	$(BCAD)$
4.	$\{i;\ 1;\ -1\}$ and $\{i;\ -1;\ 1\}$	$(BDCE)$
5.	$\{-1;\ i;\ 1\}$ and $\{1;\ i;\ -1\}$	$(CDAE)$
6.	$\{1;\ -1;\ i\}$ and $\{-1;\ 1;\ i\}$	$(ADBE)$
7.	$\{1;\ 1+i;\ 1-i\}$ and $\{1;\ 1-i;\ 1+i\}$	$(CABE)$
8.	$\{1-i;\ 1;\ 1+i\}$ and $\{1+i;\ 1;\ 1-i\}$	$(ABCE)$
9.	$\{1+i;\ 1-i;\ 1\}$ and $\{1-i;\ 1+i;\ 1\}$	$(BCAE)$
10.	$\{1+i;\ 1;\ 2\}$ and $\{1-i;\ 1;\ 2\}$	$(BDEC)$
11.	$\{1+i;\ 2;\ 1\}$ and $\{1-i;\ 2;\ 1\}$	$(CDEB)$
12.	$\{2;\ 1+i;\ 1\}$ and $\{2;\ 1-i;\ 1\}$	$(CDEA)$
13.	$\{1;\ 1+i;\ 2\}$ and $\{1;\ 1-i;\ 2\}$	$(ADEC)$
14.	$\{1;\ 2;\ 1+i\}$ and $\{1;\ 2;\ 1-i\}$	$(ADEB)$
15.	$\{2;\ 1;\ 1+i\}$ and $\{2;\ 1;\ 1-i\}$	$(BDEA)$

A generating substitution of the cycle of four, for which $[A; B; C; D; E]$ is invariant, is given with each number. The cycle of course contains the substitution for which the whole conic is invariant and is determined by it.

There are 20 *sextuple points*, which lie in pairs on 3 of the double lines and conics; they are, when $\varepsilon = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$,

4, 5, 6	$\{1;\ \varepsilon;\ \varepsilon^2\}$ and $\{1;\ \varepsilon^2;\ \varepsilon\}$	$(ABC), (AB)(DE)$
1, 4, 7	$\{1;\ -\varepsilon;\ -\varepsilon^2\}$ and $\{1;\ -\varepsilon^2;\ -\varepsilon\}$	$(ADE), (BC)(AD)$
2, 5, 8	$\{1;\ -\varepsilon^2;\ \varepsilon\}$ and $\{1;\ -\varepsilon;\ \varepsilon^2\}$	$(BDE), (AC)(BD)$
3, 6, 9	$\{1;\ \varepsilon;\ -\varepsilon^2\}$ and $\{1;\ \varepsilon^2;\ -\varepsilon\}$	$(CDE), (AB)(CD)$
1, 12, 15	$\{1-\varepsilon;\ 1;\ -\varepsilon\}$ and $\{1-\varepsilon^2;\ 1;\ -\varepsilon^2\}$	$(BCE), (BC)(AD)$
2, 11, 14	$\{1;\ 1-\varepsilon;\ -\varepsilon\}$ and $\{1;\ 1-\varepsilon^2;\ -\varepsilon^2\}$	$(ACE), (AC)(BD)$
3, 10, 13	$\{1;\ -\varepsilon;\ 1-\varepsilon\}$ and $\{1;\ -\varepsilon^2;\ 1-\varepsilon^2\}$	$(ABE), (AB)(CD)$
7, 13, 14	$\{1;\ 1-\varepsilon;\ 1-\varepsilon^2\}$ and $\{1;\ 1-\varepsilon^2;\ 1-\varepsilon\}$	$(BCD), (BC)(AE)$
8, 10, 15	$\{1-\varepsilon;\ 1;\ 1-\varepsilon^2\}$ and $\{1-\varepsilon^2;\ 1;\ 1-\varepsilon\}$	$(ACD), (AC)(BE)$
9, 12, 11	$\{1-\varepsilon;\ 1-\varepsilon^2;\ 1\}$ and $\{1-\varepsilon^2;\ 1-\varepsilon;\ 1\}$	$(ABD), (AB)(CE)$

Two generating substitutions of the group of six, for which $[A; B; C; D; E]$ is invariant, are given with each pair; so e.g.

$$(ABC),\ (AB)\ (DE)$$

gives rise to the permutations:

ABCDE, CABDE, BCADE, BACED, CBAED, ACBED.

Each double conic (or line) contains 4 sextuple points.

There are 12 *decuple points*, which lie in pairs on 5 of the double lines and conics, viz.

1, 5, 9, 10, 14 $\{\sqrt{5}+1; 2; \sqrt{5}-1\}$ and $\{\sqrt{5}-1; -2; \sqrt{5}+1\}$
$(ABCDE), (AD)(BC)$

1, 6, 8, 11, 13 $\{\sqrt{5}+1; \sqrt{5}-1; 2\}$ and $\{\sqrt{5}-1; \sqrt{5}+1; -2\}$
$(ACBDE), (AD)(BC)$

2, 4, 9, 13, 15 $\{2; \sqrt{5}+1; \sqrt{5}-1\}$ and $\{-2; \sqrt{5}-1; \sqrt{5}+1\}$
$(BACDE), (BD)(AC)$

2, 6, 7, 10, 12 $\{\sqrt{5}-1; \sqrt{5}+1; 2\}$ and $\{\sqrt{5}+1; \sqrt{5}-1; -2\}$
$(BCADE), (BD)(AC)$

3, 4, 8, 12, 14 $\{2; \sqrt{5}-1; \sqrt{5}+1\}$ and $\{-2; \sqrt{5}+1; \sqrt{5}-1\}$
$(CABDE), (AB)(CD)$

3, 5, 7, 11, 15 $\{\sqrt{5}-1; 2; \sqrt{5}+1\}$ and $\{\sqrt{5}+1; -2; \sqrt{5}-1\}$
$(CBADE), (AB)(CD)$.

Two generating substitutions of the group of ten, for which $[A; B; C; D; E]$ is invariant, are given with each pair, so e.g.

$$(ABCDE), (AD)(BC)$$

gives rise to the permutations

ABCDE, BCDEA, CDEAB, DEABC, EABCD,
DCBAE, CBAED, BAEDC, AEDCB, EDCBA.

Each double conic (or line) contains 4 decuple points.

The 12 decuple points are *real*, and can be easily constructed, preserving projective generality, by taking A, B, C, in the vertices, D in the centre of a regular triangle. The lines **1**, **2**, **3**, join the middle points resp. of AB and AC, BA and BC, CA and CB, the conics **4**, **5**, **6**, are circles touching resp. AB and AC in B and C, BA and BC in A and C, CA and CB in A and B.

The 12 decuple points are the points of intersection of **1** with **5** and **6**, **2** with **4** and **6**, **3** with **4** and **5**. Any of these points forms with A, B, C, D a set of five points, which is in ten ways, including identity, projective with itself.