Mathematics. — *On the differential geometry of contact transformations.* By HWA-CHUNG LEE. (Communicated by Prof. J. A. SCHOUTEN).

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1. *Introduction.*

In two papers of these Proceedings SCHOUTEN[1]) has shown that a doubly homogeneous contact transformation in the $2n+2$ variables $x^{\varkappa}, p_{\lambda}$ $(\varkappa, \lambda, \ldots, \tau = 0, 1, \ldots, n)$:

$$x^{\varkappa'} = x^{\varkappa'}(x^{\varkappa}, p_{\lambda}), \quad p_{\lambda'} = p_{\lambda'}(x^{\varkappa}, p_{\lambda}), \quad \ldots \quad (1)$$

where $x^{\varkappa'}$ and $p_{\lambda'}$ are homogeneous functions of degrees (1, 0) and (0, 1) in $x^{\varkappa}, p_{\lambda}$ respectively, can always be modified (without changing the geometrical meaning) in such a way that these functions satisfy the equations

$$\left.\begin{aligned} & p_{\varkappa'} \frac{\partial x^{\varkappa'}}{\partial x^{\lambda}} = p_{\lambda}, \quad p_{\varkappa'} \frac{\partial x^{\varkappa'}}{\partial p_{\varkappa}} = 0, \\ & x^{\lambda'} \frac{\partial p_{\lambda'}}{\partial x^{\lambda}} = 0, \quad x^{\lambda'} \frac{\partial p_{\lambda'}}{\partial p_{\varkappa}} = x^{\varkappa}, \end{aligned}\right\} \quad \ldots \quad (2)$$

for all values of $x^{\varkappa}$ and p_{λ}, though the latter only have a geometric meaning when they are bound by the relation

$$p_{\lambda} x^{\lambda} = 0 \quad \ldots \quad (3)$$

Transformations of the form (1) with the mentioned homogeneity property and satisfying (2) evidently form a group[2]) which we call $\mathfrak{K}_{2n+2}$. Let us consider the $2n+1$ dimensional manifold with the homogeneous coordinates $x^{\varkappa}, p_{\lambda}$, endowed with the group $\mathfrak{K}_{2n+2}$ of *coordinate* transformations and the group $\mathfrak{F}$ of point transformations

$$'x^{\varkappa} = \varrho\, x^{\varkappa}, \quad 'p_{\lambda} = \varrho\, p_{\lambda}, \quad \ldots \quad (4)$$

where ϱ is a homogeneous function of degree (0,0) in $x^{\varkappa}, p_{\lambda}$. A point $(x^{\varkappa}, p_{\lambda})$ of this manifold is called an *element* if its coordinates satisfy the

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[1]) J. A. SCHOUTEN: Zur Differentialgeometrie der Gruppe der Berührungstransformationen. Vol. **40** (1937), 100—107, 236—245.

[2]) Strictly speaking this is a pseudogroup. Cf. O. VEBLEN and J. H. C. WHITEHEAD: The Foundations of Differential Geometry, Cambr. Tracts, p. 38.

relation (3). The totality of elements constitutes a $2n$-dimensional space K_{2n} whose defining equation (3) is invariant under the group $\mathfrak{K}_{2n+2}$ and the group $\mathfrak{F}$.' It is only the elements of this space with which we shall be concerned, and the purpose of this paper is to establish a formalism for the invariant theory of the groups $\mathfrak{K}_{2n+2}$ and $\mathfrak{F}'$ by constructing certain fundamental projective tensors which we shall call contact projectors.

