Mathematics. — *On* KUMMER's *solutions of the hypergeometric differential equation.* By J. G. VAN DER CORPUT.

(Communicated at the meeting of October 31, 1936).

Consider the analytic function

$$\psi(\alpha, \beta, \gamma; z) = \frac{z^{\alpha+1/6}(1-z)^{\beta+1/6}}{\Gamma(1+2\alpha)} F(\tfrac{1}{2}+\alpha+\beta+\gamma, \tfrac{1}{2}+\alpha+\beta-\gamma; 1+2\alpha; z), \quad . \quad (1)$$

where $|\arg z| < \pi$ and $|\arg (1-z)| < \pi$; $\pm 2\alpha$, $\pm 2\beta$ and $\pm 2\gamma$ are supposed not to be integers or zero.

This function is unchanged when γ is replaced by $-\gamma$.

Theorem 1: *The function $\psi(\alpha, \beta, \gamma; z)$ is a solution of the homogeneous linear differential equation of the second order* [1])

$$\frac{d^2 w}{dz^2} + p(z)\frac{dw}{dz} + q(z) w = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which has every point except 0, 1 and ∞ as an ordinary point, these three points being regular points with exponents $\frac{1}{6} \pm \alpha$, $\frac{1}{6} \pm \beta$ and $\frac{1}{6} \pm \gamma$. This equation has also the solution $\psi(-\alpha, \beta, \gamma; z)$ and any solution has the form

$$A\,\psi(\alpha, \beta, \gamma; z) + B\,\psi(-\alpha, \beta, \gamma; z), \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where A and B are constants.

The Function F occurring in (1) satisfies the homogeneous linear differential equation of the second order, which has every point except 0, 1 and ∞ as an ordinary point, these three points being regular points; the exponents in 0 are 0 and $1-(1+2\alpha) = -2\alpha$; those in 1 are 0 and

$$(1+2\alpha) - (\tfrac{1}{2}+\alpha+\beta+\gamma) - (\tfrac{1}{2}+\alpha+\beta-\gamma) = -2\beta$$

and finally in ∞

$$\tfrac{1}{2}+\alpha+\beta+\gamma \text{ and } \tfrac{1}{2}+\alpha+\beta-\gamma.$$

Consequently the function ψ defined by (1) is a solution of the homo-

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1) It is not necessary to observe that in this equation

$$p(z) = \frac{4z-2}{3z(z-1)}$$

and

$$q(z) = \frac{1-(1+36\beta^2)z+z^2}{36z^2(z-1)^2} + \frac{\alpha^2}{z^2(z-1)} - \frac{\gamma^2}{z(z-1)}.$$

geneous linear differential equation of the second order with the exponents

$$(0 \text{ or } -2\alpha)+\alpha+\tfrac{1}{6}=\tfrac{1}{6}\pm\alpha \qquad \text{in } 0,$$

$$(0 \text{ or } -2\beta)+\beta+\tfrac{1}{6}=\tfrac{1}{6}\pm\beta \qquad \text{in } 1$$

and

$$\tfrac{1}{2}+\alpha+\beta\pm\gamma-(\alpha+\beta+\tfrac{1}{3})=\tfrac{1}{6}\pm\gamma \quad \text{in } \infty,$$

i.e. equation (2).

This equation, that is unchanged when α is replaced by $-\alpha$, has accordingly also the solution $\psi(-\alpha, \beta, \gamma; z)$. These two solutions are not multiples of each other and form so a fundamental system, so that any solution of (2) has the form (3).

Theorem 2: *Equation* (2) *has the solutions*

$$\left\{\begin{array}{l} \psi(\pm\alpha, \pm\beta, \gamma; z),\ \psi(\pm\beta, \pm\alpha, \gamma; 1-z),\ \psi\left(\pm\gamma, \pm\beta, \alpha; \dfrac{1}{z}\right), \\ \psi\left(\pm\alpha, \pm\gamma, \beta, \dfrac{z}{z-1}\right),\ \psi\left(\pm\beta, \pm\gamma, \alpha, \dfrac{z-1}{z}\right),\ \psi\left(\pm\gamma, \pm\alpha, \beta, \dfrac{1}{1-z}\right). \end{array}\right.$$

For any of the transformations $Z=z,\ 1-z,\ \dfrac{1}{z},\ \dfrac{z}{z-1},\ \dfrac{z-1}{z}$ and $\dfrac{1}{1-z}$ transforms the system $(0, 1, \infty)$ in itself. These 24 solutions give by (1) KUMMER's 24 solutions of the hypergeometric differential equation.

