Physics. — *On the scattering of neutrons in matter.* (IV). By Prof. L. S. ORNSTEIN. (Communication from the Physical Institute of the University of Utrecht).

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In this paper we will treat the problem of the mean free path in an infinite layer of matter limited by a plane on which q neutrons pro unit of square enter in a direction perpendicular to the surface since the time $-\infty$ and the law of distribution in space and velocity for these neutrons.

We introduce a probability $\alpha(v)$ of collision and a probability $w(v)$ of not to be captured.

If we follow a neutron entering at the time ξ, it will suffer collisions at the times $t_1\, t_2 \ldots t_n$. Now we wish to know the mean distance from the surface at the time t.

Now if v_0 is the original velocity and $v_1\; v_2\; \ldots\; v_n$ the velocity after the impacts, $\cos\vartheta_1\; \cos\vartheta_2\; \ldots\; \cos\vartheta_n$ the $\cos$ of the angle of the path with the axis of x, the projection of the path on this direction amounts to:

$$v_0\,(t_1-\xi)+v_1\cos\vartheta\,(t_2-t_1)+\ldots \qquad v_n\cos\vartheta_n\,(t-t_n). \quad . \quad . \quad (1)$$

If we indicate the direction of the path after the collision by the angle ψ_n of the new path with the old and by an azimuth φ_n of the plane through the last path and the x axis with a fixed plane through the axis of x, we have:

$$\cos\vartheta_n=\cos\vartheta_{n-1}\cos\psi_n+\sin\vartheta_{n-1}\sin\psi_n\cos\varphi_n .$$

With the help of the quantities introduced we can express the probability of the path described by the formula

$$\left.\begin{array}{r}\dfrac{1}{2^n}\,q\,e^{+(\xi-t_1)\alpha_0}\,\alpha_0\,d\xi\,w_0\sin\vartheta_1\cos\vartheta_1\,d\vartheta_1\,e^{-(t_2-t_1)\alpha_1}\,\alpha_1\,w_1\cos\psi_1\sin\psi_1\,d\psi_1\,d\varphi_1\\ \ldots\,e^{-(t-\tau_{n-1})\alpha_{n-1}}\sin\psi_n\cos\psi_n\,d\psi_n\,d\varphi_n\end{array}\right\}. \quad (2)$$

From this formule we can find the total number of neutrons which suffered n collisions at the time t. It amounts to

$$N_n=\frac{q\,\omega_0}{a^n\,v_0^2}\int\limits_0^{v_0}\frac{v\,dv}{a}\int\limits_v^{v_0}\frac{w\,(\xi_1)\,d\xi_1}{\xi_1}\int\limits_{\xi_1}^{v_0}\frac{w\,(\xi_2)\,d\xi_2}{\xi_2}\cdots\int\limits_{\xi_n}^{v_0}\frac{w\,(\xi_n)\,d\xi_n}{\xi}$$

which is identical with the formula given in paper III (p. 1051) if we introduce ε instead of v.

Determining the total path described by the neutron under consideration, we have to multiply expression (1) with expression (2) and to integrate for all possible values of the angles and the times. In this way we find n parts arising from the separate terms of (1). The first is:

$$\frac{v_0}{a_0} N_n$$

the second

$$\frac{q\,\omega_0}{2^n}\int\limits_0^{v_0}\frac{v\,dv}{a}\int\limits_{\xi}^{v_0}\frac{w(\xi_1)\,d\xi_1}{\xi_1}\int\limits_{\xi}^{v_0}\cdots\int\limits_{\xi}^{v_0}\frac{v_1\,w_1\cos^2\vartheta_1\sin\vartheta_1\,d\vartheta_1}{v_1^2\,a(v_1)}$$

or

$$\frac{q\,w_0}{v_0^3\,2^n}\int\limits_0^{v}\frac{v\,dv}{a}\int\limits_{\xi}^{v_0}\cdots\int\limits_{\xi}^{v_0}\frac{\omega_1}{a_1}\,v_1\,dv_1.$$

In order to find the general form, it is good still to determine the third term. We find

$$\frac{q\,w_0}{2^n}\int\limits_0^{v_0}\frac{v\,dv}{a}\int\limits_{v}^{v_0}\frac{w_n\,d\xi_n}{\xi_n}\int\limits_{\xi_n}^{v_0}\cdots\left[\int\limits_{\xi_2}^{v_0}\xi_2\frac{w_2}{\xi_2^2}\frac{\sin\psi_2\cos^2\psi_2\,d\psi_2}{a_2}\int\limits_{\xi_1}^{v_0}w_1\sin\psi_1\cos^2\psi_1\,d\psi_1\right].$$

