In the same way we get

$$N_2 = q \int_0^{\varepsilon_0} d\varepsilon_2 \int_{\varepsilon_2}^{\varepsilon_0} \frac{d\varepsilon_1}{\varepsilon_1} \; .$$

$$. \int_{-\infty}^{t} d\xi \; e^{-\alpha(\varepsilon_0)(t_1-\xi)} \; a(\varepsilon_0) \; v(\varepsilon_0) \; dt_1 \; e^{-\alpha(\varepsilon_1)(t_2-t_1)} \; a(\varepsilon_1) \; v(\varepsilon_1) \; dt_2 \; . \; e^{-\alpha(\varepsilon_2)(t-t_2)}$$

which integrated gives for N_2 the same expression as we have found with the first method.

Mathematics. — *Generalization of an inequality of* KNOPP. By J. G. VAN DER CORPUT.

(Communicated at the meeting of October 31, 1936).

In this note $a_0, a_1, \ldots$ are supposed to be $\geqq 0$, not all zero, $q > 0$ and $0 < u \leqq 1$ and I write

$$F(u) = \sum_{n=0}^{\infty} u^n \left(a_0^{\binom{n}{0} q^n} \; a_1^{\binom{n}{1} q^{n-1}} \ldots a_n^{\binom{n}{n} q^0} \right)^{\frac{1}{(q+1)^n}} .$$

KNOPP [1]) has deduced the inequality

$$F(1) < (q+1) \sum_{n=0}^{\infty} a_n , \quad \ldots \ldots \ldots \quad (1)$$

provided that the last series converges; for any fixed q the constant $q+1$ is the best possible.

As $(q+1)v^q - q v^{q+1}$ is in the interval $0 \leqq v \leqq 1$ a continuous monotonic increasing function of v, assuming for $v=0$ and $v=1$ the values 0 and 1, there is one and only one v $(0 \leqq v \leqq 1)$ such that

$$(q+1)\, v^q - q\, v^{q+1} = u^{q+1} . \quad \ldots \ldots \ldots \quad (2)$$

In this paper I will prove:

$$F(u) \leqq \frac{(q+1)\, v^q}{u^{q+1}} \sum_{n=0}^{\infty} a_n \quad \ldots \ldots \ldots \quad (3)$$

provided that the last series converges; for any fixed u und q the constant $\dfrac{(q+1)\, v^q}{u^{q+1}}$ is the best possible.

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[1]) K. KNOPP, Ueber Reihen mit positiven Gliedern (Zweite Mitteilung), Journal Lond. Math. Soc. **5** 13—21, (1930).

For $u=1$ relation (3) is always valid with the sign of inequality. If $0<u<1$, the sign of equality in (3) holds when and only when

$$a_n = p\,\frac{v^{(q+1)n}}{u^{(q+1)n}} \qquad (p>0).$$

In the special case $q=1$ relation (2) takes the form $2v-v^2=u^2$,

$$v=1-\sqrt{1-u^2}=\frac{u^2}{1+\sqrt{1-u^2}} \text{ and } \frac{(q+1)\,v^q}{u^{q+1}}=\frac{2}{1+\sqrt{1-u^2}},$$

hence

$$\sum_{n=0}^{\infty} u^n \left(a_0^{\binom{n}{0}}\, a_1^{\binom{n}{1}} \dots a_n^{\binom{n}{n}}\right)^{\frac{1}{2^n}} \leqq \frac{2}{1+\sqrt{1-u^2}} \sum_{n=0}^{\infty} a_n \quad . \quad . \quad (4)$$

and the constant $\dfrac{2}{1+\sqrt{1-u^2}}$ is the best possible. If $u=1$ relation (4) is true with the sign of inequality. If $u<1$ the sign of inequality holds when and only when

$$a_n = p\left(\frac{u}{1+\sqrt{1-u^2}}\right)^{2n} \qquad (p>0).$$

Relation (4) involves many inequalities, for instance integration gives

$$\sum_{n=0}^{\infty} \frac{1}{n+1}\left(a_0^{\binom{n}{0}}\, a_1^{\binom{n}{1}} \dots a_n^{\binom{n}{n}}\right)^{\frac{1}{2^n}} < (\pi-2) \sum_{n=0}^{\infty} a_n,$$

but $\pi-2$ is not the best possible constant.

The proof runs as follows:

Putting $w=\dfrac{v}{u}$ we obtain

$$F(u)=\sum_{n=0}^{\infty} u^n \prod_{m=0}^{n} a_m^{\binom{n}{m}\frac{q^{n-m}}{(q+1)^n}}$$

$$=\sum_{n=0}^{\infty} u^n\, w^{-nq} \prod_{m=0}^{n} \left(a_m\, w^{(n-m)(q+1)}\right)^{\binom{n}{m}\frac{q^{n-m}}{(q+1)^n}}$$

in virtue of

$$\sum_{m=0}^{n} (n-m)\binom{n}{m} q^{n-m} = n\,q\,(q+1)^{n-1}.$$

The theorem of the arithmetic and geometric means gives

$$F(u) < \sum_{n=0}^{\infty} u^n\, w^{-nq} \sum_{m=0}^{n} \binom{n}{m} \frac{q^{n-m}}{(q+1)^n}\, a_m\, w^{(n-m)(q+1)} \quad . \quad . \quad . \quad (5)$$

unless the $n+1$ numbers $a_m\, w^{(n-m)(q+1)}$ $(m=0, 1, \ldots, n)$ are equal to one another; in this last case the sign of inequality has to be replaced by the sign of equality. The right side of (5) is

$$\sum_{m=0}^{\infty} u^m\, w^{-mq}\,(q+1)^{-m}\, a_m \sum_{n=m}^{\infty} \binom{n}{m} \left(\frac{q\,u\,w}{q+1}\right)^{n-m}$$

$$= \sum_{m=0}^{\infty} u^m\, w^{-mq}\,(q+1)^{-m}\, a_m \left(1-\frac{q\,u\,w}{q+1}\right)^{-m-1}$$

for (2) implies

$$1-\frac{q\,u\,w}{q+1}=1-\frac{q\,v}{q+1}=\frac{u^{q+1}}{v^q}>0,$$

hence $\frac{q\,u\,w}{q+1}<1$. Consequently

$$F(u)<\frac{(q+1)\,w^q}{u}\sum_{m=0}^{\infty} a_m=\frac{(q+1)\,v^q}{u^{q+1}}\sum_{n=0}^{\infty} a_n$$

unless $a_m=p\,w^{(q+1)m}$ $(p>0)$; in this last case the sign of inequality is to be replaced by the sign of equality. If $u<1$, the left side of (2) has for $v=u$ the value

$$(q+1)\,u^q-q\,u^{q+1}>(q+1)\,u^{q+1}-q\,u^{q+1}=u^{q+1},$$

so that $v<u$, $w<1$ and the case $a_m=p\,w^{(q+1)m}$ is possible. If $u=1$, we have $w=1$, the case $a_m=p\,w^{(q+1)m}$ is excluded and (1) is true; if we choose $a_m=s^{m(q+1)}$ $(0<s<1)$,

$$F(1)=\sum_{n=0}^{\infty} s^n=\frac{1}{1-s}$$

and the right side of (1) is

$$(q+1)\sum_{n=0}^{\infty} s^{n(q+1)}=\frac{q+1}{1-s^{q+1}}.$$

If $s\to 1$

$$\frac{1-s^{q+1}}{(q+1)(1-s)}\to 1;$$

hence it follows that $q+1$ is in (1) the best possible constant.