We may get a correspondence between the space K_{2n} and another space H_n if we regard $x^\varkappa, p_\lambda$ as two separate sets of homogeneous coordinates, the first being homogeneous coordinates of the points of H_n [3]) and the second being homogeneous coordinates of the hyperplanes in the local spaces of H_n with $\lfloor x^\varkappa \rfloor$ [4]) as points of contact. The elements of K_{2n} correspond to the hyperplanes in the local spaces of H_n passing through the points of contact, owing to the relation (3). The configuration formed by a point of H_n and a hyperplane through it is also called an element of H_n. Since the two identical elements $(x^\varkappa, p_\lambda)$ and $(\varrho x^\varkappa, \sigma p_\lambda)$ of H_n for $\sigma \neq \varrho$ are regarded as two distinct elements of K_{2n} because of (4), the elements of K_{2n} and those of H_n are then not in one-to-one correspondence.

2. *Contact projectors.*

Besides the index-type $\varkappa$, we introduce the bracketed index-type $(\varkappa)$ defined by $(\varkappa) = (n+1) + \varkappa$, so that when $\varkappa$ takes the range $0, 1, \ldots, n$, $(\varkappa)$ will take the continuation range $n+1, \ldots, 2n+1$. We shall write $x^{(\lambda)}$ for p_λ so that the $2n+2$ variables $x^\varkappa, p_\lambda$ can be written as $x^\varkappa, x^{(\lambda)}$ or x^α, where $\alpha, \beta, \ldots, \iota$ take the two ranges of $\varkappa, (\varkappa)$ successively, that is to say, $\alpha, \beta, \ldots, \iota = 0, 1, \ldots, 2n+1$.

Equations (1) and (4) may now be written

$$\mathfrak{K}_{2n+2} : x^{\alpha'} = x^{\alpha'}(x^\alpha) , \quad \ldots \ldots \quad (5)$$

$$\mathfrak{F}' : {}'x^\alpha = \varrho x^\alpha , \quad \ldots \ldots \quad (6)$$

where $x^{\alpha'}$ are homogeneous functions of degree one in x^α and ϱ is a homogeneous function of degree zero in x^α.

The transformations (5) being supposed to be non-singular, we define contact projectors with the aid of the quantities

$$\left.\begin{aligned} \mathscr{A}^{\alpha'}_{\beta} &= \partial_\beta x^{\alpha'} , \quad \partial_\beta = \frac{\partial}{\partial x^\beta} , \\ \mathscr{A}^{\alpha}_{\beta'} &= \partial_{\beta'} x^\alpha , \quad \partial_{\beta'} = \frac{\partial}{\partial x^{\beta'}} . \end{aligned}\right\} \quad \ldots \ldots \quad (7)$$

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[3]) H_n is the generalized projective space with VAN DANTZIGs homogeneous coordinates. Cf. D. VAN DANTZIG: Theorie des projektiven Zusammenhangs n-dimensionaler Räume. Math. Ann. **106** (1932), 400.

[4]) $\lfloor x^\varkappa \rfloor$ means $x^\varkappa$ determined up to an arbitrary factor.

For example a contra- or covariant (projective) contact vector v^α or w_β of degree r is defined by the laws

$$\mathfrak{R}_{2n+2}: \begin{cases} v^{\alpha'} = \mathcal{A}^{\alpha'}_{\alpha}\, v^\alpha , \\ w_{\beta'} = \mathcal{A}^{\beta}_{\beta'}\, w_\beta , \end{cases} \qquad \mathfrak{F}': \begin{cases} 'v^\alpha = \varrho^r v^\alpha , \\ 'w_\beta = \varrho^r w_\beta . \end{cases} \quad . \quad . \quad . \quad (8)$$

Let us now write equations (2) in the form

$$\left. \begin{aligned} P_\beta &= P_{\beta'} \mathcal{A}^{\beta'}_{\beta} , \\ X_\beta &= X_{\beta'} \mathcal{A}^{\beta'}_{\beta} , \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where X_β and P_β are defined in each coordinate system by

$$\left. \begin{aligned} X_\lambda &= 0 \ , \ X_{(\varkappa)} = x^\varkappa , \\ P_\lambda &= p_\lambda \ , \ P_{(\varkappa)} = 0 \ , \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

i. e. the components of X_β only involve $x^\varkappa$, and those of P_β only involve p_λ. Then according to the definition (8), X_β and P_β are *covariant contact vectors of degree one.*