For the part between the brackets we can write

$$\int\limits_{\xi_2}^{v_0}\frac{w_2\,\xi_2}{a_2}\,d\xi_2\int\limits_{\xi_1}^{v_0}\frac{w_1}{\xi_1^3}\sin\psi_1\cos^2\psi_1\,d\psi=\frac{1}{v_0^3}\int\limits_{\xi_0}^{v_0}\frac{w_2\,\xi_2}{a_2}\,d\xi_2\int\limits_{\xi_1}^{v_0}\frac{w_1}{\xi_1}\,d\xi_1$$

or

$$\frac{q\,w_0}{2^n\,v_0^3}\int\limits_0^{v_0}\frac{v\,dv}{a}\int\limits_{v}^{v_0}\frac{w_n\,d\xi_n}{\xi_n}\cdots\int\limits_{\xi_1}^{v_0}\frac{w_2\,\xi_2\,d\xi_2}{a_2}\int\limits_{\xi_1}^{v_0}\frac{w_1}{d\xi_1}\,d\xi_1.$$

The contribution of $v_v\cos\vartheta_n\,(t_{v+1}-t_v)$ to the total path can be found by replacing $\frac{1}{\xi_v}$ by $\frac{\xi_v}{a_v}$ in the integral giving N_v.

With the help of the formula given we can deduce S_n for a path with n collisions. The mean free path is given by:

$$\frac{\Sigma\,S_n}{\Sigma\,N_n}$$

where the sum ought to be taken for all values of n from zero to infinite.

It is easily seen that we get an infinite series of which the first term is

$$\frac{q}{a_0} v_0.$$

Formula (2) enables us to calculate the distribution of the neutrons in space and velocity. With its help we can find the number of neutrons present between x and $x + dx$ having suffered a given number of collisions and showing a velocity given in magnitude and direction.

For the group which has suffered no collisions we find, if we put the condition

$$x < v_0 (t - \xi) < x + du$$

$$\frac{q}{v_0} e^{-\frac{\alpha_0}{v_0}} dx.$$

Let us consider next the group which suffered only one collision. We have to take the sum of

$$q\, a_0\, w_0\, e^{-(t_1-\xi)\alpha_0}\, d\xi\, e^{-(t-t_1)\alpha_1} \frac{\sin\vartheta \cos\vartheta\, d\vartheta}{2}$$

with the condition that

$$x < v_0 (t_1 - \xi) + v_0 \cos^2\vartheta\, (t - t_1) < x + dx.$$

We then have to eliminate ξ to take $\frac{dx}{v_0}$ instead of $d\xi$ and have to integrate with respect to t_1 from $-\frac{x - v_0 \cos^2\vartheta\, t}{r_0 \cos\vartheta}$ to t. In this way we get for the number under consideration

$$\frac{q\, a_0\, w_0}{2\, v_0} dx \frac{\sin\vartheta \cos\vartheta\, d\vartheta}{a_1 - a_0 \cos^2\vartheta} \left(-e^{-\frac{\alpha_0 x}{v_0 \cos^2\vartheta}} + e^{-\frac{\alpha_0 x}{v_0}} \right)$$

which formula is valid for ϑ from 0 to ½; for ϑ from $\frac{\pi}{2}$ to π the number is zero.

We will further calculate the distribution function for those neutrons which suffered two collisions.

In order to obtain this number we have to integrate the form

$$q\, a_0\, \omega_0\, e^{-(t_1-\xi)\alpha_0}\, d\xi\, e^{-(t_1-t_2)\alpha_1} \frac{\sin\vartheta \cos\vartheta\, d\vartheta}{2}\, a_1\, \omega_1\, e^{-(t-t_2)} \frac{\sin\psi \cos\psi\, d\varphi}{4\pi}$$

with the condition that

$$x < v_0 (t_1 - \xi) + v_1 \cos\vartheta_1 (t_2 - t_1) + v_2 \cos\vartheta_2 (t - t_2) < x + dx.$$

We eliminate ξ and integrate with respect to t_1 from

$$t_1 = t_2 \left(1 - \frac{v_2 \cos \vartheta_2}{v_1 \cos \vartheta_1}\right) - \frac{x}{v_2 \cos \vartheta_1} + \frac{v_2 \cos \vartheta_2}{v_1 \cos \vartheta_1} t$$

to t.