Theorem 3: *The four functions* $\psi(\alpha, \pm\beta, \gamma, z)$ *and* $\left(\alpha, \pm\gamma, \beta, \dfrac{z}{z-1}\right)$ *are equal to one another throughout a suitable chosen domain.*

For any of this four functions is a solution of (2) and has therefore form (3). Investigation of the behaviour of the functions in the vicinity of 0 gives $B=0$ and $A=1$.

In theorem 3 we may replace z and $\dfrac{z}{z-1}$ by $1-z$ and $\dfrac{z-1}{z}$, or by $\dfrac{1}{z}$ and $\dfrac{1}{1-z}$.

In this manner we can group the 24 functions mentioned in theorem 2 (accordingly also KUMMER's 24 solutions of the hypergeometric differential equation) into six sets of four, such that the members of the same set are constant multiples of one another throughout a suitably chosen domain.

Theorem 4: *If* $|\arg z|<\pi$ *and* $|\arg(1-z)|<\pi$, *we have*

$$\frac{1}{\pi}(\sin 2\pi\alpha)\,\psi(\beta, \alpha, \gamma; 1-z)=$$

$$\frac{\psi(-\alpha, \beta, \gamma; z)}{\Gamma(\frac{1}{2}+\alpha+\beta+\gamma)\,\Gamma(\frac{1}{2}+\alpha+\beta-\gamma)}-\frac{\psi(\alpha, \beta, \gamma; z)}{\Gamma(\frac{1}{2}-\alpha+\beta+\gamma)\,\Gamma(\frac{1}{2}-\alpha+\beta-\gamma)}.$$

The left side is a solution of (2), consequently

$$\frac{1}{\pi}(\sin 2\pi a)\,\psi(\beta, a, \gamma; 1-z) = A\,\psi(a, \beta, \gamma; z) + B\,\psi(-a, \beta, \gamma; z) \quad . \quad . \quad (4)$$

If $\Re\, a < 0$, $0 < z < 1$ and $z \to 0$, we obtain by (1)

$$z^{-a-1/6}\,\psi(\beta, a, \gamma; 1-z) \to \frac{F(\tfrac{1}{2}+a+\beta+\gamma, \tfrac{1}{2}+a+\beta-\gamma; 1+2\beta; 1)}{\Gamma(1+2\beta)}$$

$$= \frac{\Gamma(-2a)}{\Gamma(\tfrac{1}{2}-a+\beta-\gamma)\,\Gamma(\tfrac{1}{2}-a+\beta+\gamma)};$$

$\frac{1}{\pi}\sin 2\pi a$ times this result is $\frac{A}{\Gamma(1+2a)}$, hence if $\Re\, a < 0$

$$A = \frac{1}{\pi}\sin 2\pi a \,.\, \frac{\Gamma(1+2a)\,\Gamma(-2a)}{\Gamma(\tfrac{1}{2}-a+\beta+\gamma)\,\Gamma(\tfrac{1}{2}-a+\beta-\gamma)}$$

$$= \frac{-1}{\Gamma(\tfrac{1}{2}-a+\beta+\gamma)\,\Gamma(\tfrac{1}{2}-a+\beta-\gamma)}.$$

Similarly if $\Re\, a > 0$

$$B = \frac{1}{\Gamma(\tfrac{1}{2}+a+\beta+\gamma)\,\Gamma(\tfrac{1}{2}+a+\beta-\gamma)}.$$

From (4) and the relation, obtained by differentiation of (4) to z, we obtain A and B as fractions whose numerators and denominators are continuous functions of a. The values found for A and B are therefore valid if $\Re a = 0$, consequently (by the princip of analytic continuation) for any a.