From (9) we have by differentiation

$$\left. \begin{aligned} \partial_{[\gamma} P_{\beta]} &= \partial_{[\gamma'} P_{\beta']}\ \mathcal{A}^{\gamma'}_{\gamma} \mathcal{A}^{\beta'}_{\beta} , \\ \partial_{[\gamma} X_{\beta]} &= \partial_{[\gamma'} X_{\beta']}\ \mathcal{A}^{\gamma'}_{\gamma} \mathcal{A}^{\beta'}_{\beta} \ ; \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and from (10) we find by actual calculation

$$-2\, \partial_{[\beta} X_{\alpha]} = \epsilon_{\beta\alpha} = +2\, \partial_{[\beta} P_{\alpha]} , \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $\epsilon_{\beta\alpha}$ is defined in each coordinate system by

$$\epsilon_{\lambda\varkappa} = 0 \ , \ \epsilon_{\lambda(\varkappa)} = -\delta^{\varkappa}_{\lambda} \ , \ \epsilon_{(\lambda)\varkappa} = \delta^{\lambda}_{\varkappa} \ , \ \epsilon_{(\lambda)(\varkappa)} = 0 . \quad . \quad . \quad (13)$$

Hence equations (11) reduce to

$$\epsilon_{\beta\alpha} = \epsilon_{\beta'\alpha'} \mathcal{A}^{\beta'}_{\beta} \mathcal{A}^{\alpha'}_{\alpha} \ ^{5)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

from which it follows that $\epsilon_{\beta\alpha}$ is a *covariant contact bivector of degree zero.*

The determinant of $\epsilon_{\beta\alpha}$ being unity, we can define a *contravariant contact bivector* $\epsilon^{\alpha\beta}$ of degree zero by the relation

$$\epsilon^{\alpha\gamma} \epsilon_{\beta\gamma} = \mathcal{A}^{\alpha}_{\beta} \begin{cases} = 1 \text{ for } \alpha = \beta , \\ = 0 \text{ for } \alpha \neq \beta . \end{cases} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

[5]) This is equivalent to the LAGRANGE parenthesis condition for contact transformations. Similarly the transformation law of $\epsilon^{\alpha\beta}$ defined by (15) is the equivalent of the POISSON parenthesis condition.

From this we find that the components of $\epsilon^{\alpha\beta}$ are

$$\epsilon^{\varkappa\lambda}=0,\ \epsilon^{\varkappa(\lambda)}=-\delta^{\varkappa}_{\lambda},\ \epsilon^{(\varkappa)\lambda}=+\delta^{\lambda}_{\varkappa},\ \epsilon^{(\varkappa)(\lambda)}=0. \quad . \quad . \quad (16)$$

We shall use the two ϵ's to raise and lower indices, convening that in lowering indices we sum with respect to the left index of $\epsilon_{\beta\alpha}$, while in raising indices we sum with respect to the right index of $\epsilon^{\alpha\beta}$, and shall identify those objects which are obtained from one another by raising and lowering indices.

With this understanding we obtain from (10) two *contravariant contact vectors of degree one*:

$$\left.\begin{aligned} X^{\varkappa}&=-x^{\varkappa},\ X^{(\lambda)}=0,\\ P^{\varkappa}&=0,\qquad P^{(\lambda)}=+p_{\varkappa},\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad (17)$$

whose laws of transformation

$$\left.\begin{aligned} X^{\alpha'}&=X^{\alpha}\,\mathcal{A}^{\alpha'}_{\alpha},\\ P^{\alpha'}&=P^{\alpha}\,\mathcal{A}^{\alpha'}_{\alpha}.\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

express analytically the fact that $x^{\varkappa'}$ and $p_{\lambda'}$ are homogeneous of degrees (1, 0) and (0, 1) in $x^{\varkappa}, p_{\lambda}$ respectively. Thus the homogeneity property of the group $\mathfrak{K}_{2n+2}$ is merely a consequence of (9), that is to say, a consequence of (2).