The result must further be integrated from

$$t_2 = t - \frac{x}{v_2 \cos \vartheta_2} \text{ to } t_2 = t.$$

In this way we get

$$q\, a_0\, w_0 \frac{\sin \vartheta_1 \cos \vartheta_1 \, d\vartheta_1}{a} a_1\, w_1 \frac{\sin \psi \cos \psi \, d\psi \, d\varphi}{4\pi} -$$

$$\left[\frac{1}{a_1 - \frac{a_0\, v_1 \cos \vartheta_1}{v_0}} \frac{1}{a_2 - \frac{a_0\, v_2 \cos \vartheta_2}{v_0}} \left(e^{-\frac{\alpha_0 x}{v_0}} - e^{-\frac{\alpha_2 x}{v_2 \cos \vartheta_2}}\right) - \right.$$

$$\left. \frac{1}{a_1 - \frac{a_0\, v_1 \cos \vartheta_1}{v_0}} \frac{1}{a_2 - \frac{a_1\, v_2 \cos \vartheta_2}{v_1}} \left(e^{-\frac{\alpha_1 x}{v_1 \cos \vartheta_1}} - e^{-\frac{\alpha_2 x}{v_2 \cos \vartheta_2}}\right) \right].$$

In order to get the definitive result we ought to bear in mind that $\cos \vartheta_2 = \cos \vartheta_1 \cos \psi + \sin \vartheta_1 \sin \psi \cos \varphi$.

$$v_1 = v_0 \cos \vartheta_1$$
$$v_2 = v_0 \cos \vartheta_1 \cos \psi.$$

As we wish to find the distribution function for $N_2\ (x_1\, \vartheta_2\, v_2)$, we ought to integrate for such values of ϑ_1 and ψ that

$$v_2 < v_0 \cos \vartheta_1 \cos \psi < v_2 + dv_2$$

ψ having values between 0 and $\pi/2$.

The quantities a_1, a_2 and w_1 are functions of $v_1\, v_2$. The calculation can therefore only be performed if we are aequainted with these functions.

As we can proceed in the same way for any of the numbers $N_n\ (x,\, \vartheta\, v)$ we have in principle solved the problem of the distribution law.

Deducing the distribution function, we can also apply the method of paper (III).

Putting $N_0\ (x)\, dx$ for the number of neutrons between x and $x + dx$ which suffered no collision, we get the equation

$$0 = -a_0\, N_0\, (x) - v_0 \frac{d\, N_0\, (x)}{dx}.$$

The solution is

$$N_0(x) = \frac{q}{v_0} e^{-\frac{\alpha_0}{v_0} x}.$$

For the number of neutrons which suffered only one collision we get

$$0 = -a_1(v) N_1(x, v) - v \cos\vartheta \frac{d N_1(x \,.\, v)}{dx} + \frac{q}{v_0} \cos\vartheta\, e^{-\frac{\alpha_0}{v_0} x}.$$

The solution is

$$N_1(x, v) = A\, e^{-\frac{\alpha(v)}{v \cos\vartheta} x} + \frac{q \cos\vartheta}{v_0 (a(v) - a_0 \cos^2\vartheta)} e^{-\frac{\alpha_0}{v_0} x}.$$

The constant A can be determined by the condition that for $x=0$, $N(x\,v)$ is zero for all values of ϑ between 0 and $\frac{\pi}{2}$ so that we get

$$N_1(x\,v) = \frac{q \cos\vartheta}{v_0 (a(v) - a_0 \cos^2\vartheta)} \left(-e^{-\frac{\alpha(v)\, x}{v_0 \cos^2\vartheta}} + e^{-\frac{\alpha_0}{v_0} x}\right).$$

The same result was attained by our first method. It is possible to calculate $N_n(x, \vartheta)$ for any value of n by the same procedure.

We get

$$0 = -a(v) N_n(x, \vartheta_2, v) - v \cos\vartheta \frac{dN(x, \vartheta_2\, v)}{dx} +$$

$$\int_v^{v_0}\!\!\int\!\!\int N_{n-1}(x_1\, \vartheta_1\, v_1)\, a_1\, w_1 \cos\psi \sin\psi\, d\psi\, d\varphi \sin\vartheta_1\, d\vartheta_1\, dv_1$$

where

$$\cos\vartheta_2 = \cos\vartheta_1 \cos\psi + \sin\vartheta_1 \sin\psi \cos\varphi$$

and integrate for ϑ_1 from 0 to $\pi/2$, for φ from 0 to 2π and for ψ from 0 to $\pi/2$.

The quantity a_1 is a function of v_1 $v_1 = v_0 \cos\vartheta_1$, a_2 a function of v_2 $v_2 = v_0 \cos\vartheta_1 \cos\psi$.