In this theorem I replace β, γ, z by $\gamma, \beta, \frac{z}{z-1}$ and find by means of theorem 3:

Theorem 5: *If* $|\arg(-z)| < \pi$ *and* $|\arg(1-z)| < \pi$, *we have*

$$\frac{1}{\pi}(\sin 2\pi a)\,\psi\left(\gamma, \beta, a; \frac{1}{z}\right) =$$

$$\frac{\psi(-a, \beta, \gamma; z)}{\Gamma(\tfrac{1}{2}+a+\beta+\gamma)\,\Gamma(\tfrac{1}{2}+a-\beta+\gamma)} - \frac{\psi(a, \beta, \gamma; z)}{\Gamma(\tfrac{1}{2}-a+\beta+\gamma)\,\Gamma(\tfrac{1}{2}-a-\beta+\gamma)}.$$

So I obtain, almost without any calculation, the well known linear relations between KUMMER's 24 solutions of the hypergeometric differential equation, for instance

$$\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}F(a, b; c; z) = \frac{\Gamma(a)\,\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a}F(a, 1-c+a; 1-b+a; z^{-1})$$

$$+ \frac{\Gamma(b)\,\Gamma(b-a)}{\Gamma(b-c)}(-z)^{-b}F(b, 1-c+b; 1-a+b; z^{-1})$$

where $|arg(-z)| < \pi$, and

$$\frac{F(a, b; c; z)}{\Gamma(c)\,\Gamma(c-a-b)} = \frac{F(a, b; a+b-c+1; 1-z)}{\Gamma(c-a)\,\Gamma(c-b)}$$

$$+ (1-z)^{c-a-b} \frac{F(c-a, c-b; c-a-b+1; 1-z)}{\Gamma(a)\,\Gamma(b)}$$

where $|arg(-z)| < \pi$ and $|arg(1-z)| < \pi$; the first result has to be modified when $a-b$ is an integer or zero, the second when $c-a-b$ is an integer or zero.

Physics. — *Ueber die konforminvariante Gestalt der relativistischen Bewegungsgleichungen.* Von J. A. SCHOUTEN und J. HAANTJES.

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1. *Einleitung.*

In einer früheren Arbeit [1]) haben wir gezeigt, dass sich die MAXWELLschen Gleichungen und die Impulsenergiegleichungen konforminvariant schreiben lassen. In dieser Arbeit wird gezeigt, dass sich auch die relativistischen Bewegungsgleichungen geladener Teilchen in eine konforminvariante Form bringen lassen, vorausgesetzt dass man die Masse so mit transformiert, dass das Produkt von Masse und Länge invariant bleibt. Es spielt dann $\frac{h}{c}$ (Dim. $[ML]$) eine ähnliche Rolle wie c in der gewöhnlichen Relativitätstheorie. Statt der Ruhmasse $\overset{\circ}{m}$ kommt eine andere Invariante, die Konformmasse $\overset{c}{m} = \overset{\circ}{m}\,(-\mathfrak{g})^{1/8}$, die die Dimension $[ML]$ hat. Wir geben hier nur die einfachen mathematischen Tatsachen und vermeiden physikalische Spekulationen.

Wir erinnern kurz an die früher erhaltenen Resultate. In einer Raumzeitwelt mit einer konformen Metrik gibt es keinen Fundamentaltensor g_{ih}, dagegen eine Tensordichte $\mathfrak{G}_{ih} = g_{ih}\,(-\mathfrak{g})^{-1/4}$ ($\mathfrak{g} = Det.\, g_{ih}$) vom Gewicht $-1/2$. Es gibt kein Linienelement $d\tau$, dagegen ein konformes (dimensionsloses) Linienelement $d\mathfrak{s}$, definiert durch:

$$(d\mathfrak{s})^2 = \mathfrak{G}_{ih}\, d\xi^i\, d\xi^h \quad \ldots \ldots \ldots \quad (1)$$

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[1]) Ueber die konforminvariante Gestalt der MAXWELLschen Gleichungen und der elektromagnetischen Impulsenergiegleichungen, Physica I (1934), 869- 872.