From (17) the two contact vectors X^{α}, P^{α} are connected by the relation

$$x^{\alpha}=P^{\alpha}-X^{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and hence the coordinates x^{α} themselves are the components of a contravariant contact vector of degree one whose law of transformation simply expresses that the $x^{\alpha'}$ are homogeneous of degree one in the x^{α}.

From (10) the contact vector defined by

$$q_{\beta}=P_{\beta}+X_{\beta},\quad (q_{\lambda}=p_{\lambda},\ q_{(\varkappa)}=x^{\varkappa}) \quad . \quad . \quad . \quad . \quad . \quad (20)$$

is seen to be the gradient of the contact scalar $q=p_{\mu}\,x^{\mu}$:

$$q_{\beta}=\partial_{\beta}\,q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Corresponding to the relations (19), (20) we have by raising and lowering suffices,

$$\left.\begin{aligned} x_{\beta}&=P_{\beta}-X_{\beta},\\ q^{\alpha}&=P^{\alpha}+X^{\alpha}.\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

It may be remarked that by taking the determinant on both sides of (14) we have $Det(\mathcal{A}^{\alpha'}_{\alpha})=\pm 1$ and hence if we confine ourselves to the subgroup of $\mathfrak{K}_{2n+2}$ for which $Det(\mathcal{A}^{\alpha'}_{\alpha})=+1$, the difference between contact projectors and projector-densities vanishes.

3. *The contact connection.*

Let a symmetric projective connection $\Pi_{\gamma\beta}^{\alpha}$ of degree -1 be introduced in the K_{2n} in such a way that we have

$$\nabla_\gamma \epsilon_{\beta\alpha} = \partial_\gamma \epsilon_{\beta\alpha} - \Pi_{\gamma\beta}^{\delta} \epsilon_{\delta\alpha} - \Pi_{\gamma\alpha}^{\delta} \epsilon_{\beta\delta} = 0 \quad . \quad . \quad . \quad . \quad (23)$$

If we multiple this by $\epsilon^{\beta\alpha}$, we obtain, on account of (15) and of the constancy of $\epsilon_{\beta\alpha}$,

$$\Pi_{\gamma\alpha}^{\alpha} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Hence there is no difference between the covariant differentiation of a contact projector and that of a projector-density.

From (15) and (23) we also have

$$\nabla_\gamma \epsilon^{\alpha\beta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The contact connection $\Pi_{\gamma\beta}^{\alpha}$ may be further particularized in the following way. The quantities x^α and q_β may be regarded as fundamental in the sense that in terms of them the two other quantities X^α, P^α (and also X_β, P_β) can be expressed owing to the relations (19), (20) and (22). We have then only to consider the two contact projectors

$$Q_{\gamma}^{\cdot\alpha} = \nabla_\gamma x^\alpha, \quad R_{\gamma\beta} = \nabla_\gamma q_\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Since $q = p_\lambda x^\lambda$ is a quadratic function of x^α, we have by (21) the condition of homogeneity

$$x^\alpha q_\alpha = 2q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

from which a relation between the quantities defined by (26) may be obtained by covariant differentiation,

$$Q_{\gamma}^{\cdot\alpha} q_\alpha + R_{\gamma\alpha} x^\alpha = 2 q_\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The simplest solution of this equation is

$$Q_{\gamma}^{\cdot\alpha} = 2 \mathcal{A}_{\gamma}^{\alpha}. \quad R_{\gamma\beta} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

which means that the connection $\Pi_{\gamma\beta}^{\alpha}$ is a *point displacement.* [6])

In terms of the two quantities $Q_{\beta}^{\cdot\alpha}$, $R_{\beta\alpha}$, the derivatives of the four vectors x, q, X, P are

$$\left.\begin{aligned}
\nabla_\gamma x^\alpha &= Q_{\gamma}^{\cdot\alpha}, & \nabla_\gamma x_\beta &= Q_{\gamma\beta}, \\
\nabla_\gamma q^\alpha &= R_{\gamma}^{\cdot\alpha}, & \nabla_\gamma q_\beta &= R_{\gamma\beta}, \\
\nabla_\gamma X^\alpha &= \tfrac{1}{2}(R_{\gamma}^{\cdot\alpha} - Q_{\gamma}^{\cdot\alpha}), & \nabla_\gamma X_\beta &= \tfrac{1}{2}(R_{\gamma\beta} - Q_{\gamma\beta}), \\
\nabla_\gamma P^\alpha &= \tfrac{1}{2}(R_{\gamma}^{\cdot\alpha} + Q_{\gamma}^{\cdot\alpha}), & \nabla_\gamma P_\beta &= \tfrac{1}{2}(R_{\gamma\beta} + Q_{\gamma\beta}),
\end{aligned}\right\} \quad . \quad . \quad (30)$$

6) J. A. SCHOUTEN und J. HAANTJES: Zur allgemeinen projektiven Differentialgeometrie. Compositio Math. 3 (1936), p. 23.

which reduce in the particular case (29) to

$$\left.\begin{array}{ll} \nabla_\gamma x^\alpha = 2\,\mathcal{A}_\gamma^\alpha\,, & \nabla_\gamma x_\beta = 2\,\epsilon_{\gamma\beta}\,, \\ \nabla_\gamma q^\alpha = 0 \quad , & \nabla_\gamma q_\beta = 0 \quad , \\ \nabla_\gamma X^\alpha = -\mathcal{A}_\gamma^\alpha, & \nabla_\gamma X_\beta = -\,\epsilon_{\gamma\beta}, \\ \nabla_\gamma P^\alpha = +\mathcal{A}_\gamma^\alpha, & \nabla_\gamma P_\beta = +\,\epsilon_{\gamma\beta}. \end{array}\right\} \quad \ldots\ldots \quad (31)$$

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Mathematics. — *Conformal representations of an n-dimensional euclidean space with a non-definite fundamental form on itself.*
By J. HAANTJES. (Communicated by Prof. W. v. D. WOUDE).

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Introduction.

It is wellknown that every real conformal point transformation in an n-dimensional $(n > 2)$ euclidean space (R_n) with a *definite* fundamental quadratic form can be brought about by a motion and an inversion or a dilatation [1]). This theorem, which for $n = 3$ is due to LIOUVILLE and is called LIOUVILLE's *theorem*, does not hold in a euclidean space with a fundamental form which is not definite.

The problem with which we are here concerned is to find the extension of the above theorem to a euclidean manifold, the fundamental form of which is not definite. This leads to a new class of conformal transformations (formula (26)). If $\mathfrak{M}$ denotes this class, then, as we shall see, the extension of LIOUVILLE's theorem may be formulated as follows. Every real conformal point transformation in an R_n $(n > 2)$ is composed of a motion and a transformation T, where T is either a dilatation or an inversion or a transformation belonging to the class $\mathfrak{M}$.

It will appear that every transformation belonging to $\mathfrak{M}$ is the product of two inversions. Thus the following theorem holds in any euclidean space. The inversions and motions in an R_n define together the conformal group of point transformations.

§ 1. *Conformal transformations of the fundamental tensor.*

Let $a_{\lambda\varkappa}$ be the fundamental tensor in an n-dimensional RIEMANNian space V_n. A transformation of the form

$$'a_{\lambda\varkappa} = \sigma\, a_{\lambda\varkappa}, \quad \ldots\ldots\ldots\ldots \quad (1)$$

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[1]) S. LIE, Ueber Komplexe, insbesondere Linien- und Kugelkomplexe, mit Anwendung auf die Theorie partieller Differentialgleichungen, Math. Ann., 5 (1872) p. 184.